Gradient Descent Code-Along

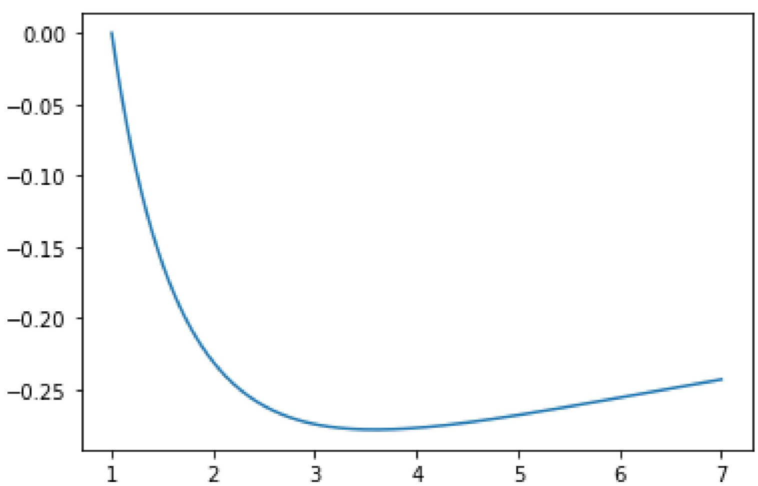
Let's walk through how gradient descent works using code.

In [1]:
```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
```

In [2]:
```
# The objective function
def f(x):
    return -np.log(x) / (1 + x) ## ln(x) / (1+x)
```

In [3]:
```
# Derivative of the objective function
def f_deriv(x):
    return -(1 + 1/x - np.log(x)) / (1 + x)**2
```

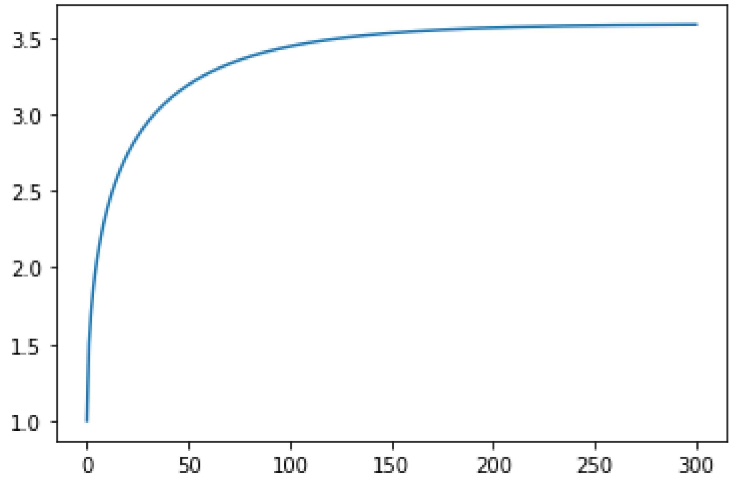
In [4]:
```
# Let's see what it looks like
xs = np.linspace(1, 7, 1000)
plt.plot(xs, f(xs));
```



In [5]:
```
# Initial value and learning rate
x = 1
alpha = 1

# Iterate and apply gradient descent
x_steps = [x]
for i in range(300):
    x = x - alpha * f_deriv(x)
    x_steps.append(x)
#     print(f"{i}: {x}")
```

In [6]:
```
plt.plot(x_steps);
```



Let's see if we can do OLS by Gradient Descent!

In [7]:
```
# Set a random seed.
np.random.seed(42)
```

In [8]:
```
# Randomly generate data from a Poisson(45) distribution.
temp = np.random.poisson(45, 100)
```

In [9]:
```
# View array.
temp
```

Out[9]:
```
array([42, 50, 37, 47, 52, 38, 41, 44, 47, 41, 44, 38, 47, 47, 41, 49, 36,
       40, 41, 46, 58, 47, 34, 29, 43, 52, 40, 37, 51, 49, 51, 42, 53, 42,
       41, 50, 55, 36, 50, 51, 45, 41, 56, 43, 39, 41, 57, 48, 52, 55, 41,
       39, 43, 36, 59, 45, 63, 45, 40, 47, 30, 56, 37, 48, 39, 42, 48, 34,
       41, 49, 45, 48, 49, 58, 42, 40, 52, 46, 55, 42, 48, 47, 35, 46, 48,
       49, 41, 48, 48, 34, 40, 55, 51, 46, 38, 40, 48, 56, 44, 41])
```

In [10]:
```
# Calculate mean and sample variance of array.
print(np.mean(temp))
print(np.var(temp, ddof = 1))
```

```
45.18
45.07838383838384
```

Ohio State Fun Facts:

1. Ohio Stadium can seat 104,944 people. (Source: Wikipedia (https://en.wikipedia.org/wiki/Ohio_Stadium).)
2. Ohio Stadium's record attendance is 110,045 people. (Source: Wikipedia (https://en.wikipedia.org/wiki/Ohio_Stadium).)
3. Ohio State is better than Michigan. (Source: It's just a fact.)
4. Ohio State students enjoy soda. (Source: first-hand knowledge.)

In [11]:
```
# sodas ~ N(200000 + 1000 * temp, 20000)
sodas_sold = 200000 + 1000 * temp + np.round(np.random.normal(0, 20000, 100)
```

In [12]:
```
sodas_sold
```

Out[12]:
```
array([233070., 267128., 241282., 222085., 255464., 245706., 223323.,
       247075., 248164., 218141., 251156., 249216., 268661., 268076.,
       213447., 230243., 246301., 250276., 251301., 323055., 269418.,
       269711., 253080., 242028., 236695., 267179., 224543., 232264.,
       241293., 250637., 297293., 204655., 266725., 209746., 231561.,
       271779., 256286., 214445., 235694., 264592., 230393., 245329.,
       256911., 229968., 281879., 253678., 216497., 251729., 238764.,
       272049., 225150., 236705., 253100., 253315., 234994., 238310.,
       253501., 231933., 275309., 255100., 204782., 274357., 279443.,
       268649., 208613., 232315., 273338., 219847., 249876., 264493.,
       226461., 246809., 184175., 237512., 236949., 215044., 284648.,
       217397., 246199., 244615., 276825., 218283., 258263., 246205.,
       228370., 258242., 244981., 235996., 249396., 226294., 242270.,
       268243., 282720., 221244., 280661., 200958., 244964., 267766.,
       249620., 228546.])
```

$$\text{sodas\_sold}_i = 200000 + 1000 * \text{temp}_i + \varepsilon_i$$

In [13]:
```
# Create dataframe with temp and sodas_sold.
df = pd.DataFrame({'temp': temp,
                   'sodas': sodas_sold})
```

In [14]:
```
# Check the first five rows.
df.head()
```

Out[14]:

	temp	sodas
0	42	233070.0
1	50	267128.0
2	37	241282.0
3	47	222085.0
4	52	255464.0

Our goal is to fit a model here.

- You and I know that our y-intercept β_0 is 200,000.
- You and I know that our slope β_1 is 1,000.
- However, our computer does not know that. Our computer has to estimate $\hat{\beta}_0$ and $\hat{\beta}_1$ from the data.
    - We might say that our **machine** has to... **learn**.

Our workflow:

1. Instantiate model.
2. Select a learning rate α.
3. Select a starting point $\hat{\beta}_{1,0}$.
4. Calculate the gradient of the loss function.
5. Calculate $\hat{\beta}_{1,i+1} = \hat{\beta}_{1,i} - \alpha * \frac{\partial L}{\partial \beta_1}$.
6. Check value of $\left|\hat{\beta}_{1,i+1} - \hat{\beta}_{1,i}\right|$.
7. Repeat steps 4 through 6 until "stopping condition" is met.

Step 1. Instantiate model.

Our model takes on the form:

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

Step 2. Select a learning rate α.

$$\alpha = 0.1$$

In [15]:
```
alpha = 0.1
```

Step 3. Select a starting point.

The zero-th iteration of $\hat{\beta}_1$ is going to start at, say, 20.

$$\hat{\beta}_{1,0} = 20$$

Two points:

- You and I know that the true value of β_1 is 1000. We need the computer to figure (machine to learn) that part out!
- We're going to pretend like the computer already knows the value for β_0. In reality, we'd have to do this for β_0 and for β_1 at the same time.

In [16]:
```
beta_1 = 20
```

Step 4. Calculate the gradient of the loss function with respect to parameter β_1.

The loss function, L, is our mean square error.

$$L = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

$$\Rightarrow L = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right)\right)^2$$

The gradient of this loss function with respect to β_1 is:

$$\frac{\partial L}{\partial \beta_1} = \frac{2}{n} \sum_{i=1}^{n} -x_i \left(y_i - \left(\hat{\beta}_1 x_i + \hat{\beta}_0 \right) \right)$$

In [17]:
```
# Calculate gradient of beta_1.

# Matt's bad code - preserved for posterity.
# def beta_1_gradient(x, y, beta_1, beta_0):
#     n = len(x)
#     # Start gradient at 0.
#     gradient = 0
#     # Begin summation.
#     for i in range(n):
#         # Add gradient for each observation.
#         gradient += -1 * x[i] * (y[i] - (beta_1 * x[i] + beta_0))
#     # Multiply gradient by 2 / n.
#     gradient *= (2 / n)
#     return gradient

# Tim's good code - to see what a differnce numpy makes
def beta_1_gradient(x, y, beta_1, beta_0):
    grads = -x * (y - (beta_1*x + beta_0))
    return 2 * np.mean(grads)
```

Step 5. Calculate $\hat{\beta}_{1,i+1} = \hat{\beta}_{1,i} - \alpha * \frac{\partial L}{\partial \beta_1}$.

In [18]:
```
# Define function to calculate new value of beta_1.
def update_beta_1(beta_1, alpha, gradient):
    beta_1 = beta_1 - alpha * gradient
    return beta_1
```

Step 6. Check value of $\left|\hat{\beta}_{1,i+1} - \hat{\beta}_{1,i}\right|$.

In [19]:
```
def check_update(beta_1, updated_beta_1, tolerance = 0.1):
    return abs(beta_1 - updated_beta_1) < tolerance
```

Step 7: Save final value of $\hat{\beta}_1$.

Putting it all together...

In [20]:

```
def gradient_descent(x, y, beta_1 = 0, alpha = 0.01, max_iter = 100):
    # Set converged = False.
    converged = False

    # Iterate through our observations.
    step = 0
    while not converged:

        # Calculate gradient.
        gradient = beta_1_gradient(x, y, beta_1, 200000)

        # Update beta_1.
        updated_beta_1 = update_beta_1(beta_1, alpha, gradient)

        # Check for convergence.
        converged = check_update(beta_1, updated_beta_1)

        # Overwrite beta_1.
        beta_1 = updated_beta_1

        # Print out current step findings.
        print(f'Iteration {step} with beta_1 value of {beta_1}.')

        # If we've converged, let us know!
        if converged:
            print(f'Our algorithm converged after {step} iterations with a b
        else:
            step += 1

        # If we exceed our step limit, break!
        if step > max_iter:
            break

    # If we didn't converge by the end of our loop, let us know!
    if not converged:
        print("Our algorithm did not converge, so do not trust the value of

    # Return beta_1.
    return beta_1
```

In [21]:
```python
# Call gradient_descent with an initial beta_1 of 20, alpha of 0.01, and 100
gradient_descent(df['temp'],
                 df['sodas'],
                 beta_1 = 20,
                 alpha = 0.01,
                 max_iter = 100)
```

```
Iteration 0 with beta_1 value of 41435.536400000005.
Iteration 1 with beta_1 value of -1644889.1423060799.
Iteration 2 with beta_1 value of 67017530.06550511.
Iteration 3 with beta_1 value of -2728723925.302785.
Iteration 4 with beta_1 value of 111106040061.21898.
Iteration 5 with beta_1 value of -4523926812130.785.
Iteration 6 with beta_1 value of 184201632837141.5.
Iteration 7 with beta_1 value of -7500174724514208.0.
Iteration 8 with beta_1 value of 3.053861142930322e+17.
Iteration 9 with beta_1 value of -1.2434467492892207e+19.
Iteration 10 with beta_1 value of 5.062966998015906e+20.
Iteration 11 with beta_1 value of -2.0614983985161323e+22.
Iteration 12 with beta_1 value of 8.393844259206108e+23.
Iteration 13 with beta_1 value of -3.4177383547094688e+25.
Iteration 14 with beta_1 value of 1.3916073613637641e+27.
Iteration 15 with beta_1 value of -5.666235525412065e+28.
Iteration 16 with beta_1 value of 2.307132451353081e+30.
Iteration 17 with beta_1 value of -9.393997344823367e+31.
Iteration 18 with beta_1 value of 3.82497268688642e+33.
Iteration 19 with beta_1 value of -1.5574217788649178e+35.
Iteration 20 with beta_1 value of 6.341385405439864e+36.
Iteration 21 with beta_1 value of -2.58203457830376e+38.
Iteration 22 with beta_1 value of 1.0513321833170987e+40.
Iteration 23 with beta_1 value of -4.280730277455898e+41.
Iteration 24 with beta_1 value of 1.742993508532273e+43.
Iteration 25 with beta_1 value of -7.096981528561027e+44.
Iteration 26 with beta_1 value of 2.88969216294725e+46.
Iteration 27 with beta_1 value of -1.1766017373715581e+48.
Iteration 28 with beta_1 value of 4.790792826090522e+49.
Iteration 29 with beta_1 value of -1.9506766965849305e+51.
Iteration 30 with beta_1 value of 7.942609319018793e+52.
Iteration 31 with beta_1 value of -3.23400812164352e+54.
Iteration 32 with beta_1 value of 1.3167975549058353e+56.
Iteration 33 with beta_1 value of -5.361630940261188e+57.
Iteration 34 with beta_1 value of 2.183105993208028e+59.
Iteration 35 with beta_1 value of -8.888996334664993e+60.
Iteration 36 with beta_1 value of 3.619350415578213e+62.
Iteration 37 with beta_1 value of -1.4736981474118123e+64.
Iteration 38 with beta_1 value of 6.000486220779624e+65.
Iteration 39 with beta_1 value of -2.443229975487281e+67.
Iteration 40 with beta_1 value of 9.948148355791073e+68.
Iteration 41 with beta_1 value of -4.0506074623241633e+70.
Iteration 42 with beta_1 value of 1.6492939416494543e+72.
Iteration 43 with beta_1 value of -6.715463128092916e+73.
Iteration 44 with beta_1 value of 2.7343485527918488e+75.
Iteration 45 with beta_1 value of -1.1133501689373623e+77.
Iteration 46 with beta_1 value of 4.5332501498656376e+78.
Iteration 47 with beta_1 value of -1.8458125300210919e+80.
Iteration 48 with beta_1 value of 7.51563179473748e+81.
Iteration 49 with beta_1 value of -3.060154829126849e+83.
```

```
Iteration 50 with beta_1 value of 1.2460093620852375e+85.
Iteration 51 with beta_1 value of -5.073401239789703e+86.
Iteration 52 with beta_1 value of 2.065746929607653e+88.
Iteration 53 with beta_1 value of -8.411143088222072e+89.
Iteration 54 with beta_1 value of 3.4247819535175585e+91.
Iteration 55 with beta_1 value of -1.3944753175776513e+93.
Iteration 56 with beta_1 value of 5.677913040087256e+94.
Iteration 57 with beta_1 value of -2.3118872083584165e+96.
Iteration 58 with beta_1 value of 9.413357384017132e+97.
Iteration 59 with beta_1 value of -3.8328555527650227e+99.
Iteration 60 with beta_1 value of 1.56063146113044e+101.
Iteration 61 with beta_1 value of -6.354454332914035e+102.
Iteration 62 with beta_1 value of 2.5873558796412734e+104.
Iteration 63 with beta_1 value of -1.053498868225297e+106.
Iteration 64 with beta_1 value of 4.289552411730306e+107.
Iteration 65 with beta_1 value of -1.7465856345890524e+109.
Iteration 66 with beta_1 value of 7.111607660068937e+110.
Iteration 67 with beta_1 value of -2.8956475141655893e+112.
Iteration 68 with beta_1 value of 1.1790265896378315e+114.
Iteration 69 with beta_1 value of -4.800666145560151e+115.
Iteration 70 with beta_1 value of 1.9546968358200174e+117.
Iteration 71 with beta_1 value of -7.958978200345081e+118.
Iteration 72 with beta_1 value of 3.240673071790908e+120.
Iteration 73 with beta_1 value of -1.3195113359872477e+122.
Iteration 74 with beta_1 value of 5.372680696965997e+123.
Iteration 75 with beta_1 value of -2.187605144745039e+125.
Iteration 76 with beta_1 value of 8.90731561996127e+126.
Iteration 77 with beta_1 value of -3.626809515610871e+128.
Iteration 78 with beta_1 value of 1.4767352840903096e+130.
Iteration 79 with beta_1 value of -6.012852590936194e+131.
Iteration 80 with beta_1 value of 2.448265215156672e+133.
Iteration 81 with beta_1 value of -9.968650441857724e+134.
Iteration 82 with beta_1 value of 4.0589553377120936e+136.
Iteration 83 with beta_1 value of -1.6526929627669086e+138.
Iteration 84 with beta_1 value of 6.729302990357276e+139.
Iteration 85 with beta_1 value of -2.7399837571897537e+141.
Iteration 86 with beta_1 value of 1.1156446663824663e+143.
Iteration 87 with beta_1 value of -4.542592701002816e+144.
Iteration 88 with beta_1 value of 1.8496165552527184e+146.
Iteration 89 with beta_1 value of -7.531120720353598e+147.
Iteration 90 with beta_1 value of 3.066461485947816e+149.
Iteration 91 with beta_1 value of -1.248577256156344e+151.
Iteration 92 with beta_1 value of 5.083856985436909e+152.
Iteration 93 with beta_1 value of -2.0700042164743167e+154.
Iteration 94 with beta_1 value of 8.428477568302805e+155.
Iteration 95 with beta_1 value of -3.43184006844099e+157.
Iteration 96 with beta_1 value of 1.3973491843472545e+159.
Iteration 97 with beta_1 value of -5.689614620890404e+160.
Iteration 98 with beta_1 value of 2.3166517644171873e+162.
Iteration 99 with beta_1 value of -9.43275732221275e+163.
Iteration 100 with beta_1 value of 3.84075466440001e+165.
Our algorithm did not converge, so do not trust the value of beta_1.
```

Out[21]:

```
3.84075466440001e+165
```

What should we do?

In [22]:

```
gradient_descent(df['temp'],
                 df['sodas'],
                 beta_1 = 20,
                 alpha = 0.0001,
                 max_iter = 100)
```

```
Iteration 0 with beta_1 value of 434.155364.
Iteration 1 with beta_1 value of 675.536706489392.
Iteration 2 with beta_1 value of 816.2205115697993.
Iteration 3 with beta_1 value of 898.214972317203.
Iteration 4 with beta_1 value of 946.0036398856907.
Iteration 5 with beta_1 value of 973.8562134272973.
Iteration 6 with beta_1 value of 990.0894731594049.
Iteration 7 with beta_1 value of 999.5506714625496.
Iteration 8 with beta_1 value of 1005.0649227471749.
Iteration 9 with beta_1 value of 1008.2787827948905.
Iteration 10 with beta_1 value of 1010.1519104187804.
Iteration 11 with beta_1 value of 1011.2436216455569.
Iteration 12 with beta_1 value of 1011.8799015164366.
Iteration 13 with beta_1 value of 1012.2507432410217.
Iteration 14 with beta_1 value of 1012.4668801816782.
Iteration 15 with beta_1 value of 1012.5928508425271.
Iteration 16 with beta_1 value of 1012.6662700708484.
Our algorithm converged after 16 iterations with a beta_1 value of 1012.6662700
708484.
```

Out[22]:

```
1012.6662700708484
```