Introduction to Graph Theory

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Contents

Preface

Graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics, and its results have applications in many areas of the computing, social, and natural sciences. The design of this book permits usage in a one-semester introduction at the undergraduate or beginning graduate level, or in a patient two-semester introduction. No previous knowledge of graph theory is assumed. Many algorithms and applications are included, but the focus is on understanding the structure of graphs and the techniques used to analyze problems in graph theory.

Many textbooks have been written about graph theory. Due to its emphasis on both proofs and applications, the initial model for this book was the elegant text by J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications* (Macmillan/North-Holland [1976]). Graph theory is still young, and no consensus has emerged on how the introductory material should be presented. Selection and order of topics, choice of proofs, objectives, and underlying themes are matters of lively debate. Revising this book dozens of times has taught me the difficulty of these decisions. This book is my contribution to the debate.

Features

Various features of this book facilitate students' efforts to understand the material. I include an early discussion of proof techniques, more than 850 exercises of varying difficulty, more than 300 illustrations, and many examples. I have tried to include the statements and illustrations that are needed in class to complete the flow of argument.

This book contains much more material than other introductions to graph theory. Collecting the advanced material as a final optional chapter of "additional topics" permits usage at different levels. The undergraduate introduction consists of the first seven chapters, leaving Chapter 8 as topical reading for interested students. The first five sections illustrate proof techniques while developing fundamental properties of graphs. This assists undergraduate students who are beginning to write proofs of their own; with advanced students, the reminder of elementary techniques can be ignored. Advanced students also may have previous exposure to graphs from a general course in combinatorics, discrete structures, or algorithms. A graduate course can treat most of Chapters 1 and 2 as recommended reading, moving rapidly to Chapter 3 in class and reaching some topics of Chapter 8. Chapter 8 can also be used as the basis for a second course in graph theory.

Most of the exercises require written proofs. Many undergraduates begin graph theory with little practice at presenting explanations, and this hinders their appreciation of graph theory and other mathematics. The intellectual discipline of justifying an argument is valuable independently of mathematics; I hope that students will become comfortable with this. In writing solutions to exercises, students should be careful in their use of language ("say what you mean"), and they should be intellectually honest ("mean what you say"), which includes acknowledging when they have left gaps.

Although many terms in graph theory suggest their definitions, the quantity of terms remains an obstacle to fluency. Mathematicians like to start with a clean list of definitions, but most students prefer to use one concept before receiving the next. By request of instructors, I have postponed many definitions to accompany their applications. For example, the definition of strongly connected digraph appears in Section 2.4 with Eulerian circuits, the definition of Cartesian product appears in 5.1 with coloring problems, and the definition of line graph appears in 4.2 with Menger's Theorem and in 6.1 with edge-coloring.

Many results in graph theory have several proofs; illustrating this can increase students' flexibility in trying multiple approaches to a problem. I include some alternative proofs as remarks and others as exercises. Many exercises have hints. Exercises marked "(−)" or "(+)" are easier or more difficult respectively than unmarked problems. Those marked "(−)" may be suitable as exam problems. Exercises marked "(!)" are especially valuable, instructive, or entertaining. Exercises that relate several concepts usually appear when the last is introduced. Many exercises are referenced in the text where relevant concepts are discussed. An exercise in the current section is cited by giving only its index exercise among the exercises of that section. Other cross-references are by Chapter.Section.Item.

Organization

I have sought a development that is intellectually coherent and displays a gradual (not monotonic) increase in difficulty of proofs and in algorithmic complexity. Most graph theorists agree that the König-Egerváry Theorem deserves an independent proof without network flow. Also, students find connectivity more abstract than matching. I therefore treat matching first and use matching to prove Menger's Theorem. Both matching and connectivity are used in the coloring material. The gradual rise in difficulty also puts Eulerian graphs early and Hamiltonian and planar graphs later.

When students discover that the coloring and Hamiltonian cycle problems lack good algorithms, they may become curious about NP-completeness. Section 6.3 can be read to satisfy this curiosity; it also can be discussed after Chapter 7. Presentation of NP-completeness via formal languages can be technically abstract, so many students appreciate a more "nuts and bolts" discussion in the context of graph problems. NP-completeness proofs also illustrate the variety and usefulness of "graph transformation" arguments.

Turán's Theorem uses only elementary ideas and hence appears in Chapter 1. The applications that motivate Sperner's Lemma involve advanced material, so this waits until Chapter 8. Trees and distance appear together (Chapter 2); many exercises relate distance and trees, and algorithms to compute distances or Eulerian circuits produce or use trees. Petersen's Theorem on 2-factors (Chapter 3) uses Eulerian circuits and bipartite matching. Menger's Theorem appears before network flow (Chapter 4), and separate applications are provided for network flow. The $k-1$-connectedness of k-color-critical graphs (Chapter 5) uses bipartite matching. Section 5.3 offers a brief introduction to perfect graphs, emphasizing chordal graphs. The main discussion of perfect graphs (with the proof of the Perfect Graph Theorem) appears in Chapter 8. Graph orientations appear in many exercises and examples, including the Gallai-Roy Theorem and Stanley's connection between the chromatic polynomial and acyclic orientations (Chapter 5). The proof of Vizing's Theorem for simple graphs (Chapter 6) is algorithmic and short. The proof of Kuratowski's Theorem (Chapter 7) uses Thomassen's approach and fits into one energetic lecture.

Chapter 8 contains highlights of advanced material and is not intended for an undergraduate course. It assumes more sophistication than earlier chapters and is written more tersely. Its sections are independent; each selects appealing results from a large topic that merits a chapter of its own. Some of these sections become more difficult near the end; an instructor may prefer to sample early material in several sections rather than present one completely. I will treat advanced graph theory more thoroughly in *The Art of Combinatorics*, of which volumes I-II are devoted to graph theory, with matroids in Volume III and random graphs in Volume IV.

Design of Courses

I intend the 23 sections in Chapters 1-7 for a pace of roughly two sections per week, skipping optional material as needed to balance the coverage of topics. With beginning students, instructors may want to spend more time on Chapters 1-2. Some items are explicitly labeled "optional".

For a slower one-semester course, the following items can be omitted without damage to continuity. 1.3: counting of subgraphs, even graphs. 1.4: the 2-switch theorem. 2.1: distance sums. 2.2: proof of the Matrix Tree Theorem. 2.3: Huffman coding. 2.4: directed Eulerian circuits and street-sweeping. 3.2: Hopcroft-Karp algorithm. 3.3: everything after Petersen's Theorem. 4.1: algorithm for blocks. 4.2: applications of Menger's Theorem. 4.3: supplies and demands. 5.1: proof of Brooks' Theorem. 5.2: everything after edge-connectivity of k-critical graphs. 5.3: inclusion-exclusion computation and acyclic orientations. 6.1: characterization of line graphs. 6.2: cycles in digraphs. 6.3: everything. 7.1: Jordan curve proof and outerplanar graphs. 7.2: bridges and planarity testing. 7.3: 4-color discussion and genus.

Courses that start in Chapter 3 may wish to include from the first two chapters such topics as Turán's Theorem, graphic sequences, the matrix tree theorem, Kruskal's algorithm, and algorithms for Eulerian circuits. Courses that introduce graph theory over two quarters can cover the first seven chapters thoroughly.

A one quarter course must aim for the highlights. 1.1: adjacency matrix and isomorphism. 1.2: all. 1.3: degree-sum formula and Turán's Theorem. 1.4: large bipartite subgraphs and Havel-Hakimi test. 2.1: through definition of distance. 2.2: through statement of Matrix Tree Theorem. 2.3: Kruskal's algorithm and possibly Dijkstra's algorithm. 2.4: Eulerian graph characterization and Chinese Postman Problem. 3.1: all. 3.2: none. 3.3: statement of Tutte's Theorem and proof of Petersen's results. 4.1: through definition of blocks, skipping Harary graphs. 4.2: through open ear decomposition, plus statement of Menger's Theorem(s). 4.3: duality between flows and cuts, statement of Max-flow = Min-cut. 5.1: through Szekeres-Wilf theorem. 5.2: Mycielski's construction only. 5.3: through chromatic recurrence, plus perfection of chordal graphs. 6.1: through Vizing's Theorem. 6.2: through Ore's condition, plus the Chvátal-Erdös condition. 6.3: none. 7.1: nonplanarity of K_5 and $K_{3,3}$, examples of dual graphs, and Euler's formula. 7.2: statement and examples of Kuratowski's Theorem and Tutte's Theorem. 7.3: the 5-color Theorem, Tait's Theorem, and Grinberg's Theorem.

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Feedback

Of course, as soon as I saw the first copies of the finished book I began to find minor typographical errors, errant cross-references, etc. Many of these were corrected for the second printing. I found in addition, partly based on feedback from users, that some definitions could be expressed more clearly. Many changes of this sort were also made, with the aim of helping students become comfortable with graphs more quickly.

I mention two global examples. 1) I now define "simple graph" as a special case of "graph" and do not use the word "multigraph"; this permits precise statement of results while minimizing confusion for beginning students. Contexts where vertex degrees are relevant but multiple edges are not, such as vertex coloring in Chapter 5, require restriction to simple graphs. 2) The convention that all graphs in the book are finite (with rare explicit exceptions) is declared early and clearly, to eliminate concern about exercises that did not state a finiteness condition.

I have tried to eliminate errors, but surely one still remains. I welcome corrections and suggestions, including comments on topics, attributions of results, suggestions for exercises, typographical errors, omissions from the glossary or index, etc. Please send these to me at

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I also maintain a web page that lists changes made between printings and changes still needing to be made. Please visit!

http://www.math.uiuc.edu/~west/igterr.html

With enough readers and printings, we can get it all right.

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