Stochastic Processes II
Exams

June 6, 2020

1 August 21, 2019

1.1 Problem 1: Poisson Processes

Let $\{N_R(t), t \geq 0\}$ be a homogeneous Poisson Process of red points on $(0, \infty)$ with rate $\lambda_R > 0$ and let $\{N_B(t), t \geq 0\}$ be a homogeneous Poisson Process of blue points on $(0, \infty)$ with rate $\lambda_B > 0$. The two Poisson processes are independent of each other. Let

$$\{N(t), t \geq 0\} = \{N_R(t) + N_B(t), t \geq 0\}$$

be the point process which contains the points of both $\{N_R(t), t \geq 0\}$ and $\{N_B(t), t \geq 0\}$ For $i \in N_0$ define

$$S_i = \min\{t \geq 0; N(t) \geq i\}$$

,

That is, $S_0 = 0$ and the points of $\{N(t), t \geq 0\}$ are denoted by $\{S_i, i \in N\}$, satisfying

$$0 = S_0 < S_1 < S_2 < S_3 < \ldots.$$

(a) Provide the distribution of S_1 and the probability that the point at S_1 is red? That is, Compute

$$P(S_1 \leq t), \quad t \geq 0$$

and

$$P(\min\{t \geq 0; N_R(t) = 1\} < \min\{t \geq 0; N_B(t) = 1\})$$

We know that S_1 is exponentially distributed and denotes the time passed until the first event.

$$S_1 = \min\{t \geq 0; N_R(t) + N_B(t) = 1\}$$

We also know that

$$P(S_1 > t) = P(N(t) < 1)$$

Hence, the probability that the time until the 1-st event is greater than t will be the same as the probability that the number of events by time t is less than 1. i.e, $N(t) = 0$.

The CDF of S_1 is given by:

$$\begin{aligned}
F_{S_1}(t) = P(S_1 \le t) &= 1 - P(S_1 > t) = 1 - P(N(t) < 1) \\
&= 1 - P(N_R(t) + N_B(t) = 0) \\
&= 1 - P(N_R(t) = 0, N_B(t) = 0) \;\; \text{By independence} \\
&= 1 - P(N_R(t) = 0)P(N_B(t) = 0) \\
&= 1 - \frac{(t\lambda)^0 e^- \lambda_R \; t}{0!} \frac{(t\lambda)^0 e^- \lambda_B \; t}{0!} \\
&= 1 - e^{-t(\lambda_R + \lambda_B)}
\end{aligned} \tag{1}$$

Consider our original Poisson process with rate $(\lambda_R + \lambda_B)$, now we would like to label events, in this case we have Type I event and Type II event, being the first one red points and the second blue points. These two type of event will be independent Poisson processes. From our initial Poisson process we know that the probability of observing red points is just $p_R = \frac{\lambda_R}{\lambda_R + \lambda_B}$, while the probability of observing blue points is just $p_B = \frac{\lambda_B}{\lambda_R + \lambda_B}$. This follows since (check **Theorem 5.2**) $\lambda_B = p_B(\lambda_R + \lambda_B)$, and so $\lambda_R = p_R(\lambda_R + \lambda_B)$
Now, the probability that the first point (or any particular point) is red is:

$$P(\min\{t \ge 0; N_R(t) = 1\} < \min\{t \ge 0; N_B(t) = 1\}) = p_R$$

b) Assume that $N(T) = n$, where $n \in N$. What is the distribution of $N_R(T)$? That is, provide

$$P(N_R(T) = k | N_R(T) + N_B(T) = n) \quad n \in N$$

$$\begin{aligned}
P(N_R(T) = k | N_R(T) + N_B(T) = n) &= P(N_R(T) = k | N_B(T) = n - k) \\
&= \frac{P(N_R(T) = k) P(N_B(T) = n - k)}{P(N(T) = n)} \\
&= \frac{(T\lambda_R)^k e^{-\lambda_R \; T}}{k!} \frac{(T\lambda_B)^{n-k} e^{-\lambda_B \; T}}{(n-k)!} \frac{n!}{(T(\lambda_R + \lambda_B))^n e^{-T(\lambda_R + \lambda_B)}} \\
&= \Big(\frac{T\lambda_R}{T(\lambda_R + \lambda_B)}\Big)^k \Big(\frac{(T\lambda_B)}{T(\lambda_R + \lambda_B)}\Big)^{n-k} \frac{n!}{k!(n-k)!} \\
&= \Big(\frac{\lambda_R}{\lambda_R + \lambda_B}\Big)^k \Big(\frac{\lambda_B}{\lambda_R + \lambda_B}\Big)^{n-k} \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k} (\pi_R)^k \quad (\pi_B)^{(n-k)}
\end{aligned} \tag{2}$$

c) Compute $E[\sum_{i=1}^{N(t)} S_i]$, Recall $\sum_{i=1}^{0} S_i = 0$ by definition.

If we apply the order statistic property where the random variable $U \sim Unif(0, T)$ then we know that $E[U] = T/2$

Order statistic property: consider S_t is a random time, we don't know, where it is and how many there are of S_t, however, if we fix a T, and give the number of S_t in the interval $(0, T)$: $N(T) = n$, That means there are $S_1, ..., S_n$ in the interval $(0, T)$. Note that n is fixed and not random. These variables $S_1, ..., S_n$ are uniformly independent in $(0, T)$. They are uniform draws from the interval $(0, T)$. However, when we solve this problem, we shouldn't write that $S_1, ..., S_n$ are uniformly independent rvs. It might be confusing given the definition of S, So we usually say that they are the same distribution as U which is independently uniform rvs.

$$
\begin{aligned}
E[\sum_{i=1}^{N(T)} S_i] &= E[E[\sum_{i=1}^{N(T)} S_i | N(T)]] = E[E[\sum_{i=1}^{N(T)} U_i | N(T)]] = E[N(T)T/2] = 1/2E[N(T)T] \\
&= 1/2E[TE[N(T)|T]] = 1/2E[TE[N(T)]] = 1/2E[T^2(\lambda_N + \lambda_B)] = T^2/2(\lambda_N + \lambda_B)
\end{aligned} \tag{3}
$$

1.2 Problem 2: Renewal Theory

Let $\{N(t), t \geq 0\}$ be a renewal process, with inter-arrival distribution function $F(t)$ and density function $\frac{d}{dt}F(t) = f(t)$. Define $m(t) = E[N(t)]$ for $t \geq 0$.

a) Justify the renewal equation. That is, show that for $t \geq 0$,

$$m(t) = F(t) + \int_0^t m(t-x) f(x) dx$$

The function $m(t)$ is known as the mean-value or the renewal-function, which uniquely determines the renewal process.

$$m(t) = E[N(t)] = \int_0^\infty E[N(t)|X_1 = x] f(x) dx$$

Where X_1 represents the time of the first renewal. Furthermore;

$$E[N(t)|X_1 = x] = 1_{\{x<t\}} E[N(t)|X_1 = x] + 1_{\{x>t\}} \underbrace{E[N(t)|X_1 = x]}_{0} = E[N(t)|X_1 = x]$$

Although a renewal process $\{N(t); t \geq 0\}$ is not a Markov process it still satisfies the Markov property at time X_1 (given the present, the future does not depend on the past). Meaning that for every x the process $\{N(t+x); t \geq 0\}$ conditioned on the event $X_1 = x$ is distributed like the process $\{1 + N(t); t \geq 0\}$, or the process $\{N(t); t \geq 0\}$ conditioned on the event $X_1 = x$ is distributed like the process $\{1 + N(t-x); t \geq 0\}$, and so

$$E[N(t)|X_1 = x] = 1 + E[N(t-x)]$$

Whenever there is a renewal and $x < t$ that means that the number of renewals by time t will have the same distribution as 1 plus the number of renewals within the remaining $t - x$ time units. Which yields to

$$m(t) = \int_0^t 1 + E[N(t-x)]f(x)dx = \int_0^t 1 + m(t-x)f(x)dx = F(t) + \int_0^t m(t-x)f(x)dx \tag{4}$$

b) Assume that the inter-arrival times are uniformly distributed on $(0,1)$, i.e.

$$f(t) = \begin{cases} 1 & \text{for} \quad t \in (0,1) \\ 0 & o.w \end{cases}$$

Show that for $t \in [0,2]$, $m(t)$ is given by

$$m(t) = \begin{cases} e^t - 1 & \text{for} \quad t \in (0,1] \\ e^t - 1 - (t-1)e^{t-1} & \text{for} \quad t \in (1,2] \end{cases}$$

Given the renewal equation we know that whenever $t \in (0,1]$

$$\begin{aligned} m(t) &= \int_0^t 1 + m(t-x)\underbrace{1}_{\text{f(x)}}\, dx = \int_0^t dx + \int_0^t m(t-x)dx \\ &= t + \int_0^t m(t-x)dx = t + \int_0^t (e^{(t-x)} - 1)dx \\ &= t + e^t - 1 - t = e^t - 1 \end{aligned} \tag{5}$$

Furthermore, whenever $t \in (0,2]$ $m(t)$ becomes

$$\begin{aligned} m(t) &= \int_0^t 1 + m(t-x)f(x)dx = \int_0^1 (1 + m(t-x))f(x)dx + \int_1^t (1 + m(t-x))f(x)dx \\ &= t + \int_0^1 m(t-x)f(x)dx \ ; \quad u = t - x \Rightarrow 1 + \int_t^{t-1} -m(u)du = 1 + \int_{t-1}^t m(u)du \\ &= 1 + \int_{t-1}^1 m(u)du + \int_1^t m(u)du \quad \iff 1 + \int_0^{t-1} m(t-x)dx + \int_{t-1}^1 m(t-x)dx \end{aligned} \tag{6}$$

$$
\begin{aligned}
m(u) &= 1 + \int_{t-1}^{1} m(u)du + \int_{1}^{t} m(u)du \\
&= 1 + \int_{t-1}^{1} e^u - 1du + \int_{1}^{t} e^u - 1 - (u-1)e^{u-1}du \\
&= 1 + \int_{t-1}^{t} e^u - 1du - \int_{1}^{t} (u-1)e^{u-1}du \\
&= 1 + (e^t - e^{t-1} - 1) - \int_{1}^{t} (u-1)e^{u-1}du
\end{aligned} \tag{7}
$$

Where the LHS of the previous equation can be solved as:

$$
\int_{1}^{t} (u-1)e^{u-1}du \quad v = u - 1 \Rightarrow \int_{0}^{t-1} v\ e^v dv = (t-1)e^{(t-1)} - (e^{t-1} - 1) = e^{(t-1)}(t-2) + 1
$$

Which yields to

$$
m(t) = 1 + (e^t - e^{t-1} - 1) - e^{(t-1)}(t-2) - 1 = e^t - 1 - e^{(t-1)}(t-1)
$$

as desired.

c) Let $U_1, U_2, ...$ be independent random variables all uniformly distributed on $(0,1)$. Define

$$
\min\{n \in N; \sum_{i=1}^{n} U_i \geq 2\}
$$

The stopping time N is an integer valued non negative random variable which represents the first moment the sum of the iid random variables U_i is at least time t, for $t = 2$. For a renewal process having inter-arrival times $U_1, U_2, ...$, let

$$
S_n = \sum_{i=1}^{n} U_i, \quad U_i \sim U(0,1)
$$

Where S_n represents the time of the n-th renewal (or n-th arrival) . We can rewrite N as

$$
N = \min\{n \geq 0 : S_n > t\}
$$

We know that $N(t) = n - 1$ if and only if $S_{n-1} \leq t$ meaning that $S_n > t$.

$$
n = N(t) + 1 \iff U_1 + ... + U_{n-1} \leq t \iff U_1, ..., U_n > t
$$

Hence,

$$E[N] = E[N(t)+1] = E[N(2)+1] = E[N(2)]+1$$

Assuming that the inter-arrival distribution F is continuous with density function f then $m(t) = N(t)$ then $m(2) = E[N(2)] = e^2 - 1 - (2-1)e$,as whenever $t \in (1,2]$ $m(t) = e^t - 1 - (t-1)e^{t-1}$ (Example 7.3) which leave us with $E[N] = e^2 - e$.

1.3 Problem 3: Queueing Theory

Consider an $M/G/1$ queue in which customers arrive according to a Poisson Process with rate λ and customers have independent workloads which are distributed as the random variable S. Let $m_1 = E[S] < \infty$ be the expected time a customer needs service and $m_2 = E[S^2] < \infty$ be the second moment of this workload.

a) Provide a necessary and sufficient condition on λ for the queue length not to go to infinity?

The $M/G/1$ model assumes (i) Poisson arrivals(M) at rate λ; (ii) a general service distribution (G); and (iii) a single server.

In order for the quantities L, L_Q and W to be finite we need the condition $\lambda E[S] < 1$ to hold, as $\frac{1}{E[S]} > \lambda$ i.e departure rate > arrival rate.

For part b) and c) assume that the condition of part a) is satisfied. For $n \in N$, let Y_n be the number of new customers arriving during the service period of the n-th customer and let X_n be the number of customers that the n-th departing customer leaves behind in the system. b) Argue that for $n \in N$

$$X_{n+1} = X_n - 1 + Y_{n+1} + 1_{\{X_n=0\}}$$

Where

$$1_{\{X_n=0\}} = \begin{cases} 1 & \text{if} \quad X_n = 0 \\ 0 & o.w \end{cases}$$

We know that the the number of customers that the n-th customer leaves behind can be expressed as:

$$X_n = (X_{n-1} - 1) + 1_{\{X_{n-1}=0\}} + Y_n$$

Where we can consider two different quantities, (i) the number of new customer that arrive at time n denoted as Y_n and (ii) the number of customers that were already in the system before customer n entered in service.

For obtaining this quantity we should consider two different scenarios, (a) there were no customers in the system before customer n entered in service meaning that prior customer n there was nobody in service, and so the n-th customer will leave behind those customer that arrived while he was in service i.e $X_n = Y_n$ and (b) all customers arrived before he entered on service, at the time that we has as well in queue i.e $X_{n-1} - 1$ as the n-th customer is included in X_{n-1} (first-come first served queue). Which is consistent with

$$X_{n+1} = X_n - 1 + Y_{n+1} + 1_{\{X_n=0\}}$$

c) Compute the long run fraction of customers that depart without leaving anybody in the system and compute the expected number of customers that the n-th departing customers leaves behind in the system for $n \to \infty$. Hint: As intermediate steps, compute

$$\lim_{n\to\infty} P(X_n = 0) = \lim_{n\to\infty} E[1_{\{X_n=0\}}]$$

by taking the expectations on both sides of equation (1) and compute $\lim_{n\to\infty} E[X_n]$ by taking expectations of the squares of both sides of equation (1).

We are being asked to compute (i) $\lim_{n\to\infty} E[X_{n+1}]$ and (ii) $\lim_{n\to\infty} E[X_n]$. Let's first make the following considerations:

- The number of customer arrivals during the n-th to the $n+1$-st is given by $Y_{n+1} \sim Po(\lambda m_1)$, as we know in a Poisson process, the number of arrivals in a time interval t has rate λt. Hence, the expected value is given by

$$E[Y_{n+1}] = \lambda E[\text{service time}] = \lambda m_1$$

- For computing the probability that the n-th customer leaves behind zero customers i.e $P(X_n = 0) = P_0$, which is the long run proportion of time the system is empty.

$$P(X_n = 0) = \frac{E[\text{idle period}]}{E[\text{idle period}] + E[\text{busy period}]} = \frac{1/\lambda}{1/\lambda + \lambda E[\text{busy period}]}$$

  Where

$$E[\text{busy period}] = \frac{E[\text{service time}]}{1 - \lambda E[\text{service time}]}$$

  yields to

$$P(X_n = 0) = 1 - \lambda E[S] = 1 - m_1$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty} E[X_{n+1}] &= \lim_{n\to\infty} E[X_n] - 1 + \lim_{n\to\infty} E[Y_{n+1}] + \lim_{n\to\infty} E[1_{\{X_n=0\}}] \\
&= \lim_{n\to\infty} E[X_n] - 1 + \lambda m_1 + 1 - \lambda m_1 \\
&= \lim_{n\to\infty} E[X_n]
\end{aligned} \tag{8}$$

Let's now use the second hint and take the square and then the expectation of the previous equation without considering the limits:

$$
\begin{aligned}
E[(X_{n+1})^2] &= E[(X_n)^2] + 1 + E[(Y_{n+1})^2] + E[(1_{\{X_n=0\}})^2] - 2E[X_n] - 2E[Y_{n+1}] \\
&- 2E[1_{\{X_n=0\}}] + 2E[X_nY_{n+1}] + 2E[X_n1_{\{X_n=0\}}] + 2E[Y_{n+1}1_{\{X_n=0\}}] \\
&= E[(X_n)^2] + 1 + E[(Y_{n+1})^2] + E[1_{\{X_n=0\}}] - 2E[X_n] - 2E[Y_{n+1}] \\
&- 2E[1_{\{X_n=0\}}] + 2E[X_n]E[Y_{n+1}] + 2E[X_n]E[1_{\{X_n=0\}}] + 2E[Y_{n+1}]E[1_{\{X_n=0\}}] \\
&= E[(X_n)^2] + 1 + E[(Y_{n+1})^2] - 2E[X_n] - 2\lambda m_1 - E[1_{\{X_n=0\}}] + 2E[X_n]\lambda m_1 \\
&+ 2E[X_n1_{\{X_n=0\}}] + 2\lambda m_1 E[1_{\{X_n=0\}}] \\
&= E[(X_n)^2] + 1 + E[(Y_{n+1})^2] - 2E[X_n] - 2\lambda m_1 - E[1_{\{X_n=0\}}] \\
&+ 2E[X_n]\lambda m_1 + 2\lambda m_1 E[1_{\{X_n=0\}}]
\end{aligned}
\tag{9}
$$

Considering that $E[X_n1_{\{X_n=0\}}] = 0$ and that

$$
\begin{aligned}
E[(Y_n)^2] &= E[E[(Y_n)^2|\text{Workload at time } n]] \\
&= E[\text{Var}(Y_n) + E[Y_n|\text{Workload at time } n]^2] \\
&= E[\lambda S_{n+1} + \lambda^2 S_{n+1}^2] = \lambda E[S_{n+1}] + \lambda^2 E[S_{n+1}^2] = \lambda m_1 + \lambda^2 m_2
\end{aligned}
\tag{10}
$$

Then

$$
\begin{aligned}
E[(X_{n+1})^2] &= E[(X_n)^2] + 1 + E[(Y_{n+1})^2] - 2E[X_n] - 2\lambda m_1 - E[1_{\{X_n=0\}}] \\
&+ 2E[X_n]\lambda m_1 + 2\lambda m_1 E[1_{\{X_n=0\}}] \\
&\text{taking the limits} \\
\lim_{n\to\infty} E[(X_{n+1})^2] &= \lim_{n\to\infty} E[(X_n)^2] + 1 + \lambda m_1 + \lambda^2 m_2 - 2\lim_{n\to\infty} E[X_n] - 2\lambda m_1 - \lim_{n\to\infty} E[1_{\{X_n=0\}}] \\
&+ 2\lim_{n\to\infty} E[X_n]\lambda m_1 + 2\lambda m_1 \lim_{n\to\infty} E[1_{\{X_n=0\}}] \\
&= \lim_{n\to\infty} E[(X_n)^2] + 1 + \lambda m_1 + \lambda^2 m_2 - 2\lim_{n\to\infty} E[X_n] - 2\lambda m_1 - (1 - \lambda m_1) \\
&+ 2\lim_{n\to\infty} E[X_n]\lambda m_1 + 2\lambda m_1 \lim_{n\to\infty} E[1_{\{X_n=0\}}] \\
0 &= +\lambda^2 m_2 - 2\lim_{n\to\infty} E[X_n](1 - \lambda m_1) + 2\lambda m_1(1 - \lambda m_1) \\
\lim_{n\to\infty} E[X_n] &= \frac{\lambda^2}{2(1 - \lambda m_1)} + \lambda m_1
\end{aligned}
\tag{11}
$$

2 May 23, 2019

2.1 Problem 1: Poisson Processes

Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process on $(0, \infty)$ with rate $\lambda > 0$.Let $\{S_i, i \in N\}$ be the points of the Poisson Process, such that $0 = S_0 < S_1 < S_2 < S_3 < \ldots$. Define $S_0 = 0$. In parts a)-c) of this question let $T > 0$ be a constant.

a) Provide the distribution of $N(T)$.

By the definition of homogeneous Poisson Process we know that $N(T)$ is Poisson distributed with rate λT having probability mass function:

$$P(N(T) = k) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}$$

And expected value

$$E[N(T)] = \sum_{k \geq 0} kP(N(T) = k) = \sum_{k \geq 0} k\frac{(\lambda T)^k e^{-\lambda T}}{k!} = \lambda T$$

b) Provide the distribution of $S_{N(T)+1} - T$, that is provide the distribution of the waiting time until the next arrival at time T.

Let $Y(T) = S_{N(T)+1} - T$ is the time until the next arrival i.e $Y(T) = \min\{u \geq 0; N(T+u) - N(T) \geq 1\}$. We know that a Poisson Process has independent and exponentially distributed arrival times, hence, memoryless.

This will mean that the time until the next arrival is also exponentially distributed and it does not depend on the past or current time, i.e the continuation of a Poisson process, remains a Poisson process with rate λ and expected value $1/\lambda$.

$$Y(t) \sim \exp(\lambda)$$

.

c) Let n be a strictly positive integer. Compute for $x \in [0, T]$

$$P[T - S_n > x | N(T) = n]$$

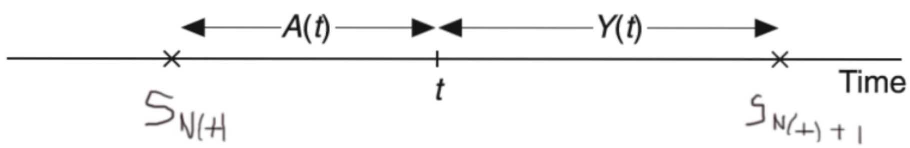
Use this (or use some other way) to compute (also for $x \in [0, T]$) $P[T - S_{N(T)} > x]$ That is, provide (one minus) the distribution of the time since the last arrival at time T. Remark: Note that $P[T - S_{N(T)} > T] = 0$, because $S_0 = 0$ and $S_i > S_0$ for all positive integers i.

$$\begin{aligned} P[T - S_n > x | N(T) = n] &= P[S_n < T - x | N(T) = n] \\ &= P(U_1, U_2, ..., U_n < T - x) \\ &= P(U_1 < T - x, U_2 < T - x, ..., U_n < T - x) \\ &= P(U_1 < T - x) P(U_2 < T - x), ..., P(U_n < T - x) \\ &= \Big(\frac{T - x}{T}\Big)^n = \Big(1 - \frac{x}{T}\Big)^n \end{aligned} \tag{12}$$

The probability of all n events happening before $T-x$, where $x\ in[0,T]$ is given by:

$$
\begin{aligned}
P[T-S_{N(T)}>T] &= \sum_{n\geq 0} P[T-S_{N(T)}>T|N(T)=n]P(N(T)=n) \\
&= \sum_{n\geq 0} P[T-S_{N(T)}>T|N(T)=n]\frac{(\lambda T)^n e^{-\lambda T}}{n!} \\
&= \sum_{n\geq 0} \left(1-\frac{x}{T}\right)^n \frac{(\lambda T)^n e^{-\lambda T}}{n!} = e^{-\lambda T}\sum_{n\geq 0}\frac{(\lambda(T-x))^n}{n!} \\
&= e^{-\lambda T}e^{\lambda(T-x)} = e^{-\lambda x}
\end{aligned}
\tag{13}
$$

d) Provide the distribution of $S_{N(T)+1}-S_{N(T)}$ for $T\to\infty$. That is, provide the distribution of the length of the inter-arrival interval at time T, for $T\to\infty$



From (b) we can notice that $Y(t)=S_{N(T)+1}-T$ and from (c) $A(t)=T-S_{N(T)}$, where $Y(T)\sim\exp(\lambda)$ and $A(T)\sim\exp(\lambda)$ as $T\to\infty$

$$
S_{N(T)+1}-S_{N(T)}=(S_{N(T)+1}-T)+(T-S_{N(T)})=Y(t)+A(t)\sim Gamma(2,\lambda)
$$

where $k=2$ independent exponentially distributed random variables.

2.2 Problem 2: Renewal Theory

In this question all limits are for $t\to\infty$. Let $\{N(t),t\geq 0\}$ be a renewal process, with inter-arrival distribution function $F(t)$. Assume that the inter-arrival time has expectation μ and variance σ^2 and that $F(0)=0$. Let $\{S_i,i=1,2,...\}$ be the times of the renewals in the renewal process, such that $0<S_1<S_2<S_3<...$ Define $S_0=0$.

a) Provide the definition of a renewal process.

b) Define the age of the process at time t as

$$A(t) = t - S_{N(t)}$$

Provide the almost sure limit of

$$1/t \int_0^t A(s)d\ s$$

and justify your answer.

Average Age of a Renewal Process

We have that $A(t) \sim \exp(\lambda)$ denotes the time from t since the last renewal. We are interest in the average value of the age. To determine this quantity, we use renewal reward theory in the following way: Let us assume that at any time we are being paid money at a rate equal to the age of the renewal process at that time. That is, at time t, we are being paid at rate $A(t)$, and so $\int_0^t A(s)d\ s$ represents our total earnings by time s. As everything starts over again when a renewal occurs, it follows that:

$$1/t \int_0^t A(s)d\ s \to \frac{E[\text{reward during a renewal cycle}]}{E[\text{time of a renewal cycle}]} = \frac{E[\int_0^X td\ t]}{E[X]} = \frac{E[X^2]}{2E[X]}$$
$$\Rightarrow \frac{\text{Var}(X) + E[X^2]}{2E[X]} = \frac{\sigma^2 + \mu^2}{2\mu}$$

where X is an inter-arrival time.

c) Provide the almost sure limit of

$$1/t \int_0^t S_{N(s)+1} - S_{N(s)}d\ s$$

Average Excess of a Renewal Process

Knowing that $A(t) + Y(t) = (S_{N(T)+1} - T) + (T - S_{N(T)}) = S_{N(T)+1} - S_{N(T)}$. then

$$1/t \int_0^t Y(t)d\ s \to \frac{E[\text{reward during a cycle}]}{E[\text{time of a cycle}]} = \frac{E[\int_0^X (X-t)d\ t]}{E[X]} = \frac{E[X^2]}{2E[X]}$$

Where

$$\int_0^X (X-t)d\ t = \int_0^X Xdt - \int_0^X td\ t = X^2 - \frac{X^2}{2} = \frac{X^2}{2E[X]} = \frac{\sigma^2 + \mu^2}{2\mu}$$

Combining the result with what we previously obtained in (b) we get:

$$1/t \int_0^t S_{N(s)+1} - S_{N(s)}d\ s \to 1/t \int_0^t A(s)d\ s + 1/t \int_0^t Y(t)d\ s = \frac{\sigma^2 + \mu^2}{\mu}$$

as $n \to \infty$

2.3 Problem 3: Queueing Theory

Consider an $M/M/2$ queue in which customers arrive according to a Poisson Process with rate λ and customers have independent workloads which are exponentially distributed with expectation $1/\mu$. Assume that $\lambda < 2\mu$. Assume further that the number of customers in the system at time 0, follows the stationary distribution of the number of customers in the system. a) For $k \in N_{\geq 0}$, let P_k be the probability that there are k customers in the system in stationarity. Show that

$$P_0 = \frac{2\mu - \lambda}{2\mu + \lambda}, \quad P_k = 2P_0\Big(\frac{\lambda}{2\mu}\Big)^k \quad k \in N_{\geq 1}$$

In the **Two-Server Exponential Queueing System** $M/M/2$

Consider a 2 server system in which customers arrive according to a Poisson process with rate λ. An arriving customer immediately enters service if any of the 2 servers are free.

If all 2 servers are busy, then the arrival joins the queue. All service times are exponential random variables with rate μ. Hence, the $M/M/2$ is a birth and death queueing model with arrival rates $\lambda_n = \lambda, \;\; n \geq 0$ and $\mu_n = n\mu \;$ if $\; n \leq 2$ and $\mu_n = k\mu \;$ if $\; n \geq 2$.

Following Example 8.6

$$\begin{aligned} P_0 &= \frac{1}{1 + \frac{\lambda}{\mu}\frac{\lambda^2}{2\mu^2} + 2\sum\limits_{n=3}^{\infty}\Big(\frac{\lambda}{2\mu}\Big)^n} = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2}\Big(\frac{\lambda}{\mu}\Big)^2 + 2\Big(\sum_{n=0}^{\infty}\Big(\frac{\lambda}{2\mu}\Big)^n - 1 - \frac{\lambda}{2\mu} - \Big(\frac{\lambda}{2\mu}\Big)^2\Big)} \\ &= \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2}\Big(\frac{\lambda}{\mu}\Big)^2 + 2\Big(\frac{1}{1-\frac{\lambda}{2\mu}} - 1 - \frac{\lambda}{2\mu} - \Big(\frac{\lambda}{2\mu}\Big)^2\Big)} = \frac{2\mu - \lambda}{2\mu + \lambda} \end{aligned} \tag{14}$$

Otherwise, Knowing customers arrive according to a Poisson process with rate λ and that the service time is exponentially distributed with rate μ $(n \leq 2)$ and 2μ $(n > 2)$. For this, let's define the **balance equations** for the following states:

$$\textbf{state} = 0, \quad \textbf{rate leave} = \lambda P_0, \; \textbf{rate enter} = \mu P_1$$

The process will leave state 0 at a rate λ via a customer arrival and will enter state 0 from state 1 with rate μ via customer departure.

$$\textbf{state} = 1, \quad \textbf{rate leave} = (\mu + \lambda)P_1, \; \textbf{rate enter} = \lambda P_0 + 2\mu P_2$$

The process will leave state 1 at a rate of $\lambda + \mu$ and will enter state 1 from state 2 by departure (at a rate 2μ) and state 0 by arrival (at a rate λ).

$$
\begin{aligned}
\lambda P_{n-1}, &= 2\mu P_n \qquad n > 1 \\
P_n &= \frac{\lambda}{2\mu} P_{n-1} \\
&= \left(\frac{\lambda}{2\mu}\right)^{n-1} P_1 \\
&= \left(\frac{\lambda}{2\mu}\right)^{n-1} \frac{\lambda}{\mu} P_0
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
1 = \sum_{i=0}^{\infty} P_n &= \sum_{i=1}^{\infty} \left(\frac{\lambda}{2\mu}\right)^{n-1} \frac{\lambda}{\mu} P_0 + P_0 \\
&= \frac{\lambda}{\mu} P_0 \sum_{i=1}^{\infty} \left(\frac{\lambda}{2\mu}\right)^{n-1} + P_0 = \frac{\lambda}{2\mu} P_0 \frac{2\mu}{2\mu - \lambda} + P_0 \\
&= P_0 \Big(\frac{2\lambda}{2\mu - \lambda} + 1\Big) = P_0 \Big(\frac{2\mu + \lambda}{2\mu - \lambda}\Big)
\end{aligned}
\tag{16}
$$

$$
\iff \quad P_0 = \frac{2\mu - \lambda}{2\mu + \lambda}
$$

b) Compute the long run average time that a customer is in the queue. That is, compute the long run average time that customers are waiting before their service start.

Recall that W_Q represents the average amount of time a customer spends waiting in queue, W the average amount of time a customer spends in the system, $E[S]$ the average amount of time a customer spends in service, L the average number of customers in the system.

$$
W_Q = W - E[S] \quad ; W = \frac{L}{\lambda_a} = \frac{\sum\limits_{n=0}^{\infty} nP_n}{\lambda}
$$

Now, the average number of customer seen by an arrival in the $M/M/2$ system is obtained a follows:

$$
\begin{aligned}
L = \sum_{n=0}^{\infty} nP_n &= \sum_{n=0}^{\infty} 2n\Big(\frac{\lambda}{2\mu}\Big)^n P_0 = 2P_0 \sum_{n=0}^{\infty} n\Big(\frac{\lambda}{2\mu}\Big)^n \\
&= \frac{\lambda}{\mu} P_0 \sum_{n=0}^{\infty} n\Big(\frac{\lambda}{2\mu}\Big)^{n-1} = \frac{\lambda}{\mu} \frac{2\mu - \lambda}{2\mu + \lambda} \left(\frac{1}{\left(1 - \frac{\lambda}{2\mu}\right)^2}\right) = \frac{4\lambda\mu}{(2\mu + \lambda)(2\mu - \lambda)}
\end{aligned}
\tag{17}
$$

We obtain that

$$
W = \frac{L}{\lambda_a} = \frac{\frac{4\lambda\mu}{(2\mu+\lambda)(2\mu-\lambda)}}{\lambda} = \frac{4\mu}{(2\mu + \lambda)(2\mu - \lambda)}
$$

Furthermore, the average amount of time a customer spends waiting in queue is obtained as:

$$W_Q^2 = W_{M/M/2} - E[S] = \frac{4\mu}{(2\mu+\lambda)(2\mu-\lambda)} - \frac{1}{\mu} = \frac{\lambda^2}{\mu(2\mu+\lambda)(2\mu-\lambda)}$$

c) Assume that Adam is the first customer arriving (strictly) after time 0, what is the Probability that Adam find no customers in the queue at his arrival?

When the system starts in stationarity, it means that at time 0 there is a random number of people already present in the system. This random number has the stationary distribution. Adam is the first customer arriving after time 0. If there have been no departures before Adam arrives, Adam finds 0 customers with probability P_0. However, if there were already people in the system at time 0, some of them might have departed before Adam arrives.

The problem is that the arrival of Adam is not a "Poisson event". in the sense that Poisson events are independent of what happened before or after it, while being "the first arrival" says that there has not been any arrival before Adam arrived and so, Adam's arrival is not independent of what happened before.

If Adam is the first customer arriving after time zero that means that before Adam arrival's there might be other customers that came first. Now, for him to not find any customer in the queue will mean that whether there is no customer on the system or there are two customers on service. Let X_A denote the number of customers in the system when Adam arrives and X_0 the number of customers in the system at time zero.

We are interested in finding $P(X_A \leq 2)$. Note that as we are in a $M/M/2$ system whenever there are ≤ 2 customers in the service, Adam will find the queue always empty. However, whenever there are $j \geq 3$ customers in the system, Adam will find the queue empty if the $j-2$ events in the system are all departures.

An exponential model that can go (in one transition) only from state n to either state $n-1$ or state $n+1$ is called a birth and death model. For such a model, transitions from state n to state $n+1$ are designated as births, and those from n to $n-1$ as deaths.
Whenever there are $j \geq 3$ customers in the system, the time until either the next arrival or the next departure occurs is an exponential random variable with rate $\lambda_n + \mu_n$ and, independent of how long it takes for this occurrence, it will be an arrival with probability $\frac{\lambda}{2\mu+\lambda}$ and departure with probability $\frac{2\mu}{2\mu+\lambda}$.

$$
\begin{aligned}
P(X_A \leq 2) &= \sum_{j=o}^{\infty} P(X_A \leq 2|X_0 = j)P(X_0 = j) \\
&= \sum_{j=0}^{2} P(X_A \leq 2|X_0 = j)P(X_0 = j) + \sum_{j=3}^{\infty} P(X_A \leq 2|X_0 = j)P(X_0 = j) \\
&= \sum_{j=0}^{2} P(X_0 = j) + \sum_{j=3}^{\infty} P(X_A \leq 2|X_0 = j)P(X_0 = j) \\
&= P_0 + P_1 + P_2 + \sum_{j=3}^{\infty} \Big(\frac{2\mu}{2\mu+\lambda}\Big)^{j-2} P(X_0 = j) \\
&= P_0 + P_1 + P_2 + \sum_{j=3}^{\infty} \Big(\frac{2\mu}{2\mu+\lambda}\Big)^{j-2} 2P_0\Big(\frac{\lambda}{2\mu}\Big)^{j} \\
&= P_0 + P_1 + P_2 + P_0\sum_{j=3}^{\infty} \Big(\frac{1}{2\mu+\lambda}\Big)^{j-2} 2^{j-2}\ \mu^{j-2}\ 2^{-j}\ \mu^{-j}\lambda^{j} \\
&= P_0 + P_1 + P_2 + 2P_0\sum_{j=3}^{\infty} \Big(\frac{1}{2\mu+\lambda}\Big)^{j-2}\ \mu^{-2}\ 2^{-2}\ \ \lambda^{j-2}\lambda^{2} \\
&= P_0 + P_1 + P_2 + 2P_0\Big(\frac{\lambda}{2\mu}\Big)^{2} \sum_{j=3}^{\infty} \Big(\frac{\lambda}{2\mu+\lambda}\Big)^{j-2} \\
&= P_0 + P_1 + P_2 + 2P_0\Big(\frac{\lambda}{2\mu}\Big)^{2} \Big(\frac{1}{1-\frac{\lambda}{2\mu+\lambda}} - 1\Big) \\
&= P_0 + P_1 + P_2 + 2P_0\Big(\frac{\lambda}{2\mu}\Big)^{3} = P_0 + 2P_0\Big(\frac{\lambda}{2\mu}\Big) + 2P_0\Big(\frac{\lambda}{2\mu}\Big)^{2} 2P_0\Big(\frac{\lambda}{2\mu}\Big)^{3}
\end{aligned} \tag{18}
$$

2.4 Problem 4: Brownian Motion and Stationary Processes

Let f$\{B(t), t \geq 0\}$ be a standard Brownian motion and for $t > 0$, let

$$M(t) := \max_{0 \leq s \leq t} B(s)$$

be the maximum of the Brownian motion up to time t. Here we assume that $B(0) = 0$ and that the variance parameter $\sigma^2 = 1$ is part of the definition of a standard Brownian motion.

a) For $y > 0$ and $x > 0$, argue that

$$P(M(t) > y, B(t) < y - x) = P(B(t) > y + x)$$

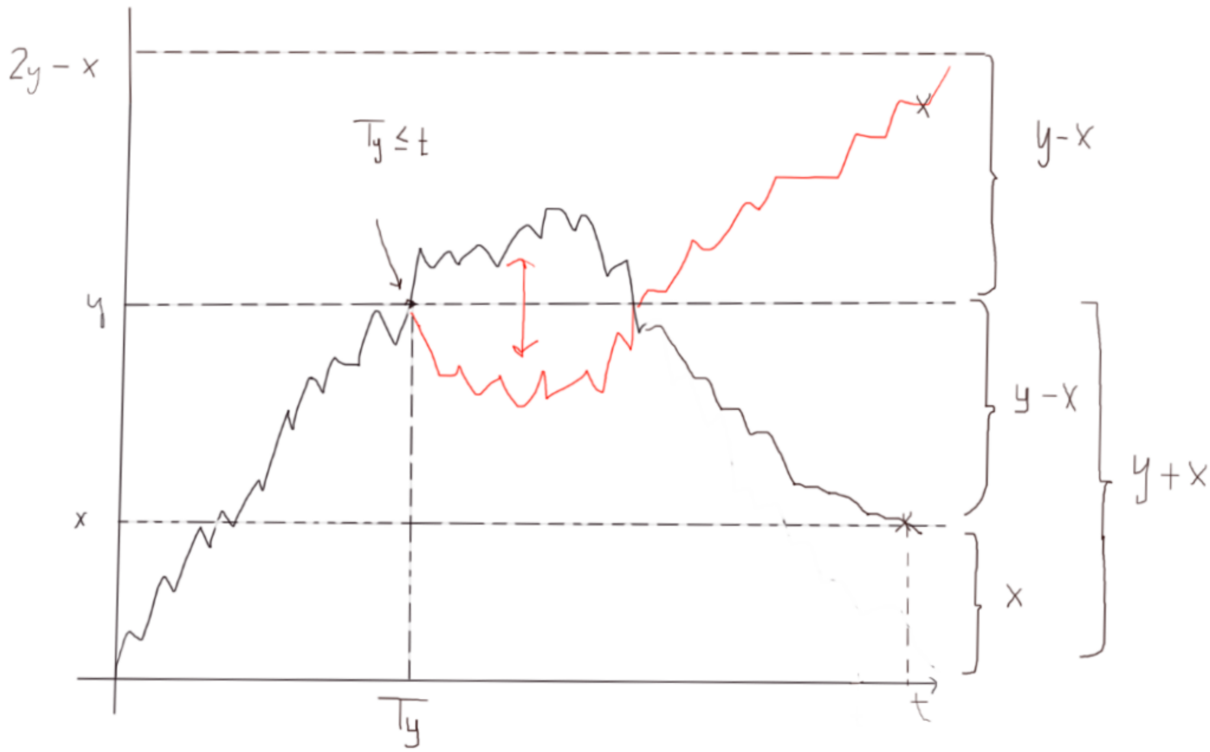
$$P(M(t) > y) = P(\max_{0 \leq s \leq t} B(s) > y) = P(T_y < t)$$

Which means that, the probability of the maximum of a Brownian motion in an interval $[0, t]$ being greater than y, is the same as the probability of the time the process hits y before time t. Where

$$T_y = \inf\{t; B(t) \geq y\}$$

We know that

$$P(\max_{0\leq s\leq t} B(s) > y) = P(B(0) > y, \dots, B(s) > y, ..., B(t) > y, B(t) < y - x) \tag{19}$$



From the figure we can see that the red path is the reflection of the Brownian motion path. The probability of the two paths is exactly the same

$$P(T_y \leq t, W(t) = x) = P(M(t) = 2y - x) = 0$$

and

$$P(T_y \leq t, W(t) \leq x) = P(M(t) \geq 2y - x) = 0$$

Now, we have that

$$\begin{aligned} P(M(t) > y, B(t) < y - x) &= P(T_y < t, B(t) < y - x) \\ &= P(M(t) > y, B(t) < y - x) \\ &\text{By the reflection principle} \\ &= P(M(t) > y, B(t) > y + x) \\ &= P(B(t) > y + x) \end{aligned} \tag{20}$$

b) For $t \in (0,1)$ and $x \in R$, provide the density of $M(t)$ conditioned on $B(t) = x$. One can deduce from part a) that the joint density function of $M(t)$ and $B(t)$ is given by

$$f_{M(t),B(t)}(y,x) = \frac{2}{\sqrt{2\pi t}} \frac{2y-x}{t} e^{-\frac{(2y-x)^2}{2t}}$$

Considering that we have a Standard Brownian motion, for $t \in (0,1), x \in R$ the density of the maximum of the Brownian motion process conditioned to the process being at x by time t is:

We know:

$$B(t) \sim \mathcal{N}(0,t)$$

$$\begin{aligned} f_{M(t)|B(t)}(y|x) &= \frac{f_{M(t),B(t)}(y,x)}{f_{B(t)}(x)} \\ &= \frac{\frac{2}{\sqrt{2\pi t}} \frac{2y-x}{t} e^{-\frac{(2y-x)^2}{2t}}}{\frac{1}{\sqrt{2\pi t}} \exp\{\frac{-x^2}{2t}\}} = 2\frac{2y-x}{t} \exp\{-\frac{(2y-x)^2}{2t} + \frac{x^2}{2t}\} \\ &= 2\frac{2y-x}{t} \exp\{-\frac{2y}{t}\Big(y-x\Big)\} \end{aligned} \tag{21}$$

For $t \in (0,1)$, compute $P(M(t) < M(1)|B(t) = x, B(1) = 0)$. That is, compute

$$P\Big(\max_{0\leq s\leq t} B(s) < \max_{t\leq s\leq 1} B(s)|B(t) = x, B(1) = 0\Big)$$

In part c) you me use without further proof that for all $x \in R$ and $y \in R$ and $t \in (0,1)$ we have

$$P\Big(\max_{0\leq s\leq t} B(s)|B(t) = x, B(1) = 0\Big) = P(M(1-t) > y|B(1-t) = x)$$

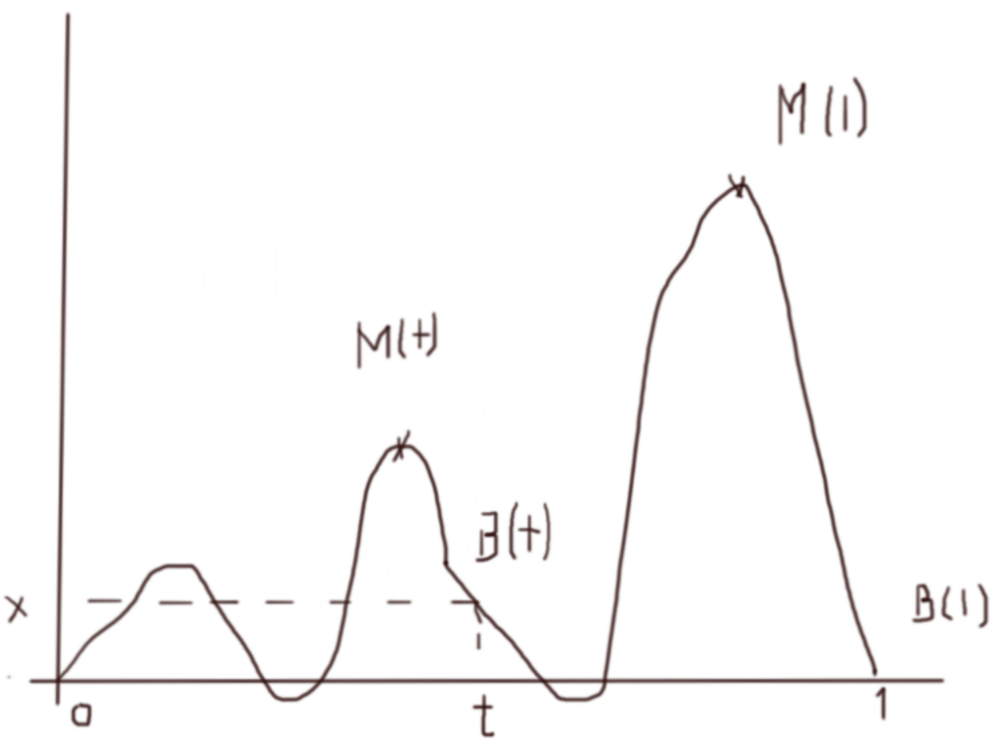
First, notice that

$$\begin{aligned} M(t) &= \max_{0\leq s\leq t} B(s) \\ M(1) &= \max_{0\leq s\leq 1} B(s) \end{aligned} \tag{22}$$

Then

$$P(M(t) < M(1)) = P(\max_{0\leq s\leq t} B(s) < \max_{0\leq s\leq 1} B(s)) = P(\max_{0\leq s\leq t} B(s) < \max_{t\leq s\leq 1} B(s))$$

But notice that $M(1) \neq \max_{t\leq s\leq 1} B(s)$.



From $P\Big(\max_{0\leq s\leq t} B(s) < \max_{t\leq s\leq 1} B(s) | B(t)=x, B(1)=0\Big)$ we can also notice that as we condition on $B(t)$ then $\max_{0\leq s\leq t} B(s)$ is independent of $\max_{t\leq s\leq 1} B(s)$

From probability theory we know that $E[1_A|B] = P(A|B)$ so let's rewrite the conditional probability as:

$$E[1\{\underbrace{\max_{t\leq s\leq 1} B(s)}_{\text{X(t)}} > \underbrace{\max_{0\leq s\leq t} B(s)}_{\text{M(t)}}\}|\underbrace{B(t)=x, B(1)=0}_{H}]$$

where H is just an event, an also notice that $M(t)$ is independent of $B(1)=0$ as t strictly smaller than 1, this yields to:

$$= \int_R \int_R 1_{m>y} \ dP_{M(t),X(t)}(y, m|H) \ dm \ \ dy$$
$$= \int_R \int_R 1_{m>y} \ f_{M(t),X(t)|H}(y, m) \ dm \ \ dy$$
$$= \int_R \int_R 1_{m>y} \ f_{M(t)|H}(y) \ f_{X(t)|H}(m) \ dm \ \ dy$$

We change the limits of the internal integral given the indicator function.

$$= \int_{y \in R} \Big(\int_{m=y}^{\infty} f_{M(t)|H}(y) \ f_{X(t)|H}(m) \ dm \Big) \ \ dy \tag{23}$$
$$= \int_{y \in R} \Big(\int_{m=y}^{\infty} f_{X(t)|H}(m) \ dm \Big) f_{M(t)|H}(y) \quad dy$$
$$= \int_{y \in R} P\Big(\max_{t \le s \le 1} B(s) > y | B(t) = x, B(1) = 0 \Big) \ \ f_{M(t)|B(t)=x,B(1)=0}(y) \quad dy$$

following the hint

$$= \int_{y \in R} P\Big(M(t-1) > y | B(1-t) = x \Big) \ \ f_{M(t)|B(t)=x}(y) \quad dy$$

2.5 Problem 5: Simulation

3 August 21, 2018

3.1 Problem 1: Poisson Processes

Let $\{N(t), t \ \geq\}$ be a homogeneous Poisson Process on $(0, \infty)$ with rate λ. Let $\{S_i, i = 1, 2, ...\}$ be the points of the Poisson Process, such that $S_1 < S_2 < S_3 < ....$ Define $S_0 = 0$.

b) Let n be a strictly positive integer. Suppose that we know that $N(1) = n$. What is the distribution of S_1? That is, compute $P(S_1 \leq t | N(1) = n)$ for $t \in [0, 1]$?

Keep in mind that if the first event occurs after time t $S_1 > t$ which implies that $N(t) = 0$

$$S_1, ..., S_n | N(t) = n \sim U_1, \dots, U_n; \quad U_i \sim Unif(0, 1)$$
$$1 - S_1, ..., 1 - S_n | N(t) = n \sim 1 - U_1, \dots, 1 - U_n; \quad 1 - U_i \sim U_1, \dots, U_n$$
$$1 - S_1 > ... > 1 - S_t > ... > 1 - S_n$$

We now apply the order statistic property with time of arrivals uniformly distributed in the interval $(0, 1)$ with expectation $1/2$ meaning that

$$P(U \geq t) = 1 - P(U \leq t) = 1 - (t \cdot 1/(1 - 0)) = 1 - (1 - t)$$

$$
\begin{aligned}
P(S_1 \leq t|N(1)=n) &= 1 - P(S_1 \geq t|N(1)=n) \\
&= 1 - P(1-S_1 \leq 1-t|N(1)=n) \\
&= 1 - P(1-S_1 \leq 1-t, 1-S_n \leq 1-t \ldots, |N(1)=n) \\
&= 1 - P(U_1 \leq 1-t, \ldots, U_n \leq 1-t \ldots) \\
&= 1 - (1-t)^n
\end{aligned} \tag{24}
$$

Define independently of $\{N(t), t \geq\}$ a second homogeneous Poisson process $\{X(t), t \geq\}$ on $(0,1)$ with rate β

c) What is the distribution of $X(S_1)$?

Given the two independent Poisson processes we know we can combine them to form a new Poisson Process $\{N(t)+X(t), t \geq 0\}$ which will have rate $\beta + \lambda$. Hence, we can label two types of events, where the probability of the N arrival is $p_N = \frac{\lambda}{\lambda+\beta}$ while the probability of the X process arrival is $p_X = \frac{\beta}{\lambda+\beta}$. Furthermore, recall that $S_1 \sim \exp(\lambda)$.

Now, for $k \in \{0,1,2,...\}$

$$
\begin{aligned}
P(X(S_1)=k) &= \int_0^\infty P(X(S_1)=t|S_1=t)P(S_1=t)dt \\
&= \int_0^\infty P(X(S_1)=k|S_1=t)\lambda e^{-\lambda t}dt \\
&= \lambda\frac{\beta^k}{k!}\int_0^\infty t^k e^{-t(\beta+\lambda)}dt = \lambda\frac{\beta^k}{k!}\Big(-t^k\frac{1}{e^{t(\beta+\alpha)}}\frac{1}{\beta+\alpha}\Big|_0^\infty + \frac{1}{\beta+\alpha}\int_0^\infty kt^{k-1}e^{-t(\beta+\alpha)}\Big) \\
&= \lambda\frac{\beta^k}{k!}\frac{1}{\beta+\alpha}\Big(\int_0^\infty kt^{k-1}e^{-t(\beta+\alpha)}\Big) \quad \text{Integrate } k-1 \text{ times} \\
&\vdots \\
&= \lambda\frac{\beta^k}{k!}\frac{k!}{(\beta+\alpha)^k}\Big(\int_0^\infty e^{-t(\beta+\alpha)}\Big) = \lambda\frac{\beta^k}{(\beta+\alpha)^{k+1}}\Big[-\frac{1}{e^{t(\beta+\alpha)}}\Big]_0^\infty = \frac{\lambda}{\beta+\alpha}\cdot\frac{\beta^k}{(\beta+\alpha)^k} \\
&= p_N(1-p_N)^k
\end{aligned} \tag{25}
$$

3.2 Problem 2: Renewal Theory

A factory has two machines. Each machine can be either broken or working. If both machines are working one is "producing", while the other is "on stand-by". If only one machine is working, that machine is "producing", while the other one is "in repair". If both machines are broken, then one machine is "in repair", while the other ones is "waiting to go in repair". So, the pair of machines can be in three states:

- A One machine "producing", the other "on stand-by".
- B One machine "producing", the other "in repair".
- C One machine "in repair", the other waiting to go "in repair"'.

Assume that a "producing" machine breaks down after an exponentially distributed time with expectation $1/\lambda$, which is independent of everything else in the process. If just before that moment the other machine was "on stand-by", it becomes "producing" immediately and the machine which broke down immediately gets "in repair". If at the moment of break down of one machine, the other machine is "in repair", then the newly broken down machine has to wait for its repair. to start again until the first machine is fully repaired. Then it gets "in repair" itself. The time needed for repair for a machine is not random and equal to exactly one time unit.

Assume that at time $S_0 = 0$ one machine just became "producing", while the other just gets "in repair" (So, the pair just enters state B at time 0). Let

$$S_k = \min\{t > S_{k-1} : \text{A machine starts producing}\} \quad k \in \{1, 2, 3, ...\}\}$$

be the k-th time one of the machines just becomes "producing" (and by the definition of the model, the other just gets "in repair"). That is S_k is the k-th time strictly after time 0, that the pair of machines enters state B. For $k \in \{1, 2, \ldots\}$, define $X_k = S_K - S_{K-1}$

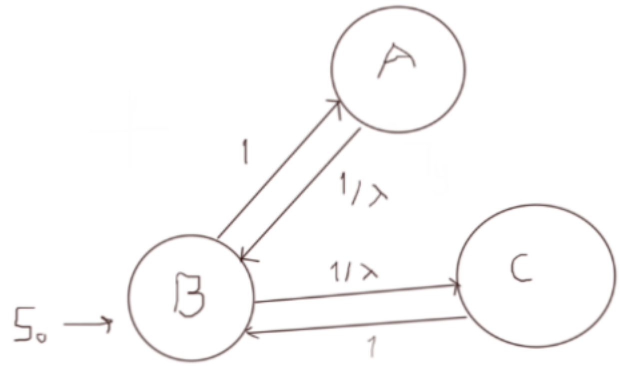
a) Argue that for $k \in \{1, 2, \ldots\}$ the random variable X_k satisfies

$$P(X_k \leq t) = 0 \quad t < 1, \quad P(X_k \leq t) = 1 - e^{-\lambda t} \quad t \geq 1$$

Furthermore. show that $E[X_k] = 1 + e^{-\lambda}/\lambda$

In order to return to state B, both machines have to break down , the repair of one of the broken machine has to finish $C \to B$, or one of the two machines has to break down $A \to B$.

At time S_{k-1} both machines are in repair, or both machines are in production, while time S_k is the time when we enter state B meaning that one is in production and the other enters in repair or both where in repair and now one enters in production. So X_k denotes the time it takes to go back to state B.



Remember that time needed to repair a machine is always 1. If we let T be the time until a breakdown, we should consider two cases,

- $T > 1$ This means that we go from $A \to B$. As 1 is the time it takes to fix a machine and the time to the next breakdown is > 1 this means that there is for sure a machine in production. So the time it takes to enter state B is the time until the next breakdown.

$$X_k = T$$

- $T < 1$ means that there is already a machine being repaired, meaning that now both machines are in repair $C \to B$. Then the time that will take to go back to state B is the time of repair

$$X_k = 1$$

Meaning that

$$X_k = \max\{T, 1\} \quad T \sim \exp(\lambda)$$

And so

$$P(X_k \leq t) = P(\max\{T,1\} \leq t) = P(T \leq t, 1 \leq t) = P(T \leq t)P(1 \leq t) = P(T \leq t) = 1 - \lambda e^{-\lambda t}$$

Furthermore,

$$
\begin{aligned}
E[X_k] &= E[\max\{T,1\}] = E[T \cdot 1_{(1<T)} + 1 \cdot 1_{(T<1)}] \\
&= TP(1<T) + P(T<1) \\
&= \int\limits_1^{\infty} s\lambda e^{-\lambda s} ds + \int\limits_0^1 \lambda e^{-\lambda s} ds
\end{aligned} \tag{26}
$$

b) Compute the long run fraction of time that none of the machines is working (that is the fraction of time the pair of machines is in state C).

Let $N(t)$ be the number of "renewals" up to time t. That is $N(t) = n$ if and only if $Sn \leq t$ and $S_{n+1} > t$.
c) For $t \to \infty$, compute $E[t - S_{N(t)}]$ That is, compute the expected time since the last renewal at time t, in the limit as $t \to 1$.

We have that $A(t) = T - S_{N(T)}$; $A(t) \sim \exp(\lambda)$ denotes the time from t since the last renewal. We are interest in the average value of the age. To determine this quantity, we use renewal reward theory in the following way: Let us assume that at any time we are being paid money at a rate equal to the age of the renewal process at that time. That is, at time t, we are being paid at rate $A(t)$, and so $\int_0^t A(s)d\ s$ represents our total earnings by time s. As everything starts over again when a renewal occurs, it follows that:

$$
1/t \int_0^t A(s)d\ s \to \frac{E[\text{reward during a renewal cycle}]}{E[\text{time of a renewal cycle}]} = \frac{E[\int_0^X td\ t]}{E[X]}
$$

Where

$$
E[\text{reward during a renewal cycle}] = \int_0^X td\ t = \frac{X^2}{2} = \frac{X^2}{2}
$$

3.3 Problem 4: Brownian Motion

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. Let $\alpha > 0$ be a strictly positive constant and let $\{B(t), t \geq 0\}$ be an Ornstein Uhlenbeck Process, defined through

$$
V(t) = e^{-\alpha t/2} B(e^{\alpha t})
$$

for $t \geq 0$
a) Compute $E[V(t)]$ for $t \geq 0$.

$$
E[V(t)] = E[e^{-\alpha t/2} B(e^{\alpha t})] = e^{-\alpha t/2} E[B(e^{\alpha t})] = 0
$$

As $B(e^{\alpha t}) \sim \mathcal{N}(0, e^{\alpha t})$

b) Compute the covariance $\mathrm{Cov}[V(t), V(t+s)]$ for $t > 0$ and $s > 0$.

$$\begin{aligned}
\mathrm{Cov}[V(t), V(t+s)] &= E[V(t)V(t+s)] - E[V(t)]E[V(t+s)] \\
&= E[e^{-\alpha t/2}B(e^{\alpha t}) \cdot e^{-\alpha(t+s)/2}B(e^{\alpha(t+s)})] - 0 \\
&= e^{\frac{-\alpha}{2}(t+(t+s))}E[B(e^{\alpha t})B(e^{\alpha(t+s)})] \\
&= e^{\frac{-\alpha}{2}(t+(t+s))}E[B(e^{\alpha t})(B(e^{\alpha(t+s)}) - B(e^{\alpha t}) + B(e^{\alpha t}))] \\
&= e^{\frac{-\alpha}{2}(t+(t+s))}E[B(e^{\alpha t})(B(e^{\alpha(t+s)}) - B(e^{\alpha t})) + B(e^{\alpha t})^2] \\
&= e^{\frac{-\alpha}{2}(t+(t+s))}E[B(e^{\alpha t})(B(e^{\alpha(t+s)}) - B(e^{\alpha t}))] + E[B(e^{\alpha t})^2] \\
&= 0 + \mathrm{Var}(B(e^{\alpha t})) = e^{\alpha t}
\end{aligned} \tag{27}$$

c) Provide the distribution of $V(1)$.

$$V(1) = e^{-\alpha/2}B(e^{\alpha})$$

Which follows a normal distribution with expectation $E[V(1)] = e^{-\alpha/2}E[B(e^{\alpha})] = 0$ and variance

$$\mathrm{Var}(V(1)) = E[V(1)^2] - E[V(1)]^2 = E[(e^{-\alpha/2}B(e^{\alpha}))^2] = e^{-\alpha}\,\mathrm{Var}(B(e^{\alpha})) = e^{-\alpha}e^{\alpha} = 1$$

d) Let $x > 0$ and $t > 1$, compute

$$P\Big(\min_{1\le s\le t} V(s) > 0 | V(1) = x\Big)$$

$$\begin{aligned}
P\Big(\min_{1\le s\le t} V(s) > 0 | V(1) = x\Big) &= P\Big(\min_{1\le s\le t} e^{-\alpha s/2}B(e^{\alpha s}) > 0 | e^{-\alpha/2}B(e^{\alpha}) = x\Big) \\
&= P\Big(\min_{1\le s\le t} B(e^{\alpha s}) > 0 | B(e^{\alpha}) = xe^{\alpha/2}\Big) \\
&= P\Big(\min_{1\le s\le t} B(e^{\alpha s}) > -xe^{\alpha/2} | B(e^{\alpha}) = 0\Big) \\
&\text{by symmetry} \\
&= P\Big(\max_{1\le s\le t} B(e^{\alpha s}) < xe^{\alpha/2} | B(e^{\alpha}) = 0\Big) \\
&\text{by stationarity} \\
&= P\Big(\max_{1\le s\le t} B(e^{\alpha s} - e^{\alpha}) \le xe^{\alpha/2} | B(0) = 0\Big) \\
&= P\Big(\max_{1\le s\le e^{\alpha t}-e^{\alpha}} B(s) \le xe^{\alpha/2} | B(0) = 0\Big) \\
&= P(T_{xe^{\alpha/2}} > e^{\alpha t} - e^{\alpha}) = 1 - P(T_{xe^{\alpha/2}} \le e^{\alpha t} - e^{\alpha}) \\
&= 2P(B(e^{\alpha t} - e^{\alpha}) \ge xe^{\alpha/2})
\end{aligned} \tag{28}$$

3.4 Problem 5: Simulation

a) Show in detail that the position of the first point of the Poisson process $\{N(t); t \geq 0\}$ can be obtained through simulating U_1 and then computing $-\log[U_1]/\lambda$

4 May 28, 2018

4.1 Problem 1: Poisson Processes

Let $\{N(t), t \geq\}$ be a homogeneous Poisson Process on $(0, 1)$ with rate λ. Let $\{S_i, i = 1, 2, ...\}$ be the points of the Poisson Process, such that $S_1 < S_2 < S_3 < ....$ Define $S_0 = 0$.

a) Provide the distribution of S_1.

Let S_1 be the time of the first event, knowing that $S_0 = 0$ an that an homogeneous Poisson process has exponentially distributed inter-arrival times, then S_1, is exponentially distributed with expectation $1/\lambda$ i.e. $S_1 \sim \exp(\lambda)$ with density function $\lambda t e^{-\lambda t}$ and CDF

$$P(S_1 \leq t) = 1 - e^{-\lambda t}$$

Meaning that the probability of not having any event on the interval $(0, t)$ is

$$P(S_1 > t) = e^{-\lambda t}$$

b) For $x \in [0, T]$ compute $P[T - S_n > x|N(T) = n]$.

Order statistic property: Let S_t be a random time, we do not know, where it is and how many there are of these random variables (S_t), however, if we fix a T, and give the number of S_t in the interval $(0, T) : N(T) = n$, That means that there are $S_1, ..., S_n$ random variables in the interval $(0, T)$. Note that n is fixed and not random. These variables $S_1, ..., S_n$ are uniformly independent in $(0, T)$. They are uniform draws from the interval $(0, T)$. However, when we solve this problem, we should not write that $S_1, ..., S_n$ are uniformly independent random variables, as it might be confusing given the definition of S, So we usually say that they are the same distribution as U which are independent uniformly distributed random variables.

$$\begin{aligned}
P[T - S_n > x|N(T) = n] &= P[S_n < T - x|N(T) = n] \\
&= P(U_1, U_2, ..., U_n < T - x) \\
&= P(U_1 < T - x, U_2 < T - x, ..., U_n < T - x) \\
&= P(U_1 < T - x)P(U_2 < T - x), ..., P(U_n < T - x) \\
&= \Big(\frac{T - x}{T}\Big)^n = \Big(1 - \frac{x}{T}\Big)^n
\end{aligned} \tag{29}$$

c) Compute $E[T - S_n | N(T) = n]$

We know that if $T - S_n : \Omega \to [0, +\infty)$ is a non negative random variable then

$$\begin{aligned}
E[T - S_n > x | N(T) = n] &= \int_0^\infty P[T - S_n > x | N(T) = n] dx \\
&= \int_0^\infty \Big(1 - \frac{x}{T}\Big)^n dx \quad u = x/T \\
&= T \int_0^T \Big(1 - u\Big)^n du \quad v = 1 - u \\
&= -T \int_0^T \Big(v\Big)^n dv \quad = -T \Big[\frac{(v)^{n+1}}{n+1}\Big]_0^T \\
&= -\frac{T}{(n+1)} \Big[1 - \frac{x}{T}\Big]_0^T = \frac{T}{(n+1)}
\end{aligned} \tag{30}$$

d) Compute $E[T - S_{N(T)}]$

Using the law of total expectations

$$\begin{aligned}
E[T - S_{N(T)}] &= \sum_{n=0}^{\infty} E[T - S_{N(T)} | N(T) = n] P(N(T) = n) \\
&= \sum_{n=0}^{\infty} \frac{T}{n!(n+1)} (\lambda T)^n e^{-\lambda T} \frac{\lambda}{\lambda} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(\lambda T)^{n+1}}{(n+1)!} e^{-\lambda T} \\
&= \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{(\lambda T)^n}{n!} e^{-\lambda T} - e^{-\lambda T} + e^{-\lambda T} = \frac{e^{-\lambda T}}{\lambda} \Big(e^{-\lambda T} - 1\Big) \\
&= \frac{1 - e^{-\lambda T}}{\lambda}
\end{aligned} \tag{31}$$

4.2 Problem 2: Renewal Theory

Alice and Bob play a match consisting of rallies, where Alice starts the first rally and the winner of a rally starts the next rally. The probability that Alice wins a rally that she starts herself is p_a (and she loses that rally with probability $1 - p_a$, while the probability that Bob wins a rally that he starts himself is p_b (and he loses the rally with probability $1 - p_b$ Conditioned on who starts the rally, the outcomes of a rally is independent of the outcomes of other rallies. Assume that $0 < p_a < 1$ and $0 < p_b < 1$. Let $\{N(t), t \in N\}$ be the number of rallies won by Alice among the first t rallies, and for $n \in N$ let $S_n = \min\{k \in N; N(k) = n\}$ be the number of rallies Alice needs to play in order to win n rallies.
a) Provide the distribution and expectation of S_1.

We know $N(t)$ is the number of rallies Alice won in the first t rallies. And that $S_1 = \min\{k \in N; N(k) = 1\}$ is the first win of Alice. Let's now denote two events:

- A, Alice is the one the starts the rally.
- B, Bob is the one starting the rally.

The probability that the first win of Alice happens only after 1 rally:

$$P(S_1 = 1) = p_a$$

The probability that the first win of Alice happens only after 2 rallies: is given by the product of the probability of Alice loosing the first rally and Bob loosing the second one.

$$P(S_1 = 2) = (1 - p_a)(1 - p_b)$$

The probability that the first win of Alice happens only after 3 rallies: she looses the first one, Bob wins the second one and looses the third one:

$$P(S_1 = 3) = (1 - p_a)p_b(1 - p_b)$$

The probability that the first win of Alice happens only after m rallies: She looses the first one, Bob wins the next $m - 2$ and looses the m one.

$$P(S_1 = m) = (1 - p_a)p_b^{m-2}(1 - p_b), \quad m \geq 2$$

$$\begin{aligned}
E[S_1] &= \sum_{m=1}^{\infty} mP(S_1 = m) \\
&= p_a + \sum_{m=2}^{\infty} mP(S_1 = m) = p_a + \sum_{m=2}^{\infty} m(1 - p_a)p_b^{m-2}(1 - p_b) \\
&= p_a + (1 - p_a)(1 - p_b) \sum_{m=2}^{\infty} m\ p_b^{m-2} \frac{p_b}{p_b} \\
&= p_a + (1 - p_a)(1 - p_b)\frac{1}{p_b}\Big[\Big(\frac{\partial}{\partial m} \sum_{m=1}^{\infty} p_b^m\Big) - 1\Big] = p_a + (1 - p_a)(1 - p_b)\frac{1}{p_b}\Big(\frac{1}{(1 - p_b)^2} - 1\Big) \\
&= p_a + (1 - p_a)(1 - p_b)\frac{1}{p_b}\Big(\frac{p_b(2 - p_b)}{(1 - p_b)^2}\Big) = p_a + \frac{(1 - p_a)(2 - p_b)}{(1 - p_b)} = 1 + \frac{(1 - p_a)}{(1 - p_b)}
\end{aligned} \tag{32}$$

b) Compute $E[N(t)]/t$ for $t \to \infty$

By applying the **Elementary Renewal Theorem** we obtain that th

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} = \lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu} = \frac{1}{E[S_1]} = \frac{(1-p_b)}{(1-p_b)+(1-p_a)} \tag{33}$$

Assume now that Alice and Bob get points for their "winning streaks" (rows of consecutive wins). If Alice wins k rallies in a row then the points for that streak are k_2.

c) Provide the (almost-sure) long run average number of points per rally for Alice.

In the long-run, the average number of points per rally for Alice is given by

$$\frac{E[\text{number of points Alice makes in a cycle}]}{E[\text{number of rallies in a cycle}]} = \frac{E[W_a^2]}{E[W_a]+E[W_b]}$$

The *number of points Alice makes in a cycle* is geometrically distributed, as we are looking for consecutive wins.

$$P(W_a = k) = (1-p_a)^{k-1} p_a \quad k = 1, 2, ...$$

Then

$$E[W_a] = \sum_{k=0}^{\infty} k\ (1-p_a)p_a^{(k-1)} = (1-p_a)\sum_{k=1}^{\infty} k\ p_a^{(k-1)} = \frac{1}{1-p_a} \Rightarrow E[W_b] = \frac{1}{1-p_b}$$

$$E[W_a^2] = \text{Var}(W_a) - E[W_a]^2 = \frac{1-(1-p_a)}{(1-p_a)^2} + \frac{1}{(1-p_a)^2} = \frac{p_a+1}{(1-p_a)^2}$$

Which yields to a long-run average of

$$\frac{(1-p_a)(1-p_a+1-p_b)}{(p_a+1)(1-p_b)}$$

4.3 Problem 4: Brownian Motion

Let f$\{B(t), t \geq 0\}$ be a standard Brownian motion and for $t > 0$, let

$$M(t) := \max_{0\leq s\leq t} B(s)$$

be the maximum of the Brownian motion up to time t. Here we assume that $B(0) = 0$ and that the variance parameter $\sigma^2 = 1$ is part of the definition of a standard Brownian motion.

a) For $y > 0$ and $x > 0$, argue that

$$P(M(t) > y, B(t) < y - x) = P(B(t) > y + x)$$

Let T_y be the first time the Brownian motion hits y, then we know that:

$$P(M(t) > y) = P(\max_{0 \leq s \leq t} B(s) > y) = P(T_y < t)$$

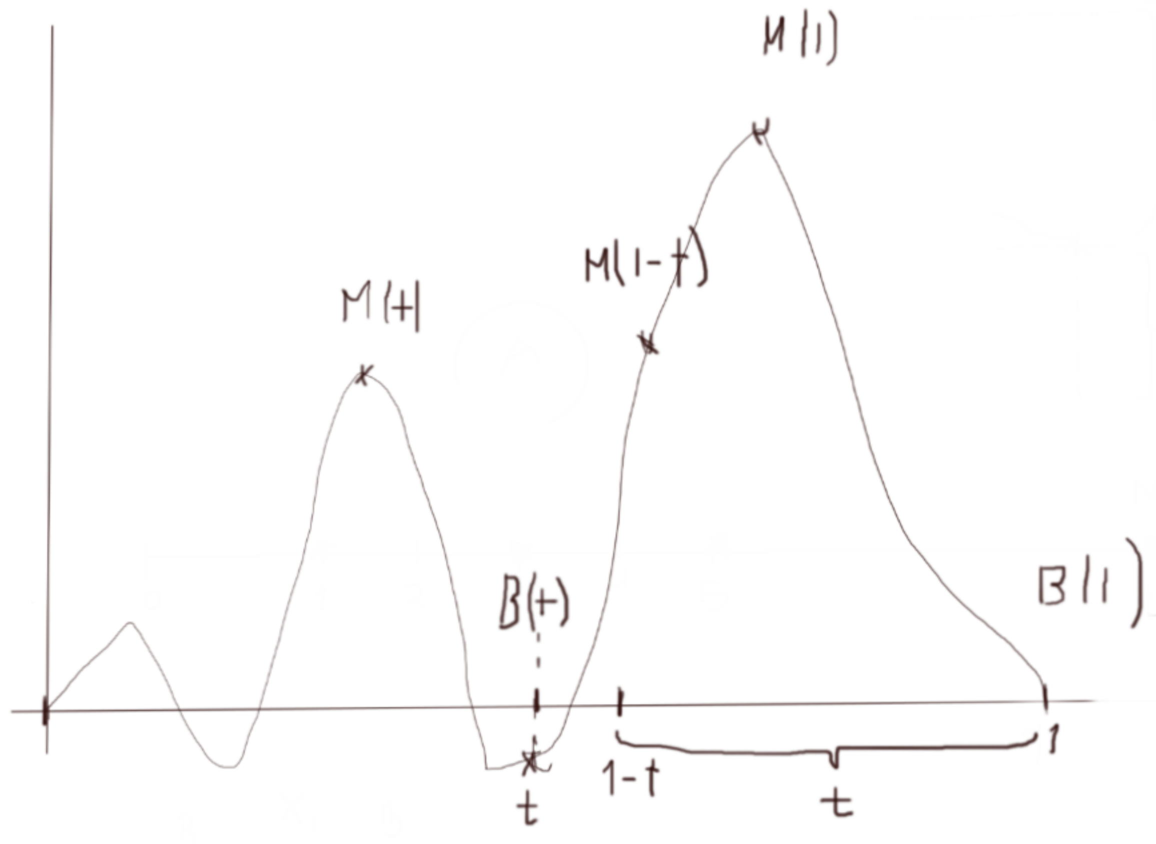
Which means that, the probability of the maximum of a Brownian motion in an interval $[0, t]$ being greater than y, is the same as the probability of the time the process hits y before time t. Where

$$T_y = \inf\{t; B(t) \geq y\}$$

Now, we have that

$$\begin{aligned} P(M(t) > y, B(t) < y - x) &= P(T_y < t, B(t) < y - x) \\ &= P(M(t) > y, B(t) < y - x) \\ &\text{By the reflection principle} \\ &= P(M(t) > y, B(t) > y + x) \\ &= P(B(t) > y + x) \end{aligned} \tag{34}$$

b) For $0 < t < 1$, show that $P(M(1) > M(t)) = \ldots$



$$
\begin{aligned}
M(1) > M(t) &= \max_{0\le s\le 1} B(s) > \max_{0\le s\le t} B(s) \\
&= \max_{0\le s\le 1} B(s) > \max_{0\le s\le t} B(s) - B(t) + B(t) \\
&= \max_{t\le s\le 1} (B(s) - B(t)) > M(t) - B(t) \\
&= M(1-t) > M(t) - B(t)
\end{aligned} \tag{35}
$$

Note that $\max_{t\le s\le 1}(B(s)-B(t))$ **is not** $M(1-t)$ but it is distributed as $M(1-t)$, meaning that it has a similar behaviour, i.e

$$
M(1-t) = \max_{0\le s\le 1-t} (B(s) - B(t))
$$

So saying that $\max_{t\le s\le 1}(B(s) - B(t)) = M(1-t)$ is "abuse of language". By the independent increments property of a Brownian Motion $\max_{t\le s\le 1}(B(s) - B(t))$ is independent of $\max_{0\le s\le t} B(s) - B(t)$.

$$
\begin{aligned}
P(M(1) > M(t)) &= P(M(1-t) > M(t) - B(t)) \\
&= \int_0^\infty \int_{-\infty}^y f_{M(t),B(t)}(x,y) P(M(1-t) > y - x) dx dy
\end{aligned} \tag{36}
$$

c) Let $T_{\max}(1) = \{t \in (0,1); B(t) = M(1)\}$ be the time when the Brownian Motion takes its maximum on the interval $(0,1)$. Provide the distribution function of $T_{\max}(1)$.

$$
\begin{aligned}
P(T_{\max}(1) \le t) &= P(\max_{0\le s\le t} B(t) = \max_{0\le s\le 1} B(s)) \\
&= P(M(t) = M(1)) = 1 - P(M(t) < M(1))
\end{aligned} \tag{37}
$$

4.4 Problem 5: Simulation

Let $U_1, U_2, \ldots$ be independent random variables taking uniform values between 0 and 1.
a) Explain in detail that $-\log(U_1)$ has density $f_1(t) = e^{-t}$ for $t \ge 0$.

Consider that

$$
Y_1 = -\log(U_1)
$$

where $U_1 \sim U(0,1)$

$$
\iff F_{Y_1}(t) = P(Y_1 < t) = P(\log U_1 > -t) = P(U_1 > e^{-t}) = \int_0^{e^{-t}} \quad dt = e^{-t}
$$

which is the density function of an exponential random variable with rate 1.

b)Show that $-\sum_{k=1}^{n} \log(U_k)$ has density

$$f_n(t) = \frac{x^{n-1}}{(n-1)!}e^{-x}$$

The sum of exponentially distributed random variables follows a gamma distribution.

c)Define

$$N = \min\left\{k \in N; \prod_{j=1}^{k+1} U_j \leq e^{-\lambda}\right\}$$

What is the distribution of N?

$$\begin{aligned} N &= \min\left\{k \in N; \prod_{j=1}^{k+1} U_j \leq e^{-\lambda}\right\} = \min\left\{k \in N; \log\left(\prod_{j=1}^{k+1} U_j\right) \leq -\lambda\right\} \\ &= \min\left\{k \in N; -\log\left(\prod_{j=1}^{k+1} U_j\right) \geq \lambda\right\} = \min\left\{k \in N; -\sum_{j=1}^{k+1} \log(U_j) \geq \lambda\right\} \end{aligned} \tag{38}$$

Which means that N the minimum number of iid exponentially distributed random variables that we will sum, all of them being at least λ. We know that $-\sum_{j=1}^{k+1} \log(U_j) \sim gamma(k+1, 1)$ while $U_j \sim \exp(1)$ which means that $N \sim 1 + X$ where $X \sim Po(\lambda)$ as N can not be zero, while k can.