Chaos and the Lorenz System

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March 17, 2021

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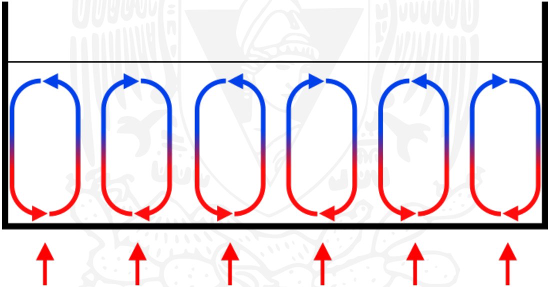
The Lorenz Equations

In 1960, Edward Lorenz proposed an innovative mathematical model intended to predict the weather. The crucial ingredient for this model was **convection**.



In this project we pretend to solve computationally the Lorenz Equations and analyze their consequences.

Las ecuaciones de Lorenz

To be able to solve the governing equations for convection using the computational power of his time, Lorenz simplified his equations and reduced them to the following system:

Lorenz Equations

$$\dot{x} = \sigma(y - x) \;\; ; \quad \dot{y} = x(\rho - z) - y \quad ; \quad \dot{z} = xy - \beta z$$

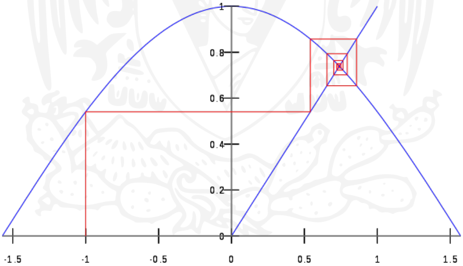
where $\rho > 0$ quantifies the difference in temperature ΔT, $\beta > 0$ quantifies the relative height of the layer of fluid and $\sigma > 0$ quantifies the loss of energy due to viscosity and thermal conduction.

Las ecuaciones de Lorenz

Before solving this equations is important to notice that we can find the fixed point of the systems. This points are:

Fixed points of the Lorenz System

$$\vec{x*} = (0,0,0) \quad , \quad C^{\pm} = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1)$$



The 4th Order Runge-Kutta method

To solve the Lorenz system, we used the 4th order Runge-Kutta method programmed in Fortran. This method takes a system of 3 equations:

ODE system

$$\frac{dx}{dt} = f(x,y,z) \quad , \quad \frac{dy}{dt} = g(x,y,z) \quad , \quad \frac{dz}{dt} = u(x,y,z)$$

with $x(t_0) = x_0$, $y(t_0) = y_0$ and $z(t_0) = z_0$, and solves it iteratively advancing the solution of x y y z to $t_i = t_{i-1} + h$ with $h = (b-a)/n$ so that the solutions are:

The 4th Order Runge-Kutta method

$$x_{i+1} = x_i + (K_0 + 2K_1 + 2K_2 + K_3)/6 \quad , \quad y_{i+1} = y_i + (L_0 + 2L_1 + 2L_2 + L_3)/6$$

The 4th Order Runge-Kutta method

The 4th Order Runge-Kutta method

$$z_{i+1} = z_i + (M_0 + 2M_1 + 2M_2 + M_3)/6$$

with the coefficients given by:

$$K_0 = hf(x_i, y_i, z_i) \quad , \quad L_0 = hg(x_i, y_i, z_i) \quad \text{y} \quad M_0 = hu(x_i, y_i, z_i)$$

$$K_n = hf(x_i + \frac{1}{2}K_{n-1}, y_i + \frac{1}{2}L_{n-1}, z_i + \frac{1}{2}M_{n-1}) \quad \forall n \in 1, 2$$

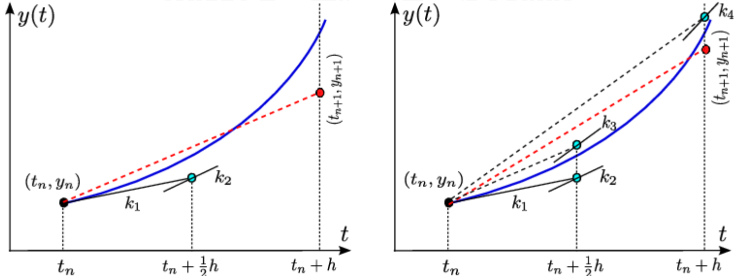
$$L_n = hg(x_i + \frac{1}{2}K_{n-1}, y_i + \frac{1}{2}L_{n-1}, z_i + \frac{1}{2}M_{n-1}) \quad \forall n \in 1, 2$$

$$M_n = hu(x_i + \frac{1}{2}K_{n-1}, y_i + \frac{1}{2}L_{n-1}, z_i + \frac{1}{2}M_{n-1}) \quad \forall n \in 1, 2$$

The 4th Order Runge-Kutta method

$$K_3 = hf(x_i + K_2, y_i + L_2, z_i + M_2) \quad ; \quad L_3 = hg(x_i + K_2, y_i + L_2, z_i + M_2)$$

$$M_3 = hu(x_i + K_2, y_i + L_2, z_i + M_2)$$

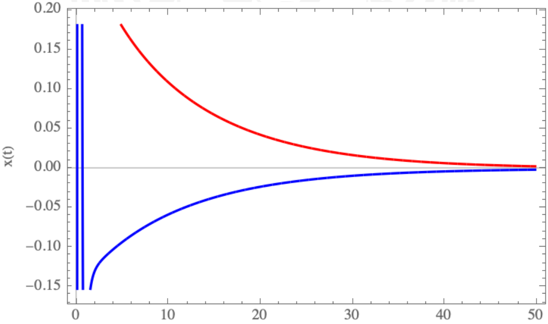


Global stability, $0 < \rho < 1$

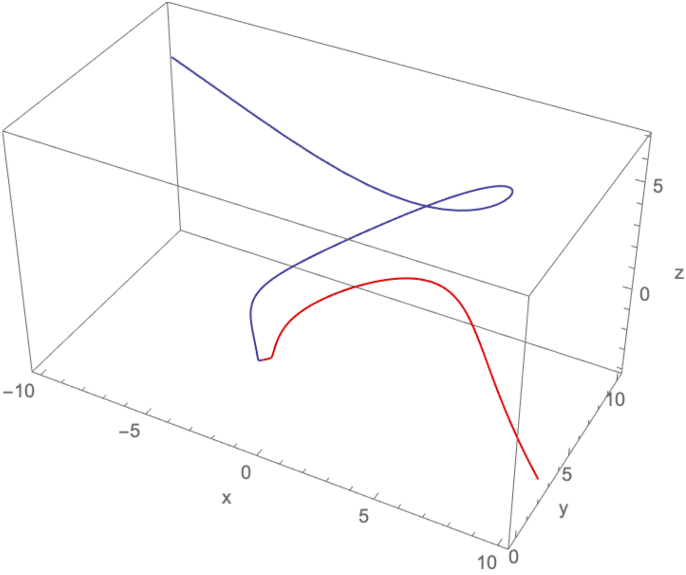
Mathematically, near the origin, the solution of the system remains stable and tends to zero. Mathematically, $z(t) \to 0$ for $t \to \infty$. If $\rho < 1$, the trajectories in x and y approximate the origin as well, thus we conclude that the origin is **globally stable**.

Two solutions

$\sigma = 10$, $\beta = 8/3$ y $\rho = 0.9$ for $\vec{x_0} = (-10, 10, 5)$ and for $\vec{x_0} = (10, 3, -4)$.



Global stability, $0 < \rho < 1$



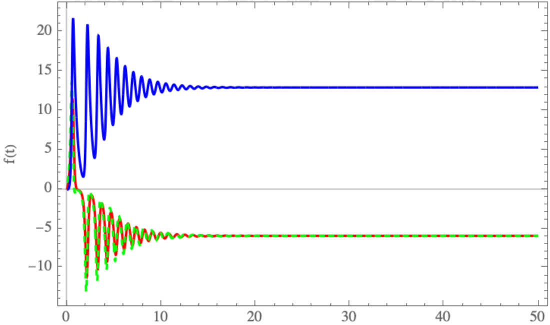
Stability in C^+ y C^- for $1 < \rho < \rho_c$

Mathematically if $\rho > 1$, the trajectories in x and y are on a cycle and the motion around C^+ and C^- is stable for:

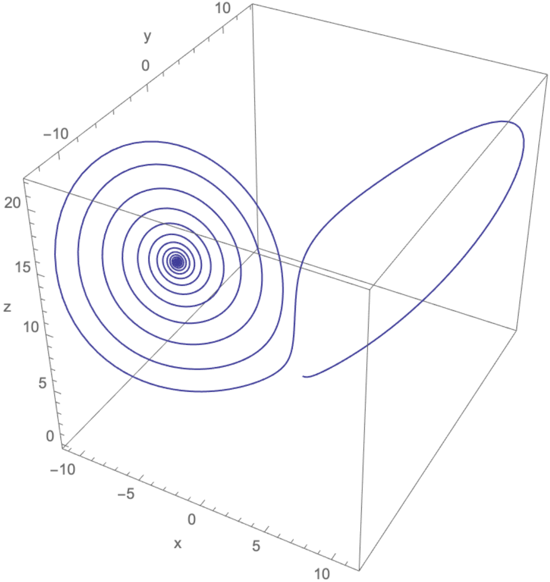
$$1 < \rho < \rho_c = \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1} \quad \text{assuming that: } \sigma - \beta - 1 > 0$$

One solution:

$$\sigma = 10,\ \beta = 8/3 \text{ and } \rho = 14 \text{ for } \vec{x_0} = (0, 1, 0).$$



Stability in C^+ y C^- for $1 < \rho < \rho_c$

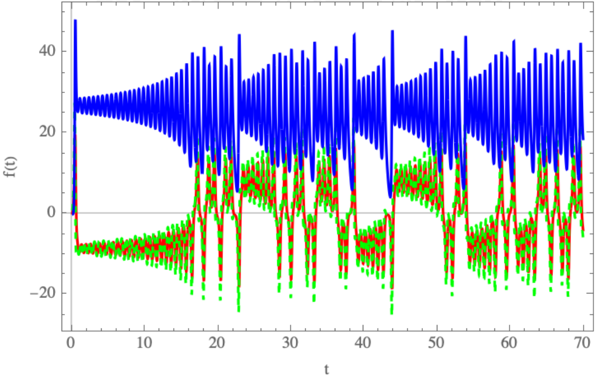


Chaos for $\rho > \rho_c$

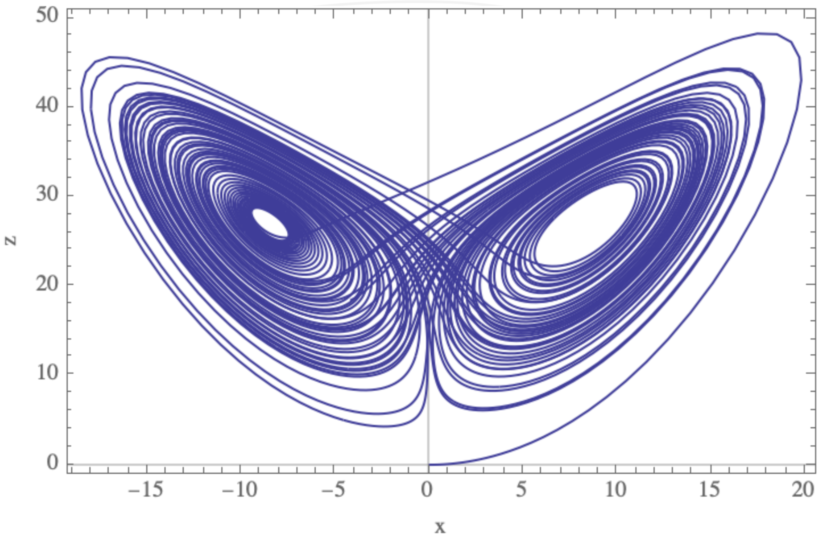
In this case, the trajectories are constantly repelled by the C^+ and C^+ points like in a pinball machine and don't converge to a stable point.

The Lorenz Attractor

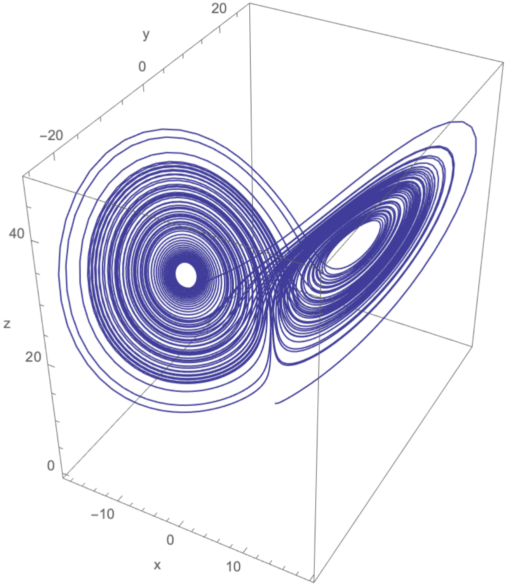
$\sigma = 10$, $\beta = 8/3$ and $\rho = 28$ for $\vec{x_0} = (0, 1, 0)$.



Chaos for $\rho > \rho_c$



Chaos for $\rho > \rho_c$

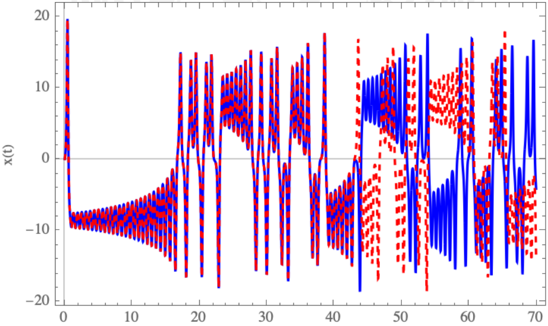


The butterfly effect

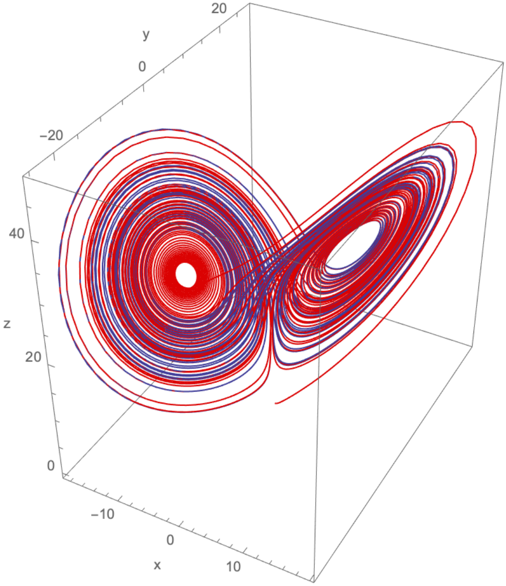
This system exhibits **extreme sensibility to initial conditions**. At first, the trajectories of very similar initial conditions, for example separated by $\delta_0 = 1 \times 10^{-12}$, are very similar. Nevertheless, when time goes by, the trajectories diverge completely.

Two solutions

$\sigma = 10$, $\beta = 8/3$ and $\rho = 28$ for $\vec{x_0} = (0, 1, 0)$ y para $\vec{x_0} = (\delta_0, 1 + \delta_0, \delta_0)$.

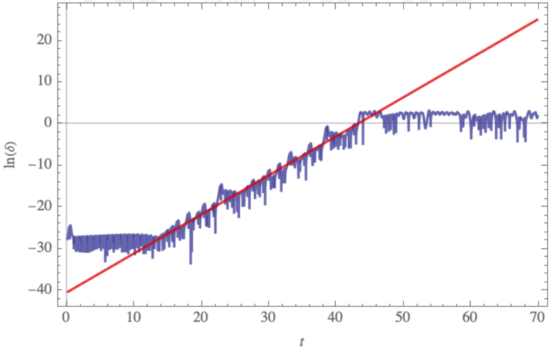


The butterfly effect



The butterfly effect

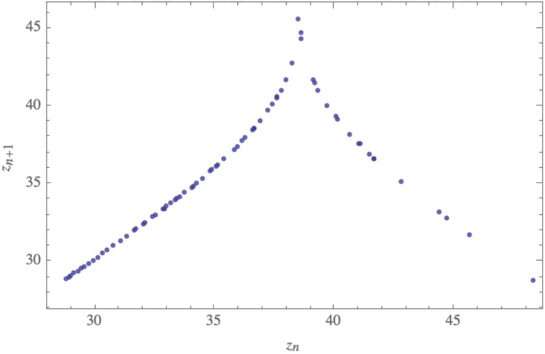
Let's suppose we have a point $x(t)$ in the attractor at time t and we take another point $x(t) + \delta(t)$ such that initially $\delta(0) = \|\delta_0\| \ll 1$. If we plot $\ln(\|\delta(t)\|)$ vs t:



We find a graph similar to a linear fit with positive slope $\lambda \approx 0.93$. This means that $\|\delta(t)\| \sim \|\delta_0\| e^{\lambda t}$. This λ is known as the **Lyapunov coefficient**.

The Lorenz map

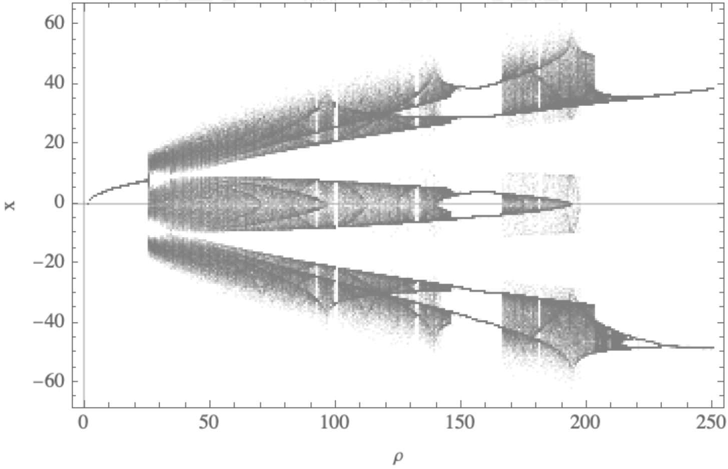
In 1963 Lorenz found a way to analyze the dynamics of his strange attractor. The idea was that the local maxima of $z(t)$, z_n should predict the z_{n+1}. If we plot z_{n+1} vs z_n:



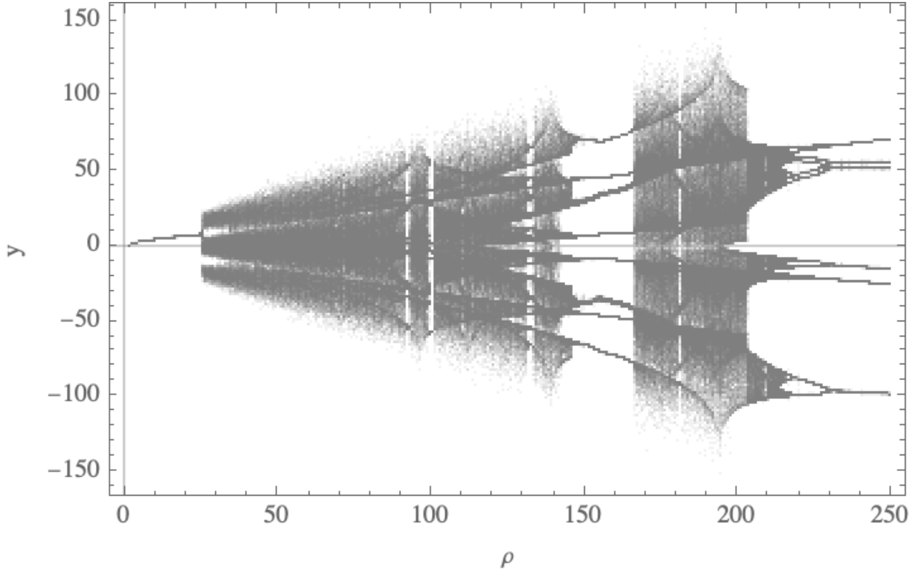
We find an almost smooth function $z_{n+1} = f(z_n)$ that is known as the **Lorenz map**.

The bifurcation diagram

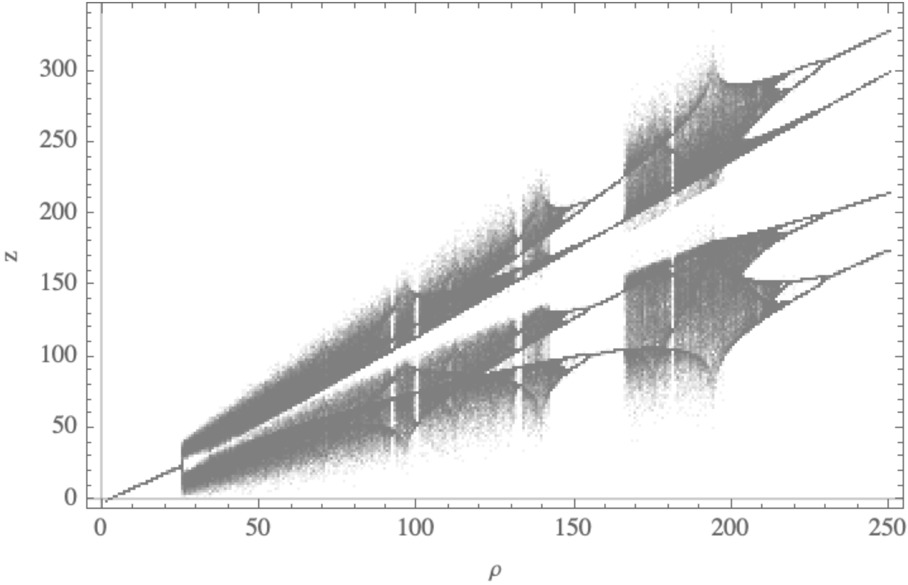
ρ plays a crucial part in the trajectory of the system. When we change this parameter we can find infinite cases: limit cycles, intermittent chaos or strange attractors, the possibilities are endless.



The bifurcation diagram



The bifurcation diagram



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¡Gracias!

谢谢!

ありがとう！

Thanks!

Grazie!

Merci!

Mulțmesc!

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