

# Lecture 4: Understanding the small world phenomena

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Social Networks (Soc 204)  
Princeton University

Wednesday, February 5, 2025



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- ▶ Precept is not a time for lecture; it is a time for new learning.
- ▶ Submit quiz 1 by 11:59pm tonight.

1. Watts, Chapter 3.
2. Watts and Strogatz (1998). Collective dynamics of 'small-world' networks. *Nature* 393, 440-442.
3. Victor (2011). *Scientific Communication As Sequential Art*.
4. Watts (1999). Networks, dynamics, and the small world phenomenon. *American Journal of Sociology*, 105(2):493-527.

Review:

- ▶ empirical vs modeling approaches

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- ▶ empirical vs modeling approaches
- ▶ empirical approach runs into difficulties

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- ▶ empirical approach runs into difficulties
- ▶ models are different
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Today we will see two different small world models and then an empirical assessment

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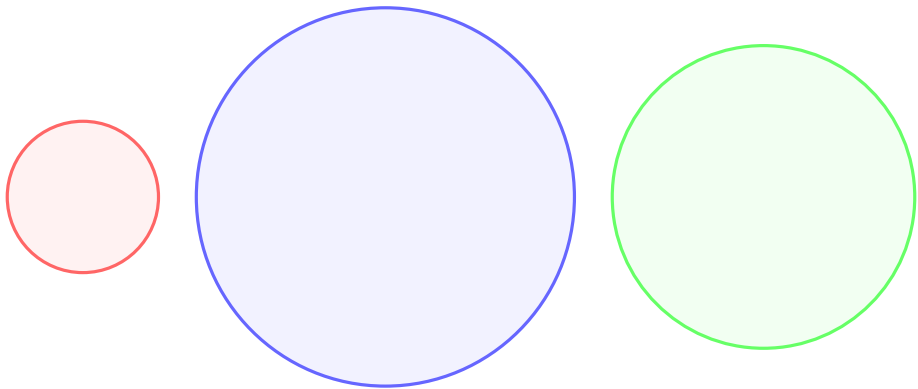
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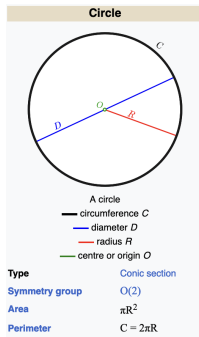
- ▶ small overlapping groups that are linked by people who belong to multiple groups
- ▶ social network evolve
- ▶ not all relationships are equally likely
- ▶ occasionally we do things that are not determined by existing network structure

Can we parameterize these ideas?

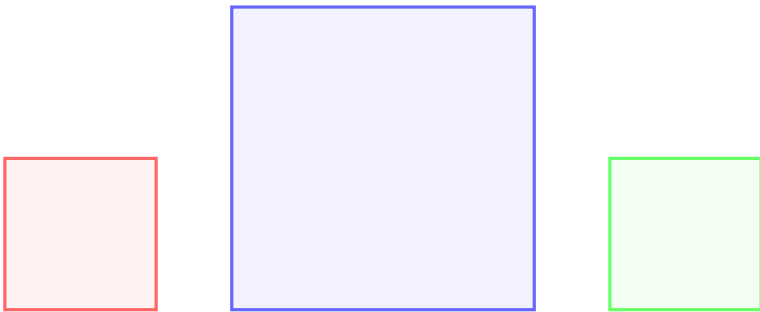


These circles are all different, but they are from the same “family”

- ▶ The family of circles is defined by one parameter: usually the radius
- ▶ If something is a circle with radius  $r$ , then you know other things about it

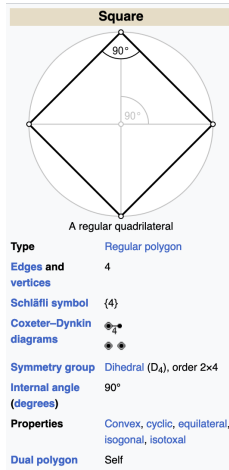


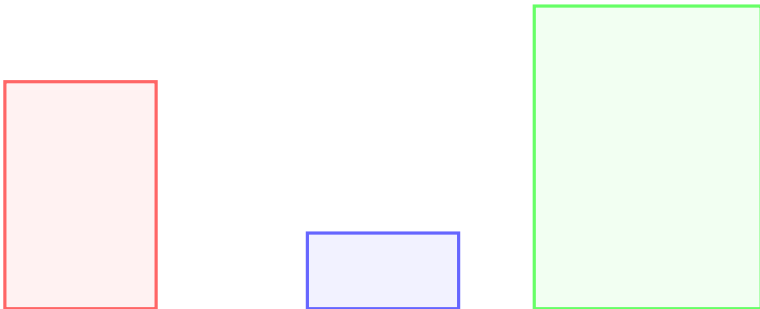




These squares are all different, but they are from the same “family”.

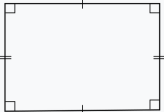
- ▶ The family of squares is defined by one parameter: usually the length
- ▶ If something is a square with length  $l$ , then you know other things about it





These rectangles are all different, but they are from the same “family”.

- ▶ The family of rectangle is defined by two parameters: length and width
- ▶ If something is a rectangle with length  $l$  and width  $w$  then you know other things about it

Rectangle	
	
Rectangle	
Type	quadrilateral, trapezium, parallelogram, orthotope
Edges and vertices	4
Schläfli symbol	$\{ \} \times \{ \}$
Coxeter–Dynkin diagrams	$\bullet \bullet$
Symmetry group	Dihedral $(D_2)$ , $[2]$ , $(^*22)$ , order 4
Properties	convex, isogonal, cyclic Opposite angles and sides are congruent
Dual polygon	rhombus

3.1

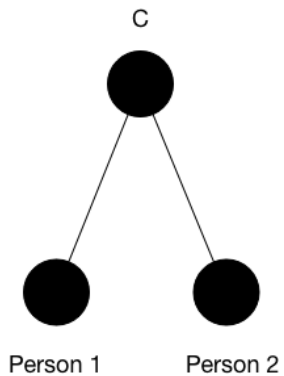
Likelihood  
that  
A meets B

Caveman World

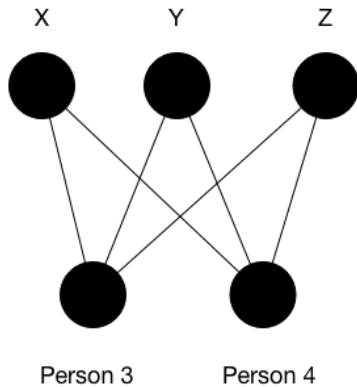
Solaria World

Number of mutual friends  
shared by A and B

The graph features a vertical y-axis and a horizontal x-axis, both ending in arrows. A smooth, S-shaped curve starts at the origin (0,0). The upper portion of the curve, which rises steeply and then levels off at a high value, is labeled 'Caveman World'. The lower portion of the curve, which remains near zero for a long distance before rising steeply at the end, is labeled 'Solaria World'. The y-axis is labeled 'Likelihood that A meets B' and the x-axis is labeled 'Number of mutual friends shared by A and B'.

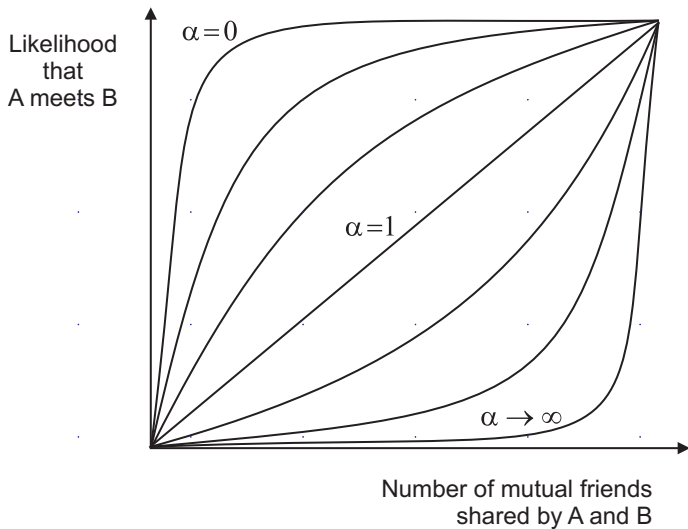


1 mutual friend

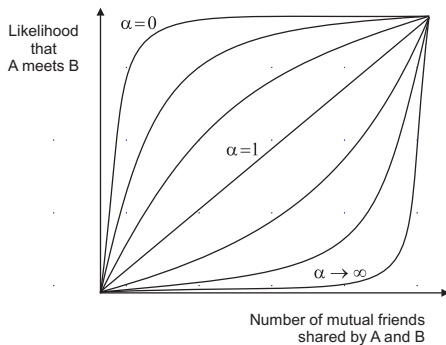


3 mutual friend

3.2



3.2

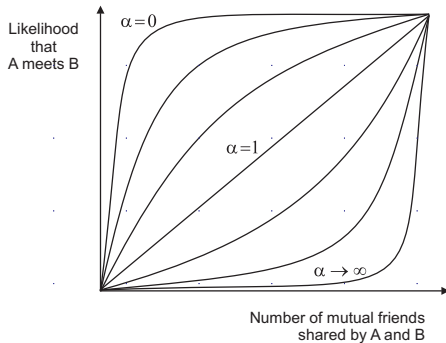


As technology changes do you think we are moving more toward:

1. caveman world ( $\alpha = 0$ )
2. solaria world ( $\alpha \rightarrow \infty$ )



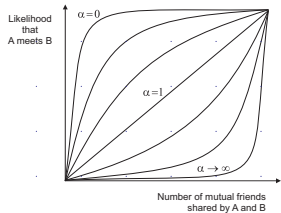
3.2



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3.2



$$R_{i,j} = \begin{cases} 1 & m_{i,j} \geq k \\ \left[ \frac{m_{i,j}}{k} \right]^\alpha (1 - p) + p & k > m_{i,j} > 0, \\ p & m_{i,j} = 0 \end{cases} \quad (5)$$

Note on this process of modeling: the graph came before the equation.

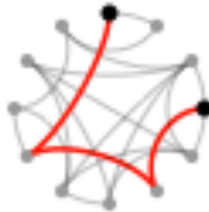
First metric:

Characteristics path length  $L$ : number of edges in shortest path, averaged over all paths

$L$  is defined as the number of edges in the shortest path between two vertices



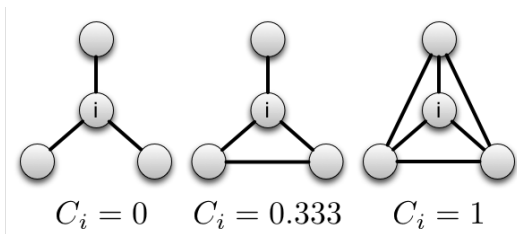
*shortest path  
is 1 edge*



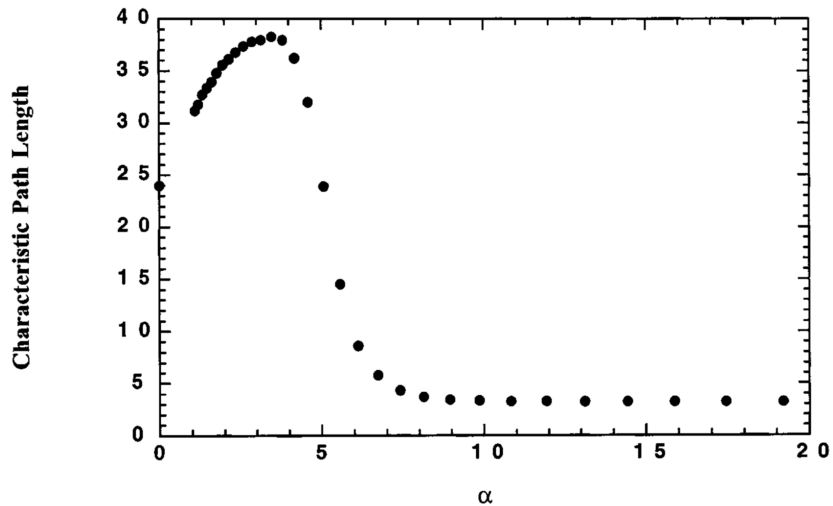
*shortest path  
is 3 edges*

Second metric:

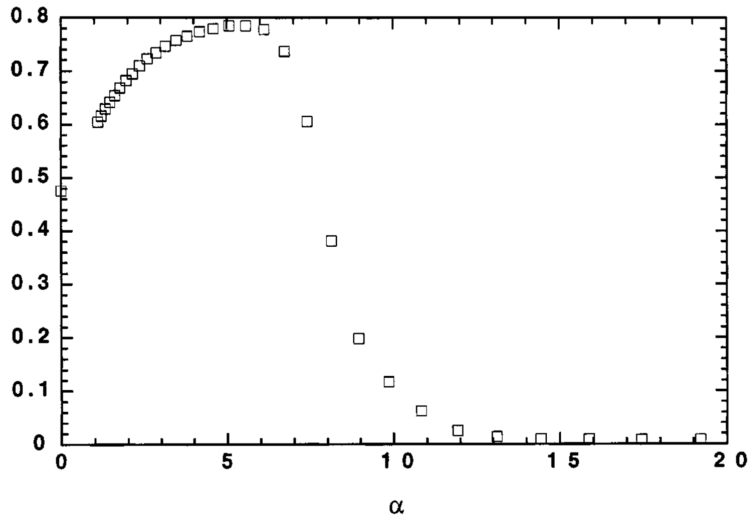
Clustering coefficient  $C$ : probability that two friends of a randomly chosen person are friends

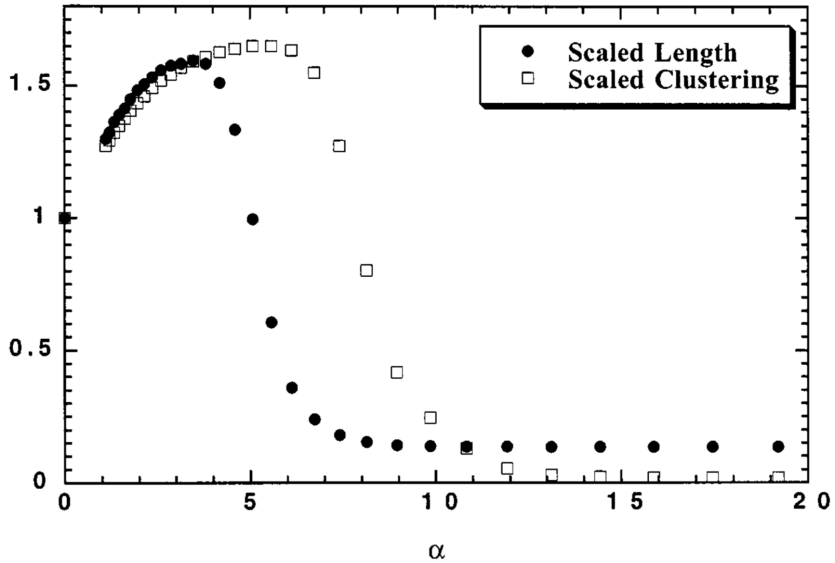


If we simulate lots of people following these rules, what kinds of networks get created?

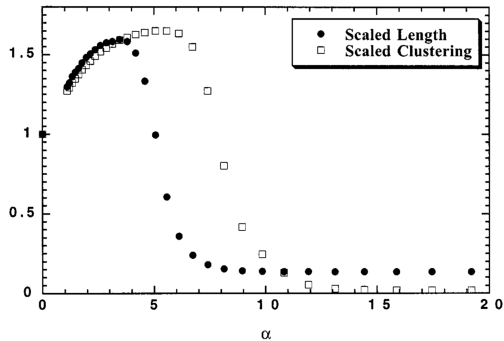
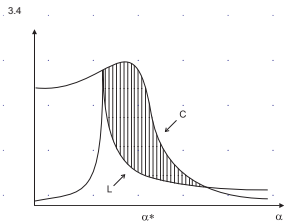


**Clustering Coefficient**

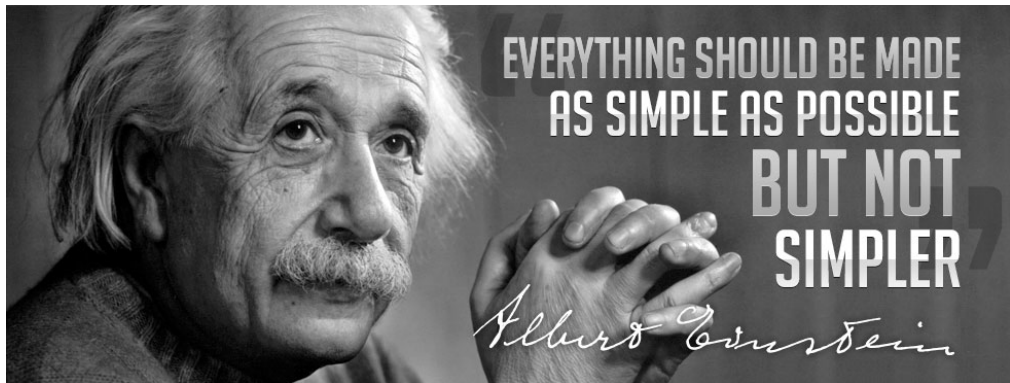






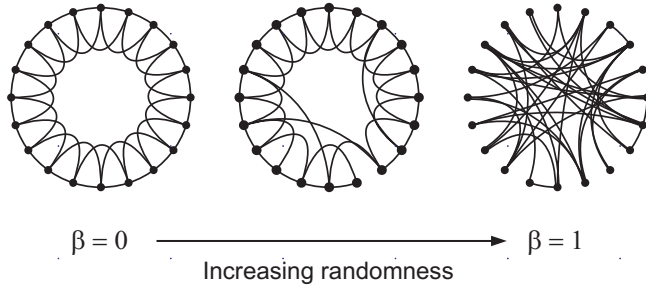


Paper and book look different because paper does not include the disconnected region

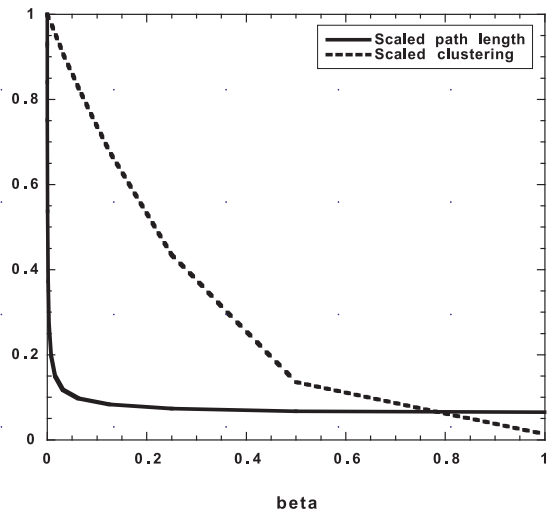


<http://vireomd.net/blog/dhc/einstein-kiss.html>

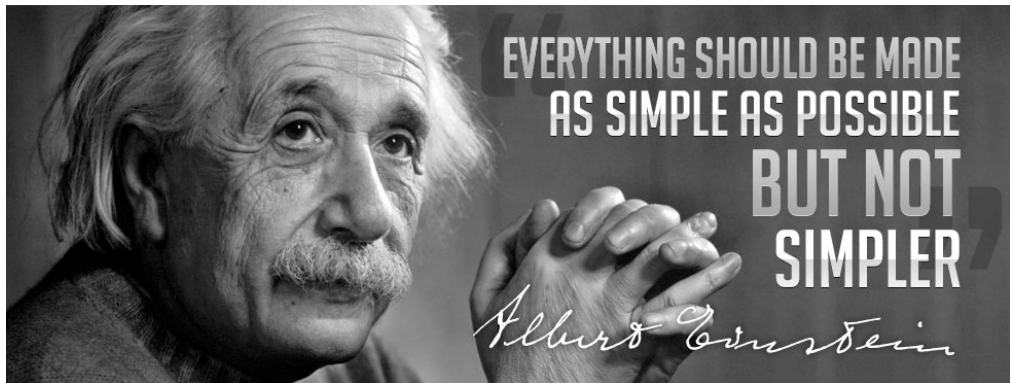
3.6



3.7

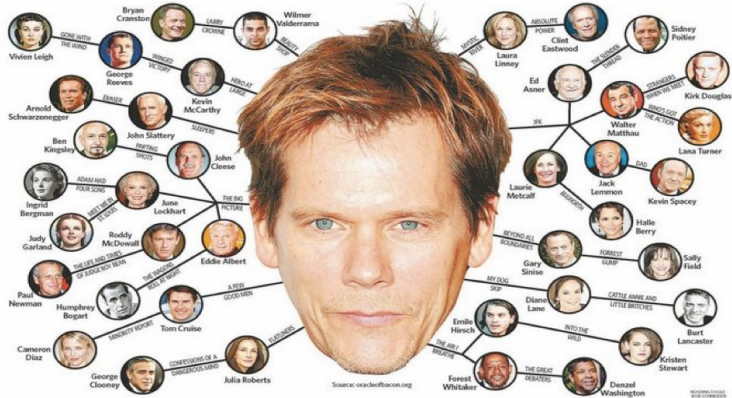


Demo: [http://mathinsight.org/small\\_world\\_network](http://mathinsight.org/small_world_network)



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# Are real networks small world networks?



Small world means:

▶  $L_{actual} \approx L_{random}$

▶  $C_{actual} \gg C_{random}$

---

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# Network ‘Small-World-Ness’: A Quantitative Method for Determining Canonical Network Equivalence

Mark D. Humphries\*, Kevin Gurney

Adaptive Behaviour Research Group, Department of Psychology, University of Sheffield, Sheffield, United Kingdom

## Abstract

**Background:** Many technological, biological, social, and information networks fall into the broad class of ‘small-world’ networks: they have tightly interconnected clusters of nodes, and a shortest mean path length that is similar to a matched random graph (same number of nodes and edges). This semi-quantitative definition leads to a categorical distinction (‘small/not-small’) rather than a quantitative, continuous grading of networks, and can lead to uncertainty about a network’s small-world status. Moreover, systems described by small-world networks are often studied using an equivalent canonical network model – the Watts-Strogatz (WS) model. However, the process of establishing an equivalent WS model is imprecise and there is a pressing need to discover ways in which this equivalence may be quantified.

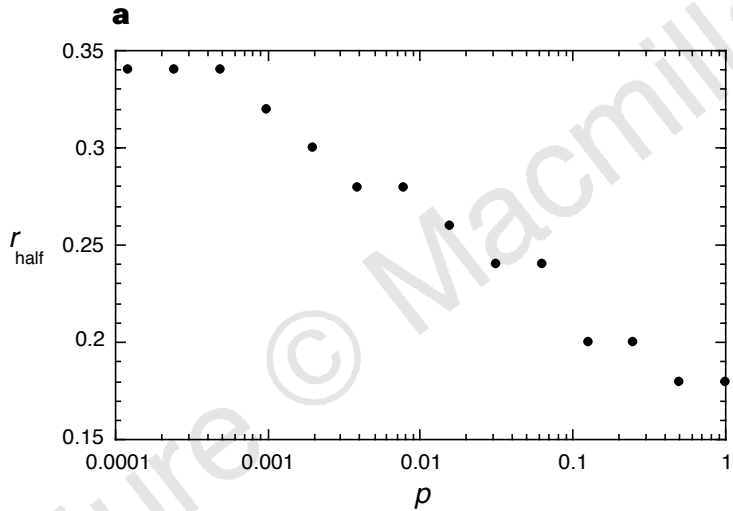
**Methodology/Principal Findings:** We defined a precise measure of ‘small-world-ness’  $S$  based on the trade off between high local clustering and short path length. A network is now deemed a ‘small-world’ if  $S > 1$  – an assertion which may be tested statistically. We then examined the behavior of  $S$  on a large data-set of real-world systems. We found that all these systems were linked by a linear relationship between their  $S$  values and the network size  $n$ . Moreover, we show a method for assigning a unique Watts-Strogatz (WS) model to any real-world network, and show analytically that the WS models associated with our sample of networks also show linearity between  $S$  and  $n$ . Linearity between  $S$  and  $n$  is not, however, inevitable, and neither is  $S$  maximal for an arbitrary network of given size. Linearity may, however, be explained by a common limiting growth process.

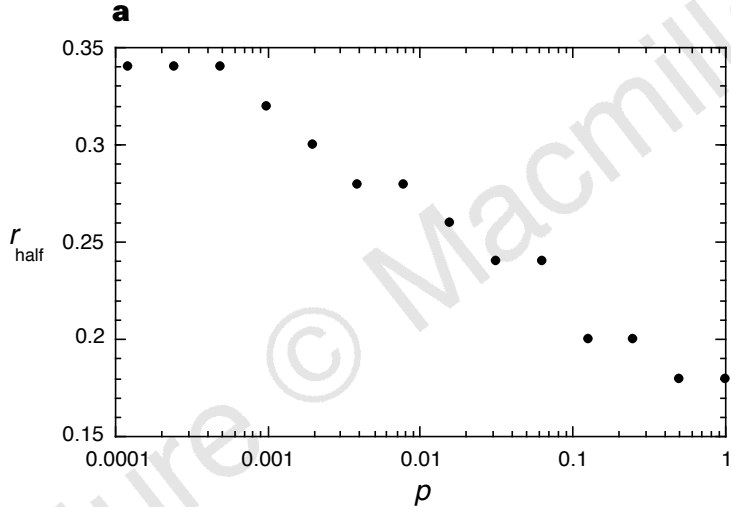
**Conclusions/Significance:** We have shown how the notion of a small-world network may be quantified. Several key properties of the metric are described and the use of WS canonical models is placed on a more secure footing.

We started in sidewalk cafe in Tunis, and we ended up in a worm brain.

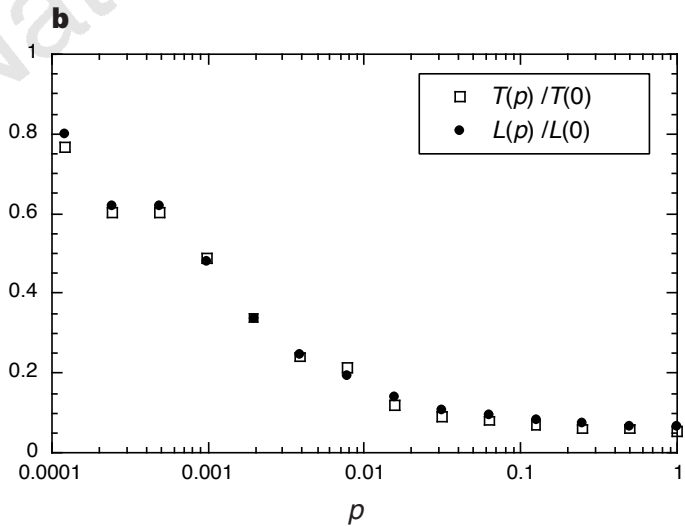


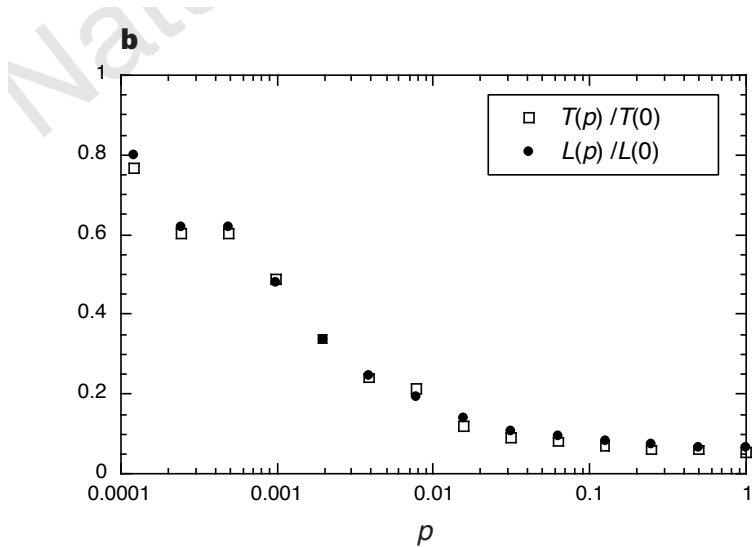
Who cares?





The more shortcuts the less infectious ( $r$ ) a disease needs to be to spread





The more shortcuts the faster a disease spreads

Making length contractions concrete . . . .

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- ▶ shortcuts are key the small world property (characteristic path length changes fast, clustering changes slow)
- ▶ small local changes can have global impacts
- ▶ similarity across networks of different types
- ▶ network structure impacts dynamics

Here's my approach to reading articles. You will get better with practice, and you will find your own way.

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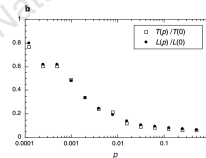
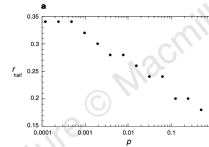
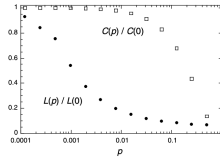
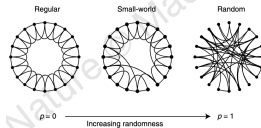
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- ▶ try to explain the paper to your friend

# A second look at the figures/tables in Watts and Strogatz (1998)



**Table 1 Empirical examples of small-world networks**

	$L_{\text{actual}}$	$L_{\text{random}}$	$C_{\text{actual}}$	$C_{\text{random}}$
Film actors	3.65	2.99	0.79	0.00027
Power grid	18.7	12.4	0.080	0.005
<i>C. elegans</i>	2.65	2.25	0.28	0.05

Reflection and feedback:

<http://rb.gy/zczj7r>

Next class. Power law networks

- ▶ Watts, Chapter 4, 101-114.
- ▶ Barabasi, A.L. and Bonabeau, E. (2003) Scale-free networks. *Scientific American*, 50-59.
- ▶ Barabasi, A.L. and Albert, R. (1999) The emergence of scaling in random networks. *Science*, 286:509-512.
- ▶ Liljeros, F. et al. (2001). The web of human sexual contacts. *Nature*, 411:907-908 with comment and rejoinder.