Appendix A
Families of Random Variables

A.1 Discrete Random Variables

Bernoulli (p)

For $0 \le p \le 1$,

$$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = 1 - p + pe^s$$

$$E[X] = p$$

$$\text{Var}[X] = p(1-p)$$

Binomial (n, p)

For a positive integer n and $0 \le p \le 1$,

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \phi_X(s) = (1 - p + pe^s)^n$$

$$E[X] = np$$

$$\text{Var}[X] = np(1-p)$$

Discrete Uniform (k, l)

For integers k and l such that $k < l$,

$$P_X(x) = \begin{cases} 1/(l-k+1) & x = k, k+1, \ldots, l \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{e^{sk} - e^{s(l+1)}}{(l-k+1)(1-e^s)}$$

$$E[X] = \frac{k+l}{2}$$

$$\text{Var}[X] = \frac{(l-k)(l-k+2)}{12}$$

Geometric (p)

For $0 < p \leq 1$,

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{pe^s}{1-(1-p)e^s}$$

$$\mathrm{E}[X] = 1/p$$

$$\mathrm{Var}[X] = (1-p)/p^2$$

Multinomial

For integer $n > 0$, $p_i \geq 0$ for $i = 1, \ldots, n$, and $p_1 + \cdots + p_n = 1$,

$$P_{X_1,\ldots,X_r}(x_1, \ldots, x_r) = \binom{n}{x_1, \ldots, x_r} p_1^{x_1} \cdots p_r^{x_r}$$

$$\mathrm{E}[X_i] = np_i$$

$$\mathrm{Var}[X_i] = np_i(1-p_i)$$

Pascal (k, p)

For positive integer k, and $0 < p < 1$,

$$P_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k} \qquad \phi_X(s) = \left(\frac{pe^s}{1-(1-p)e^s}\right)^k$$

$$\mathrm{E}[X] = k/p$$

$$\mathrm{Var}[X] = k(1-p)/p^2$$

Poisson (α)

For $\alpha > 0$,

$$P_X(x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = e^{\alpha(e^s - 1)}$$

$$\mathrm{E}[X] = \alpha$$

$$\mathrm{Var}[X] = \alpha$$

Zipf (n, α)

For positive integer $n > 0$ and constant $\alpha \geq 1$,

$$P_X(x) = \begin{cases} \dfrac{c(n,\alpha)}{x^\alpha} & x = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

where

$$c(n,\alpha) = \left(\sum_{k=1}^{n} \frac{1}{k^\alpha} \right)^{-1}$$

A.2 Continuous Random Variables

Beta (i, j)

For positive integers i and j, the beta function is defined as

$$\beta(i,j) = \frac{(i+j-1)!}{(i-1)!(j-1)!}$$

For a $\beta(i,j)$ random variable X,

$$f_X(x) = \begin{cases} \beta(i,j)x^{i-1}(1-x)^{j-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \frac{i}{i+j}$$

$$\text{Var}[X] = \frac{ij}{(i+j)^2(i+j+1)}$$

Cauchy (a, b)

For constants $a > 0$ and $-\infty < b < \infty$,

$$f_X(x) = \frac{1}{\pi} \frac{a}{a^2 + (x-b)^2} \qquad \phi_X(s) = e^{bs - a|s|}$$

Note that $E[X]$ is undefined since $\int_{-\infty}^{\infty} x f_X(x)\, dx$ is undefined. Since the PDF is symmetric about $x = b$, the mean can be defined, in the sense of a principal value, to be b.

$$E[X] \equiv b$$

$$\text{Var}[X] = \infty$$

Erlang (n, λ)

For $\lambda > 0$, and a positive integer n,

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^n$$

$$\mathrm{E}[X] = n/\lambda$$

$$\mathrm{Var}[X] = n/\lambda^2$$

Exponential (λ)

For $\lambda > 0$,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{\lambda}{\lambda - s}$$

$$\mathrm{E}[X] = 1/\lambda$$

$$\mathrm{Var}[X] = 1/\lambda^2$$

Gamma (a, b)

For $a > -1$ and $b > 0$,

$$f_X(x) = \begin{cases} \dfrac{x^a e^{-x/b}}{a! b^{a+1}} & x > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{1}{(1 - bs)^{a+1}}$$

$$\mathrm{E}[X] = (a+1)b$$

$$\mathrm{Var}[X] = (a+1)b^2$$

Gaussian (μ, σ)

For constants $\sigma > 0$, $-\infty < \mu < \infty$,

$$f_X(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \qquad \phi_X(s) = e^{s\mu + s^2\sigma^2/2}$$

$$\mathrm{E}[X] = \mu$$

$$\mathrm{Var}[X] = \sigma^2$$

Laplace (a, b)

For constants $a > 0$ and $-\infty < b < \infty$,

$$f_X(x) = \frac{a}{2} e^{-a|x-b|} \qquad \phi_X(s) = \frac{a^2 e^{bs}}{a^2 - s^2}$$

$$\mathrm{E}[X] = b$$

$$\mathrm{Var}[X] = 2/a^2$$

Log-normal (a, b, σ)

For constants $-\infty < a < \infty$, $-\infty < b < \infty$, and $\sigma > 0$,

$$f_X(x) = \begin{cases} \dfrac{e^{-(\ln(x-a)-b)^2/2\sigma^2}}{\sqrt{2\pi}\sigma(x-a)} & x > a \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{E}[X] = a + e^{b+\sigma^2/2}$$

$$\mathrm{Var}[X] = e^{2b+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

Maxwell (a)

For $a > 0$,

$$f_X(x) = \begin{cases} \sqrt{2/\pi}\, a^3 x^2 e^{-a^2x^2/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{E}[X] = \sqrt{\frac{8}{a^2\pi}}$$

$$\mathrm{Var}[X] = \frac{3\pi - 8}{\pi a^2}$$

Pareto (α, μ)

For $\alpha > 0$ and $\mu > 0$,

$$f_X(x) = \begin{cases} (\alpha/\mu)\,(x/\mu)^{-(\alpha+1)} & x \geq \mu \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{E}[X] = \frac{\alpha\mu}{\alpha - 1} \qquad (\alpha > 1)$$

$$\mathrm{Var}[X] = \frac{\alpha\mu^2}{(\alpha-2)(\alpha-1)^2} \qquad (\alpha > 2)$$

Rayleigh (a)

For $a > 0$,

$$f_X(x) = \begin{cases} a^2 x e^{-a^2 x^2/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{E}[X] = \sqrt{\frac{\pi}{2a^2}}$$

$$\mathrm{Var}[X] = \frac{2 - \pi/2}{a^2}$$

Uniform (a, b)

For constants $a < b$,

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{e^{bs} - e^{as}}{s(b-a)}$$

$$\mathrm{E}[X] = \frac{a+b}{2}$$

$$\mathrm{Var}[X] = \frac{(b-a)^2}{12}$$