Stochastic Processes II
Homework exercises

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1 Session 1: Ross-Chapter 5

Exercise 5.3: Let X be an exponential random variable. Without any computations, tell which one of the following is correct. Explain your answer.

- (a) $E[X^2|X > 1] = E[(X + 1)^2]$
- (b) $E[X^2|X > 1] = E[X^2] + 1$
- (c) $E[X^2|X > 1] = (1 + E[X])^2$

X is exponentially distributed and therefore memoryless. This implies that for all $t, s > 0$, we have:

$$P(X > t + s|X > t) = P(X > s)$$

In particular, this implies that for any continuous function $f : R_+ \to R_+$ (or any other nice enough function), we have:

$$E[f(X)|X > t] = E[f(X + t)]$$

- (a) is correct, as the unconditional distribution of X given $X > 1$ is $X + 1$. Where

$$E[(X + 1)^2] = E[X^2] + 2E[X] + 1$$

- (b) is incorrect as

$$E[X^2] + 1 \neq E[X^2] + 2E[X] + 1$$

  $E[X] = 1/\lambda$ where $\lambda > 0$, hence $E[X] > 0$

- (c) is incorrect, as $\mathrm{var}(X) > 0$

$$E[X]^2 + 2E[X] + 1 \neq E[X^2] + 2E[X] + 1 - (E[X^2] - E[X]^2)$$

Exercise 5.8: If X and Y are independent exponential random variables with respective rates λ and μ, what is the conditional distribution of X given that $X < Y$?

$X \sim \exp(\lambda), \quad Y \sim \exp(\mu)$. Then $f_X(t) = \lambda e^{-\lambda t}$ and $f_Y(t) = \mu e^{-\mu t}$ As both X and Y are independent the joint probability density function is :

$$f_{X,Y}(x, y) = f_X(t) f_Y(t) = \lambda\mu e^{-t(\lambda+\mu)}$$

We would like to find $P(X > t|X < Y)$:

$$f(X > t|X < Y) = \frac{f(X > t, X < Y)}{f(X < Y)}$$

$$\begin{aligned}
f(X > t, X < Y) &= \int_t^{\infty}\int_x^{\infty} f(x, y)dy\ dx = \int_t^{\infty} \lambda e^{-\lambda x}\int_x^{\infty} \mu e^{-\mu y}dy\ dx \\
&= \int_t^{\infty} \lambda e^{-\lambda x}\mu\left[\lim_{x\to\infty}\frac{1}{\mu e^{x\mu}} + \frac{e^{-\mu x}}{\mu}\right] dx \\
&= \int_t^{\infty} \lambda e^{-\lambda x} e^{-\mu x}\ dx = \int_t^{\infty} \lambda e^{-x(\lambda+\mu)}\ dx = \frac{\lambda}{\lambda+\mu} e^{-t(\lambda+\mu)}
\end{aligned} \tag{1}$$

$$\begin{aligned}
f(X < Y) = \int_0^{\infty}\int_0^{y} f(x, y)dx\ dy = \int_0^{\infty} \lambda e^{-\lambda x}\int_x^{\infty} \mu e^{-\mu y}dy\ dx = \int_0^{\infty} \lambda e^{-x(\lambda+\mu)}\ dx \\
= \frac{\lambda}{\mu+\lambda}
\end{aligned} \tag{2}$$

$$f(X > t|X < Y) = \frac{\frac{\lambda}{\lambda+\mu}e^{-t(\lambda+\mu)}}{\frac{\lambda}{\mu+\lambda}} = e^{-t(\lambda+\mu)} = P(\min(X, Y) > t)$$

□

Exercise 5.36: Let $S(t)$ denote the price of a security at time t. A popular model for the process $S(t), t \geq 0$ supposes that the price remains unchanged until a "shock" occurs, at which time the price is multiplied by a random factor. If we let $N(t)$ denote the number of shocks by time t, and let X_i denote the i^{th} multiplicative factor, then this model supposes that:

$$S(t) = S(0)\prod_{i=0}^{N(t)} X_i$$

Where $\prod_{i=0}^{N(t)} X_i$ is equal to 1 when $N(t) = 0$. Suppose that the X_i are independent exponential random variables with rate μ; that $N(t), t \geq 0$ is a Poisson process with rate λ; that $N(t), t \geq 0$ is independent of the X_i ; and that $S(0) = s$.

- Find $E[S(t)]$.
- Find $E[S^2(t)]$.

First: We have the processes $\{S(t), t \geq 0\}$ where $S(t) \sim \text{Gamma}(n, \lambda)$

$$S(t) = s \prod_{i=0}^{N(t)} X_i$$

and the Poisson Process $\{N(t), t \geq 0\}$ with rate λ. We know that the multiplicative factor $X_i \overset{iid}{\sim} \exp(\mu)$ and that they are independent of the counting process $\{N(t), t \geq 0\}$. Hence $S(t) = E[S(t)|N(t)]$:

- (a)

$$\begin{aligned} E[S(t)] = E[E[S(t)|N(t)]] &= E\left[E\left[s \prod_{i=0}^{N(t)} X_i \middle| N(t)\right]\right] = E\left[s \prod_{i=0}^{N(t)} E\left[X_i \middle| N(t)\right]\right] \\ &= E\left[s \prod_{i=0}^{N(t)} E[X_i]\right] = E\left[s \prod_{i=0}^{N(t)} \mu^{-1}\right] = E\left[s(\mu)^{-N(t)}\right] \end{aligned} \tag{3}$$

$$\begin{aligned} E\left[s(\mu)^{-N(t)}\right] &= \sum_{n \geq 0} S(t) P(N(t) = n) = \sum_{n \geq 0} s\mu^{-n} e^{-\lambda t} \frac{(\lambda t)^n}{n!} = se^{-\lambda t} \sum_{n \geq 0} \frac{(\lambda t/\mu)^n}{n!} \\ &= se^{-\lambda t} e^{(\lambda t/\mu)} = se^{-\lambda t(1-1/\mu)} \end{aligned} \tag{4}$$

$\square$

- (b)

$$\begin{aligned} E[S(t)^2] &= E[E[S(t)^2|N(t)]] \\ &= E\left[E\left[s^2 \prod_{i=0}^{N(t)} X_i^2 \middle| N(t)\right]\right] \\ &\text{by independence of the } X_i\text{'s and known } s \\ &= E\left[s^2 \prod_{i=0}^{N(t)} E\left[X_i^2 \middle| N(t)\right]\right] \\ &\text{by independence of the } X_i\text{'s and the counting process N(t)} \\ &= E\left[s^2 \prod_{i=0}^{N(t)} E[X_i^2]\right] = E\left[s^2 (E[X_i^2])^{N(t)}\right] = E\left[s^2 (2/\mu^2)^{N(t)}\right] \end{aligned} \tag{5}$$

$$\begin{aligned} E\left[s^2(2/\mu^2)^{N(t)}\right] &= \sum_{n \geq 0} S^2(t) P(N(t) = n) = \sum_{n \geq 0} s^2 (2/\mu^2)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} = s^2 e^{-\lambda t} \sum_{n \geq 0} \frac{(2\lambda t/\mu^2)^n}{n!} \\ &= s^2 e^{-\lambda t} e^{(2\lambda t/\mu^2)} = s^2 e^{-\lambda t(1-2/\mu^2)} \end{aligned} \tag{6}$$

$\square$

Exercise 5.40: Show that if $\{N_i(t), t \geq 0\}$ are independent Poisson processes with rate λ_i, $i = 1, 2$, then $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$ where $N(t) = N_1(t) + N_2(t)$.

Combining and Splitting Poisson Processes: We know already that $\{N_i(t), t \geq 0\}$ are independent Poisson processes then we want to prove that $\{N_1(t) + N_2(t), t \geq 0\}$ is also a Poisson process with rate $\lambda_1 + \lambda_2$. For this we will need to show that the following axioms hold:

- (i) $N(0) = 0$

Given that $N_1(0) = 0$ and $N_2(0) = 0$, then $N_1(0) + N_2(0) = 0$

- (ii) $\{N_i(t), t \geq 0\}$ has independent increments,

The process $\{N(t);\ t \geq 0\}$ has independent increments, in the sense that for all $0 \leq t_1 < t_2 < t_3 < t_4 < \infty$, $N(t_4) - N(t_3)$ is independent of $N(t_2) - N(t_1)$.

Hence, the increment $N_1(t_2) - N_1(t_1)$ is independent of $N_1(t_4) - N_1(t_3)$ and $N_2(t_2) - N_2(t_1)$ is also independent of $N_2(t_4) - N_2(t_3)$ meaning that $N_1(t_2) + N_2(t_2) - N_1(t_1) - N_2(t_1)$ and $N_1(t_4) + N_2(t_4) - N_1(t_3) - N_2(t_3)$ are also independent.

- (iii) $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$

Using the fact that $N(t) = N_1(t) + N_2(t)$. We can prove the third axiom as:

$$\begin{aligned}
P(N(t+h) - N(t) = 1) &= P(N_1(t+h) - N_1(t) = 1, N_2(t+h) - N_2(t) = 0) \\
&\quad + P(N_1(t+h) - N_1(t) = 0, N_2(t+h) - N_2(t) = 1) \\
&= P(N_1(t+h) - N_1(t) = 1)P(N_2(t+h) - N_2(t) = 0) \\
&\quad + P(N_1(t+h) - N_1(t) = 0)P(N_2(t+h) - N_2(t) = 1) \\
&= (\lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) + (\lambda_2 h + o(h))(1 - \lambda_1 h + o(h)) \\
&= \lambda_1 h + \lambda_2 h - 2\lambda_1\lambda_2 h^2 + 2o(h) + o(h)^2 \\
&= h(\lambda_1 + \lambda_2) - o(h) + o(h) + o(h) = h(\lambda_1 + \lambda_2) + o(h)
\end{aligned} \tag{7}$$

- (iv) $P(N(t+h) - N(t) \geq 2) = o(h)$

$$\begin{aligned}
P(N(t+h) - N(t) \geq 2) &= P(N_1(t+h) - N_1(t) = 1, N_2(t+h) - N_2(t) = 1) \\
&\quad + P(N_1(t+h) - N_1(t) \geq 2 \cup N_2(t+h) - N_2(t) \geq 2) \\
&\leq P(N_1(t+h) - N_1(t) = 1, N_2(t+h) - N_2(t) = 1) \\
&\quad + P(N_1(t+h) - N_1(t) \geq 2, N_2(t+h) - N_2(t) \geq 2) \\
&= (\lambda_1 h + o(h))(\lambda_2 h + o(h)) + o(h) + o(h) \\
&= o(h)o(h) + o(h) = o(h)
\end{aligned} \tag{8}$$

Using the fact that any finite linear combination of functions, each of which is $o(h)$, is $o(h)$. $\square$

Exercise 5.45: Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ that is independent of the non-negative random variable T with mean μ and variance σ^2. Find

- (a) $Cov(T, N(T))$,
- (b) $Var(N(T))$.

We will first compute the moment generating function of $N(t)$ so we can find the first and second moment:
$M_{N(t)}(\sigma) = E[e^{\sigma N(t)}] = \sum_{n=1}^{\infty} e^{\sigma n} P(N(T) = n) = \sum_{n=1}^{\infty} e^{\sigma n} e^{-\lambda T} \frac{(\lambda T)^n}{n!} = e^{-\lambda T} \sum_{n=1}^{\infty} \frac{(e^{\sigma} \lambda T)^n}{n!} = e^{-\lambda T} e^{e^{\sigma} \lambda T} = e^{\lambda T(e^{\sigma}-1)}$

$$
\begin{aligned}
E[N(T)] &= \left.\frac{\partial M_{N(t)}(\sigma)}{\partial \sigma}\right|_{\sigma=0} \\
&\text{apply chain rule, where } f = e^u \text{ and } u = \lambda T(e^{\sigma} - 1) \\
&= \frac{\partial}{\partial u} e^u \frac{\partial}{\partial \sigma} \lambda T(e^{\sigma} - 1) \\
&= \left.\lambda T e^{\lambda T(e^{\sigma}-1)+\sigma}\right|_{\sigma=0} \\
&= \lambda T
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
E[N(T)^2] &= \left.\frac{\partial^2 M_{N(t)}(\sigma)}{\partial^2 \sigma}\right|_{\sigma=0} \\
&= \lambda T \frac{\partial}{\partial \sigma} e^{\lambda T(e^{\sigma}-1)+\sigma} \\
&\text{apply chain rule, where } f = e^u \text{ and } u = \lambda T(e^{\sigma} - 1) + \sigma \\
&= \frac{\partial}{\partial u} e^u \frac{\partial}{\partial \sigma} \lambda T(e^{\sigma} - 1) + \sigma \\
&= \left.\lambda T e^{\lambda T(e^{\sigma}-1)+\sigma} (\lambda T e^{\theta} + 1)\right|_{\sigma=0} \\
&= (\lambda T)^2 + \lambda T
\end{aligned} \tag{10}
$$

$E[TN(T)] = E[E[TN(T)|T]] = E[T\ E[N(T)|T]] = E[T\ E[N(T)]] = E[T\ \lambda T] = \lambda E[T^2]$
$E[N(T)] = E[E[N(T)|T]] = E[E[N(T)]] = E[\lambda T] = \lambda E[T]$
$E[N(T)^2] = E[E[N(T)^2|T]] = E[E[N(T)^2]] = \lambda^2 E[T^2] + \lambda E[T]$

- (a)
  $Cov(T, N(T)) = E[TN(T)] - E[T]E[N(T)] = \lambda E[T^2] - \lambda E[T]^2 = \lambda(E[T^2] - E[T]^2) = \lambda \sigma^2$

- (b)
  $Var(N(T)) = E[N(T)^2] - E[N(T)]^2 = E[E[N(T)^2|T]] - E[E[N(T)|T]]^2 = \lambda^2 E[T^2] + \lambda E[T] - \lambda^2 E[T]^2 = \lambda E[T] + \lambda^2(E[T^2] - E[T]^2) = \lambda\mu + \lambda\sigma^2$

$\square$

Exercise 5.49: Events occur according to a Poisson process with rate λ. Each time an event occurs, we must decide whether or not to stop, with our objective being to stop at the last event to occur prior to some specified time T , where $T > \frac{1}{\lambda}$. That is, if an event occurs at time $t, 0 \leq t \leq T$, and we decide to stop, then we win if there are no additional events by time T , and we lose otherwise. If we do not stop when an event occurs and no additional events occur by time T , then we lose. Also, if no events occur by time T , then we lose. Consider the strategy that stops at the first event to occur after some fixed time $s, 0 \leq s \leq T$.

- (a) Using this strategy, what is the probability of winning?
- (b) What value of s maximizes the probability of winning?
- (c) Show that one's probability of winning when using the preceding strategy with the value of s specified in part (b) is 1/e.

- (a) We win if there is exactly one event in the time interval (s, T) We would like to find $P(N(T) - N(s) = 1)$. Let $\{N(t),\ t \geq 0\}$ be a Poisson Process with rate λ $N(t) \sim \text{Poisson}(t\lambda)$.

$$f(s) = P(N(T-s) = 1) = \lambda(T-s)e^{-\lambda(T-s)} \tag{11}$$

- (b) For obtaining the value of s that maximizes $f(s)$ we will differentiate $f(s)$ with respect to s and set it to zero.

$$\begin{aligned} \frac{\partial f(s)}{\partial s} &= \lambda \frac{\partial}{\partial s}(T-s)e^{-\lambda(T-s)} \\ &\text{applying the product rule} \\ &= \lambda\big(-e^{-\lambda(T-s)} + \lambda(T-s)e^{-\lambda(T-s)}\big) \end{aligned} \tag{12}$$

Set to zero for finding the optimal time:

$$\lambda\big(-e^{-\lambda(T-s)} + \lambda(T-s)e^{-\lambda(T-s)}\big) = 0$$

$$(T-s) = \frac{1}{\lambda} \tag{13}$$

- If we now plug the results obtained in eq (13) into eq (11) we obtain that the probability of winning

$$P(N(T) - N(s) = 1) = 1/e$$

$\square$

Exercise 5.60: Customers arrive at a bank at a Poisson rate λ. Suppose two customers arrived during the first hour. What is the probability that

- (a) both arrived during the first 20 minutes?
- (b) at least one arrived during the first 20 minutes?

Let $\{N(t), t \geq 0\}$ be a Poisson Process with rate λ where the time unit t is an hour, we suppose that $N(1) = 2$. $N(t) \sim Pois(t\lambda)$.

- (a) We would like to find $P(N(1/3) = 2|N(1) = 2)$. Supposing that two customers arrive during the first hour we will have to compute: (a) the probability of two customers arriving during the first 20 minutes and (b) the probability of having no customers on the remaining 40 minutes. i.e $P(N(s) = 2, N(t) - N(s) = 0)$

$$\begin{aligned}
P(N(1/3) = 2|N(1) = 2) &= \frac{P(N(1/3) = 2, N(1) = 2)}{P(N(1) = 2)} \\
&\text{independent increments} \\
&= \frac{P(N(1/3) = 2)P(N(2/3) = 0)}{P(N(1) = 2)} \\
&= \frac{\frac{(\lambda/3)^2 e^{-\lambda/3}}{2!}\frac{(2\lambda/3)^0 e^{-2\lambda/3)}}{0!}}{\frac{(\lambda)^2 e^{-\lambda}}{2!}} = \frac{\lambda^2 e^{-\lambda/3 - 2\lambda/3}}{9\lambda^2 e^{-\lambda}} = \frac{1}{9}
\end{aligned} \tag{14}$$

□

- (b) We would like to find $P(N(1/3) \geq 1|N(1) = 2)$ for this we will need to consider two possibilities (a) exactly one person arrived on the first 20 minutes and the second person arrived on the remaining 40 minutes or (b) the two persons arrived on the first 20 minutes and nobody arrived on the remaining 40 minutes. Possibility (b) has already been computed on the previous question, hence we will only compute $P(N(s) = 1, N(t) - N(s) = 1)$

$$\begin{aligned}
P(N(1/3) = 1|N(1) = 2) &= \frac{P(N(1/3) = 1, N(1) = 2)}{P(N(1) = 2)} \\
&\text{independent increments} \\
&= \frac{P(N(1/3) = 1)P(N(2/3) = 1)}{P(N(1) = 2)} \\
&= \frac{\frac{(\lambda/3)^1 e^{-\lambda/3}}{1!}\frac{(2\lambda/3)^1 e^{-2\lambda/3)}}{1!}}{\frac{(\lambda)^2 e^{-\lambda}}{2!}} = \frac{4\lambda^2 e^{-\lambda/3} e^{-2\lambda/3}}{9\lambda^2 e^{-\lambda}} = \frac{4}{9}
\end{aligned} \tag{15}$$

Hence: $P(N(1/3) \geq 1|N(1) = 2) = \frac{5}{9}$

□

2 Session 2: Ross-Chapter 5

Exercise 5.46: Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ that is independent of the sequence $X1, X2, ...$ of independent and identically distributed random variables with mean μ and variance σ^2. Find

$$Cov\left(N(t), \sum_{i=1}^{N(t)} X_i\right)$$

$$Cov\left(N(t), \sum_{i=1}^{N(t)} X_i\right) = E\left[N(t)\sum_{i=1}^{N(t)} X_i\right] - E[N(t)]E\left[\sum_{i=1}^{N(t)} X_i\right]$$

$$\begin{aligned} E\left[\sum_{i=1}^{N(t)} X_i\right] &= E\left[E\left[\sum_{i=1}^{N(t)} X_i \middle| N(t)\right]\right] = E\left[\sum_{i=1}^{N(t)} E\left[X_i \middle| N(t)\right]\right] = E\left[\sum_{i=1}^{N(t)} E\Big[X_i\Big]\right] \\ &= E\Big[\sum_{i=1}^{N(t)} \mu\Big] = E\Big[N(t)\mu\Big] = \mu E\Big[N(t)\Big] = \mu\lambda t \end{aligned} \tag{16}$$

$$\begin{aligned} E\left[N(t)\sum_{i=1}^{N(t)} X_i\right] &= E\left[E\left[N(t)\sum_{i=1}^{N(t)} X_i \middle| N(t)\right]\right] = E\left[EN(t)\sum_{i=1}^{N(t)}\Big[X_i\Big]\right] = E\Big[N(t)^2\mu\Big] = \mu E\Big[N(t)^2\Big] \\ &= \mu((\lambda t)^2 + \lambda t) \end{aligned} \tag{17}$$

$$Cov\left(N(t), \sum_{i=1}^{N(t)} X_i\right) = \mu((\lambda t)^2 + \lambda t) - (\lambda t)^2\mu = \mu\lambda t$$

□

Exercise 5.78: A store opens at 8 A.M. From 8 until 10 A.M. customers arrive at a Poisson rate of four an hour. Between 10 A.M. and 12 P.M. they arrive at a Poisson rate of eight an hour. From 12 P.M. to 2 P.M. the arrival rate increases steadily from eight per hour at 12 P.M. to ten per hour at 2 P.M.; and from 2 to 5 P.M. the arrival rate drops steadily from ten per hour at 2 P.M. to four per hour at 5 P.M.. Determine the probability distribution of the number of customers that enter the store on a given day.

The intensity function $\lambda(t)$ is given by:

$$\lambda(t) = \begin{cases} 0, & 0 \leq t \leq 8 \\ 4, & 8 \leq t \leq 10 \\ 8, & 10 \leq t \leq 12 \\ 8 + (t-12), & 12 \leq t \leq 14 \\ 10 - 2(t-14), & 14 \leq t \leq 17 \\ 0, & 17 \leq t \leq 24 \end{cases}$$

The mean value function $m(t)$ of the non homogeneous Poisson Process gives us the probability distribution of the number of customers that entered the store in a day:

$$m(24) = \int_0^{24} \lambda(t)\ dt = \int_8^{10} 4\ dt + \int_{10}^{12} 8\ dt \int_{12}^{14} t - 4\ dt + \int_{14}^{17} 38 - 2t\ dt = 63$$

$\square$

Exercise 5.81

- (a) Let $\{N(t), t \geq 0\}$ be a non-homogeneous Poisson process with mean value function $m(t)$. Given $N(t) = n$, show that the unordered set of arrival times has the same distribution as n independent and identically distributed random variables having distribution function:

$$F(x) = \begin{cases} \frac{m(x)}{m(t)}, & x \leq t \\ 1, & x \geq t \end{cases}$$

- (b) Suppose that workmen incur accidents in accordance with a non-homogeneous Poisson process with mean value function $m(t)$. Suppose further that each injured man is out of work for a random amount of time having distribution F. Let $X(t)$ be the number of workers who are out of work at time t. By using part (a), find $E[X(t)]$.

- (a)
  If $\{N(t), t \geq 0\}$ is a non-homogeneous Poisson Process with intensity function $\lambda(t)$, then $N(t+s) - N(s)$ is a Poisson random variable with mean

$$\lambda(t)h + o(h) = (t+h) - m(s) = \int_s^{t+s} \lambda(v)d\ v \quad \text{and} \quad m(t) = \int_0^t \lambda(s)d\ s$$

  . Let S_i denote the time of the i-th event, $i \geq 1$ and let $t_i + h_i < t_{i+1}, t_n + h_n \leq t$. Then $t_i < t_i + h_i, t_n < t_n + h_n \leq t$.

$$
\begin{aligned}
&P(t_i < S_i < t_i + h_i, i = 1, ..., n | N(t) = n) \\
&\qquad = \frac{P(t_i < S_i < t_i + h_i, i = 1, ..., n, N(t) = n)}{P(N(t) = n)} \\
&\qquad = \frac{P(t_i < S_i < t_i + h_i, N(t) = n) \cdot ... \cdot P(t_n < S_n < t_n + h_n, N(t) = n)}{P(N(t) = n)} \\
&\qquad = \frac{\prod_{i=1}^{n} P(t_i < S_i < t_i + h_i, N(t) = n)}{P(N(t) = n)} \\
&\qquad = \frac{\prod_{i=1}^{n} (m(t_i + h_i) - m(t_i)) e^{-(m(t_i+h_i) - m(t_i))}}{\frac{(m(t))^n}{n!} e^{-m(t)}} \\
&\qquad = \frac{n! \prod_{i=1}^{n} [m(t_i + h_i) - m(t_i)]}{[m(t)]^n} \frac{e^{-\sum_{i=1}^{n} [m(t_i+h_i) - m(t_i)]}}{e^{-m(t)}} \\
&\qquad = \frac{n! \prod_{i=1}^{n} [m(t_i + h_i) - m(t_i)]}{[m(t)]^n} \\
&\qquad = \frac{n! \prod_{i=1}^{n} \lambda(t_i) h_i + o(h_i)}{[m(t)]^n}
\end{aligned}
\tag{18}
$$

Where, $\exp\{-\sum_{i=1}^{n} [m(t_i+h_i) - m(t_i)]\} = \exp\{-\sum_{i=1}^{n} \int_{t_i}^{t_i+h_i} \lambda(v)dv\} = \exp\{-\int_{t_1}^{t_n+h_n} \lambda(v)dv\} = \exp\{-m(t)\}$ As the intensity function takes values of zero between 0 and t_1.

If we now take the limit of $\frac{f(h_i)}{h_i}$ for $i = 1, ..., n$ and $h_i \to \infty$ we obtain the following result:

$$
\begin{aligned}
&\lim_{h_i \to 0} P(t_i < S_i < t_i + h_i, i = 1, ..., n | N(t) = n)/h_i, \qquad i = 1, ..., n \\
&\qquad\qquad\qquad\qquad = \frac{n!}{[m(t)]^n} \prod_{i=1}^{n} \lim_{h_i \to 0} \frac{\lambda(t_i) h_i}{h_i} + \lim_{h_i \to 0} \frac{o(h)}{h_i} \\
&f_{S_1, S_2, ...}(t_1, ..., t_n | N(t) = n) = n! \prod_{i=1}^{n} \frac{\lambda(t_i)}{m(t)}
\end{aligned}
\tag{19}
$$

- (b)

Injuries happen with an intensity rate $m(t) = \int_0^t \lambda(s)ds$ up to time t where the sick-leave length of an injured worker is distributed as F. $X(t)$ represents the number of workers on sick-leave at time t and let $N(t)$ be the number of injured workers at time t. $(N(t) \sim Po(m(t)))$

$$
E[X(t)] = E[E[X(t)|N(t)]] \tag{20}
$$

□

Exercise 5.95: Let $\{N(t), t \geq 0\}$ be a conditional Poisson process with a random rate L.

- (a) Derive an expression for $E[L|N(t) = n]$.

  Given the event $\{N(t) = n\}$ we know that $N(t) \sim Po(\lambda)$ is a discrete random variable and $L \sim \Gamma(m, \theta)$. By definition, the conditional expectation is given by:

$$E[L|N(t) = n] = \frac{E[L\mathbb{1}_{\{N(t)=n\}}]}{E[\mathbb{1}_{\{N(t)=n\}}]} = \frac{E[LE[\mathbb{1}_{\{N(t)=n\}}|L]]}{E[E[\mathbb{1}_{\{N(t)=n\}}|L]]} = \frac{E[LP(N(t) = n|L)]}{E[P(N(t) = n|L)]} = \frac{E[L^n e^{-Lt}]}{E[L^{n+1}e^{-Lt}]}$$

  Knowing that:

$$E[P(N(t) = n|L)] = E\left[\frac{(Lt)^n}{n!}e^{-Lt}\right] \quad \text{and} \quad P(N(t) = n) = \int_0^\infty f(n; \lambda t) g_L(\lambda)\, d\lambda$$

$$E[LP(N(t) = n|L)] = E\left[L^{n+1}\frac{t^n}{n!}e^{-Lt}\right] \quad \text{and} \quad [L\mathbb{1}_{\{N(t)=n\}}] = \int_0^\infty \lambda f(n; \lambda t) g_L(\lambda)\, d\lambda$$

- (b) Find, for $s > t$, $E[N(s)|N(t) = n]$.

$$\begin{aligned} E[N(s)|N(t) = n] &= E[N(s) - N(t)|N(t) = n] + E[N(t)|N(t) = n] \\ &= E[E[N(s) - N(t)|L, N(t) = n]|N(t) = n] + n \\ &= E[(s - t)L|N(t) = n] + n \\ &= (s - t)E[L|N(t) = n] + n \end{aligned} \tag{21}$$

- (c) Find, for $s < t$, $E[N(s)|N(t) = n]$.

$$\begin{aligned} E[N(s)|N(t) = n] &= E[N(s)|N(t) = n] \\ &= E[E[N(s)|L, N(t) = n]|N(t) = n] \\ &= E[ns/t|N(t) = n] = ns/t \end{aligned} \tag{22}$$

  Notice that $E[N(s)|L, N(t) = n]$ is computed using the order statistic property, as $s < t$ this means that for any value of L given that there have been n events by time t i.e $(N(t) = n)$, those n events are random variables uniformly distributed in the interval $(0, t)$.

  □

3 Session 3:Renewal Theory ; Ross-Chapter 7

Exercise 7.1 Is it true that

- (a) $N(t) < n$ if and only if $S_n > t$?
- (b) $N(t) \leq n$ if and only if $S_n \geq t$?
- (c) $N(t) > n$ if and only if $S_n < t$?

- (a) **true**.
  Knowing that $S_n > t$ means that the n-th event occurred after time t, will be the same as saying that the total number of "events" that occur by time t i.e. $N(t)$ is less than n.

- (b) **false**
  Whenever the total number of events that occur by time t is n, i.e $N(t) = n$, this will mean that the n-th event occurred at the latest at time t then $S_n \leq t$.

- (c) **false**
  Notice that when the number of "events" that occur by time t is greater than n i.e $N(t) > n$, implies that that n-th arrival occurred before time t, i.e $S_{n+1} < t$, while saying that $S_n < t$ means that the n-th arrival occurred before time t but this does not imply that the $n+1$ event also occurred before time t.

Exercise 7.3 Let S_n denote the time of the n-th event of the renewal process $\{N(t), t \geq 0\}$ having inter-arrival distribution F.

S_n denotes the time of the n-th renewal having inter-arrival times $X_1, X_2, ...$ Hence,

$$S_n = \sum_{i=1}^{n} X_i, \quad X_i \sim \exp(\lambda) \quad \text{and} \quad S_n \sim \Gamma(n, \lambda)$$

.

- (a) What is $P(N(t) = n | S_n = y)$?

  (i) If we consider that $t \geq y$ and using the fact $N(t) = n$. We have that when the number of "events" that occur by time t is of n given that the n-th event occurred at time y this will mean that $S_n \leq t$ and $S_{n+1} > t$ i.e no event occurred in the time elapsed between y and t. We can compute the probability as:

$$\begin{aligned} P(N(t) = n|S_n = y) &= P(S_n \leq t < S_{n+1}|S_n = y) = P(S_n \leq t < S_n + X_{n+1}|S_n = y) \\ &= P(X_{n+1} < t|X_1 + ... + X_n = y) = P(X_{n+1} < t - y) \\ &= 1 - P(X_{n+1} \geq t - y) = 1 - F(t - y) \end{aligned}$$

  Notice that $F(t - y)$ denotes the inter-arrival distribution.

$$\begin{aligned} F_{X_{n+1}}(t - y) = P(X_{n+1} < t - y) &= \int_{-\infty}^{t-y} f_{X_{n+1}}(s)\ ds = \int_{-\infty}^{0} f_{X_{n+1}}(s)\ ds + \int_{0}^{t-y} f_{X_{n+1}}(s)\ ds \\ &= \int_{0}^{t-y} f_{X_{n+1}}(s)\ ds = \int_{0}^{t-y} \lambda e^{-\lambda s}\ ds = \left[\lambda \frac{e^{-\lambda s}}{-\lambda}\right]_0^{t-y} = 1 - e^{-\lambda(t-y)} \end{aligned}$$

$$P(N(t) = n|S_n = y) = e^{-\lambda(t-y)}$$

(ii) If we now consider that $t < y$ then $S_n > t$ i.e the n-th event happens after time t which will be the same as saying that $N(t) < n$ and so $P(N(t) = n | S_n = y) = 0$.

- (b) Starting with

$$P(N(t) = n) = \int_0^{\infty} P(N(t) = n | S_n = y) f_{S_n}(y) \, dy$$

And using that the sum of n independent exponentials with rate λ has the Gamma (n, λ) distribution, derive $P(N(t) = n)$ when $F(y) = 1 - e^{-\lambda y}$.

In a **renewal process** the times between successive events are independent and identically distributed with an **arbitrary distribution**. In this case we assume that the inter-arrival times $X_1, X_2, ...$ are exponentially distributed, meaning that the sum of these iid. random variables will follow an Erlang distribution with density function:

$$S_n \sim E(n, \lambda)$$

The probability density function is of the form:

$$f_{S_n}(y) = \frac{e^{-\lambda y} \lambda^n y^{n-1}}{(n-1)!} = e^{-\lambda y} \lambda \frac{(\lambda y)^{n-1}}{(n-1)!}$$

As $y \leq t$ we can rewrite the probability as

$$\begin{aligned} P(N(t) = n) &= \int_0^t e^{-\lambda(t-y)} e^{-\lambda y} \lambda \frac{(\lambda y)^{n-1}}{(n-1)!} \, dy \\ &= \frac{e^{-\lambda t} \lambda^n}{(n-1)!} \int_0^t y^{n-1} d\ y = \frac{e^{-\lambda t} (t\lambda)^n}{n(n-1)!} = e^{-\lambda t} \frac{(t\lambda)^n}{n!} \end{aligned} \tag{23}$$

Which is just the probability mass function of the Poisson distribution, which gives us an homogeneous Poisson process with mean λt.

□

Exercise 7.5 Let $U_1, U_2, ...$ be independent uniform $(0, 1)$ random variables, and define N by

$$N = \min\{n : U_1 + U_2 + ... + U_n > 1\}$$

What is $E[N]$?

The stopping time N is an integer valued non negative random variable which represents the first moment the sum of the iid random variables U_i is greater than time t, for $t = 1$. For a renewal process having inter-arrival times $U_1, U_2, ...$, let

$$S_n = \sum_{i=1}^{n} U_i, \quad U_i \sim U(0,1)$$

Where S_n represents the time of the n-th renewal (or n-th arrival) . We can rewrite N as

$$N = \min\{n \geq 0 : S_n > t\}$$

We know that $N(t) = n - 1$ if and only if $S_{n-1} \leq t$ meaning that $S_n > t$.

$$n = N(t) + 1 \iff U_1 + ... + U_{n-1} \leq t \iff U_1, ..., U_n > t$$

Hence,

$$E[N] = E[N(t) + 1] = E[N(1) + 1] = E[N(1)] + 1$$

Assuming that the inter-arrival distribution F is continuous with density function f then $m(t) = N(t)$ then $m(1) = E[N(1)] = e^1 - 1$ (Example 7.3) which leave us with $E[N] = e$. □

Exercise 7.6 Consider a renewal process $\{N(t), t \geq 0\}$ having a gamma (r, λ) inter-arrival distribution. That is, the inter-arrival density is

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{(r-1)!}, \quad x > 0$$

- (a) Show that

$$P\{N(t) \geq n\} = \sum_{i=nr}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

- (a)

  In this case we assume that the inter-arrival times $X_1, X_2, ...$ are gamma distributed, meaning that the renewal S_n at time n will also be gamma distributed. $(X_i) \overset{\text{iid}}{\sim} \Gamma(r_i, \lambda)$ and $S_n \sim \Gamma(\sum_{i=1}^{n} r_i, \lambda) = \Gamma(nr, \lambda)$.

  We know that $N(t) \geq n \iff S_n \leq t$ and so $P(N(t) \geq n) = P(S_n \leq t) = F_{S_n}(t)$

$$F_{S_n}(t) = \sum_{i=nr} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

□

Exercise 7.9 A worker sequentially works on jobs. Each time a job is completed, a new one is begun. Each job, independently, takes a random amount of time having distribution F to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker discontinues working on the present job and starts a new one. In the long run, at what rate are jobs completed?

Lest assume that a renewal starts whenever a job is completed. Notice that a renewal can occur given two type of events; (i) a job is completed, (ii) a shock occurs.

Let S_n denote the n-th renewal, where $S_n = \sum_{i=1}^{n} X_i$ where the random variables X_i denotes the inter-arrival times (time between renewals). Notice that the inter-arrival times depend on the events that (i) a job is finished, (having distribution F) and (ii) a crash occurs (according to a Poisson Process).

Let $A(t)$ = shock before t , and $B(t)$ = finished job before t. Then the probability of a renewal happening before t is given by:

$$P(X \leq t) = P(A(t) \cup B(t)) \ , P(X > t) = \overline{P(X \leq t)} = \overline{P(A(t) \cap B(t))} = \overline{P(A(t))} \ \ \overline{P(B(t))}$$

Thus, the probability of a renewal happening after time t is given by:

$$P(X > t) = P(\text{no shock before time } t)P(\text{no finished job before } t)$$

The **renewal rate** is given by the sum of the shock rate $(1/\lambda)$ and the rate of completed jobs, hence in the long run the renewal rate is:

$$\frac{1}{\mu} = \frac{1}{E[X]}$$

where:

$$\begin{aligned} E[X] &= \int_0^\infty \overline{P(A(t))} \ \ \overline{P(B(t))} \ \ dt = \int_0^\infty \Big(\int_s^\infty f(s)\lambda e^{-\lambda s} \ ds \Big) \ \ dt \\ &= \int_0^\infty (1 - P(X \leq t)) \int_t^\infty \lambda e^{-\lambda s} \ \ ds \ \ dt = \int_0^\infty (1 - F(t))e^{-\lambda t} dt \\ &= \int_0^\infty e^{-\lambda t} dt - \int_0^\infty F(t) e^{-\lambda t} \ \ dt = \frac{1}{\lambda} - \frac{1}{\lambda} \int_0^\infty f(t)e^{-\lambda t} \ \ dt \end{aligned} \tag{24}$$

and so;

$$\frac{1}{E[X]} = \frac{\lambda}{1 - \int_0^\infty f(t)e^{-\lambda t}}$$

Hence, the **rate of completed jobs** is given by the renewal rate minus the shock rate which is just:

$$\frac{\lambda}{1 - \int_0^\infty f(t)e^{-\lambda t}} - \lambda$$

□

Exercise 7.12 Events occur according to a Poisson process with rate λ. Any event that occurs within a time d of the event that immediately preceded it is called a d-event. For instance, if $d = 1$ and events occur at times $2, 2.8, 4, 6, 6.6, ...$, then the events at times 2.8 and 6.6 would be d-events.

Let X be the time between successive d events, whether they are (i) d-events, (ii) events. Hence, if $t > d$ then this event is not a d-event and the time to the next event will be $t+E[T]$ where T represents the time between 2 d- events.

$$\begin{aligned}
\mu &= \int_0^\infty \lambda e^{-\lambda t}(t(\mathbb{1}_{\{t\leq d\}} + (t+\mu)\mathbb{1}_{\{t>d\}}) \;\; dt \\
&= \int_0^d t\lambda e^{-\lambda t} \;\; dt + \int_d^\infty \lambda e^{-\lambda t}(t+\mu) \;\; dt = \int_0^\infty t\lambda e^{-\lambda t} \;\; dt + \int_d^\infty \lambda e^{-\lambda t}(t+\mu) \;\; dt \\
&= \int_0^\infty t\lambda e^{-\lambda t} \;\; dt + \int_d^\infty \lambda e^{-\lambda t} t \;\; dt + \mu\int_d^\infty \lambda e^{-\lambda t} \;\; dt \\
&= \int_0^\infty t\lambda e^{-\lambda t} \;\; dt + \mu\int_d^\infty \lambda e^{-\lambda t} \;\; dt = \frac{1}{\lambda}\frac{1}{1-e^{-\lambda d}}
\end{aligned} \tag{25}$$

Hence, the rate $\frac{1}{\mu} = \lambda(1-e^{-\lambda d})$ □

4 Session 4:Renewal Reward Theory; Ross-Chapter 7

Exercise 7.15 Consider a miner trapped in a room that contains three doors. Door 1 leads him to freedom after two days of travel; door 2 returns him to his room after a four-day journey; and door 3 returns him to his room after a six-day journey. Suppose at all times he is equally likely to choose any of the three doors, and let T denote the time it takes the miner to become free.

- (a) Define a sequence of independent and identically distributed random variables $X_1, X_2, ...$ and a stopping time N such that

  $$T = \sum_{i=1}^{N} X_i$$

  Note: You may have to imagine that the miner continues to randomly choose doors even after he reaches safety.

- (b) Use Wald's equation to find $E[T]$.
- (c) Compute $E[\sum_{i=1}^{N} X_i | N = n]$ and note that it is not equal to $E[\sum_{i=1}^{N} X_i]$
- (d) Use part (c) for a second derivation of $E[T]$.

- (a)

  The sequence of random variables $X_1, X_2, ...$ can take the following values:

$$X_i = \begin{cases} 2 & w.p \;\; 1/3 \\ 4 & w.p \;\; 1/3 \\ 6 & w.p \;\; 1/3 \end{cases}, \quad \text{for } i \in N$$

Furthermore, the stopping time N is a non-negative integer valued random variable defined as

$$N = min\{i \in N; X_i = 2\} \quad N \sim Geo(1/3)$$

such that $T = \sum_{i=1}^{N} X_i$, being T the time that takes the miner to be free. Hence $N = i$ only depends on X_i.

- (b)
Wald's equation : If $X_1, X_2, ...,$ is a sequence of independent and identically distributed random variables with finite expectation $E[X]$, and if N is a stopping time for this sequence such that $E[N] < \infty$, then $E[\sum_{n=1}^{N} X_n] = E[N]E[X]$

$$E[T] = E[X]E[N] = 4 \cdot (\frac{1}{1/3}) = 12$$

- (c)
Notice that

$$\begin{aligned} E[\sum_{i=1}^{N} X_i | N = n] &= E[X_1 + X_2 + ... + X_{n-1} + X_n | N = n] \\ &= E[X_1|N = n] + E[X_2|N = n] + ... + E[X_{n-1}|N = n] + E[X_n|N = n] \\ &\text{Consider that } E[X_n|N = n] = 2 \\ &= \sum_{i=1}^{n-1} E[X_i|N > i] + E[X_i|N = i] = \sum_{i=1}^{n-1} E[X_i|X_i \neq 2] + E[X_i|X_i = 2] \\ &= \sum_{i=1}^{n-1} \frac{E[X_i \mathbb{1}_{\{X_i \neq 2\}}]}{P(X_i \neq 2)} + \frac{E[X_i \mathbb{1}_{\{X_i = 2\}}]}{P(X_i = 2)} \\ &= (n-1)\Big(\frac{2 \cdot 1/3 \cdot 0 + 4 \cdot 1/3 \cdot 1 + 6 \cdot 1/3 \cdot 1}{2/3}\Big) + \frac{2 \cdot 1/3}{1/3} \\ &= (n-1)5 + 2 \end{aligned} \tag{26}$$

Hence;

$$E[\sum_{i=1}^{N} X_i | N = n] \neq E[\sum_{i=1}^{n} X_i] = 4n$$

- (d)
From (c) we know that

$$E[\sum_{1=1}^{N} X_i] = E[E[\sum_{1=1}^{N} X_i | N = n]] = E[5N - 3] = 5E[N] - 3 = 12$$

$\square$

Exercise 7.16 A deck of 52 playing cards is shuffled and the cards are then turned face up one at a time. Let X_i equal 1 if the i-th card turned over is an ace, and let it be 0 otherwise, $i = 1, ..., 52$. Also, let N denote the number of cards that need be turned over until all four aces appear. That is, the final ace appears on the N-th card to be turned over. Is the equation

$$E[\sum_{i=1}^{N} X_i] = E[N]E[X_i]$$

valid? If not, why is Wald's equation not applicable?

The sequence of random variables $X_1, X_2, ...,$ are identically distributed taking values:

$$X_i = \begin{cases} 1 & \text{if card is an ace} \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, N is the number of cards needed to have all 4 aces. Hence, the random variable N a stopping time following a geometric distribution.

$$N = \min\{n; \sum_{i=1}^{n} X_i = 4\}$$

A deck of 52 playing cards has 4 aces, meaning that

$$P(X_i = 1) = 1/13 = 1 - P(X_i = 0)$$

Meaning that $\sum_{i=1}^{N} X_i = 4$ and so $E[\sum_{i=1}^{N} X_i] = 4$, hence

$$E[X_i] = E[\mathbb{1}_{\{\text{card is an ace}\}}] = P(X_i = 1) = 1/3$$

Wald's Inequality holds true if

$$4 \neq E[N]\frac{1}{3} \quad \Longleftrightarrow \quad E[N] = \frac{1}{(1/52)} = 52$$

Hence, in this case Wald's Inequality does not hold true, and this is because the random variable X_i is not independent of the rest. We can see that $P(X_2 = 1|X_1 = 0) = 4/51$ but $P(X_2 = 1|X_1 = 0) = 3/51$. $\square$

Exercise 7.19 For the renewal process whose inter-arrival times are uniformly distributed over $(0, 1)$, determine the expected time from $t = 1$ until the next renewal.

If $X_1, X_2, ...,$ are the inter-arrival times of a renewal process $\{N(t), t \geq 0\}$ then given that

$$N(t)+1 = n \iff N(t) = n-1 \iff S_{n-1} \leq t \iff X_1, ..., X_{n-1} \leq t \iff X_1, ..., X_n > t$$

Using Wald's equation:

$$E[S_n] = E[S_{N(t)+1}] = E[N(t)+1]E[X] = (E[N(t)]+1)\mu = (m(t)+1)\mu = \frac{(m(t)+1)}{2}$$

Making use of the **elementary renewal theorem** we know that because $S_{N(t)+1}$ is the time of the first renewal after t, it follows that

$$S_{N(t)+1} = t + Y(t)$$

Where $Y(t)$ is the excess time between time t until the next renewal. Which will mean that

$$(m(t)+1)\mu = E[t+Y(t)] = t + E[Y(t)]$$

Whenever $t = 1$ (Using example 7.3)

$$\frac{e}{2} - 1 = E[Y(1)]$$

$\square$

Exercise 7.20 For a renewal reward process consider

$$W_n = \frac{R_1 + R_2 + ... + R_n}{X_1 + X_2 + ... + X_n}$$

where W_n represents the average reward earned during the first n cycles. Show that $W_n \to E[R]/E[X]$ as $n \to \infty$.

Consider the renewal process $\{N(t), t \geq 0\}$ having inter-arrival times $X_n, \quad n \geq 1$. At the time of the n-th renewal we receive a reward R_n, which is iid of the rest of rewards, although it might (usually will) dependent on X_n. Hence, the total reward earned by time t. is denoted by

$$R(t) = \sum_{n=1}^{N(t)} R_n$$

If $E[R] < \infty$ and $E[X] < \infty$ then with probability 1 (i) $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R]}{E[X]}$ and (ii) $\lim_{t\to\infty} \frac{E[R(t)]}{t} = \frac{E[R]}{E[X]}$.

- (i)
  We know that $\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \left(\frac{\sum_{n=1}^{N(t)} R_n}{N(t)}\right)\left(\frac{N(t)}{t}\right)$ By the **strong law of large numbers** we obtain that $\frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \to E[R]$ as $t \to \infty$ We know that if $S_{N(t)} = \sum_{n=1}^{N(t)} X_i$ while for $\frac{N(t)}{t}$ we know that:

  $$S_{N(t)} \iff n = N(t) \iff S_{N(t)} \leq t \quad , \quad S_{N(t)+1} \iff N(t) = n-1 \iff S_{N(t)+1} > t$$

$$S_{N(t)} \leq t < S_{N(t)+1} \iff \frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

where $\frac{\sum_{n=1}^{N(t)} X_i}{N(t)} \to \mu$ as $t \to \infty$ hence $\frac{S_{N(t)+1}}{N(t)} = \left(\frac{S_{N(t)+1}}{N(t)+1}\right)\left(\frac{N(t)+1}{N(t)}\right) \to \mu$ as $t \to \infty$ which will mean that

$$\frac{t}{N(t)} \to \mu; \quad t \to \infty \iff \frac{N(t)}{t} \to \frac{1}{\mu} = \frac{1}{E[x]}; \quad t \to \infty$$

Hence,

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E[R]}{E[X]}$$

Now, in this exercise we want to prove that $\lim_{n \to \infty} W_n = E[R]/E[X]$ where

$$W_n = \frac{\sum_{i=1}^n R_i}{\sum_{i=1}^n X_i} = \left(\frac{\sum_{i=1}^n R_i}{n}\right)\left(\frac{n}{\sum_{i=1}^n X_i}\right) = \left(\frac{R}{n}\right)\left(\frac{n}{S_n}\right) \to E[R]\frac{1}{E[X]}, n \to \infty$$

□

Exercise 7.26 Consider a train station to which customers arrive in accordance with a Poisson process having rate λ. A train is summoned whenever there are N customers waiting in the station, but it takes K units of time for the train to arrive at the station. When it arrives, it picks up all waiting customers. Assuming that the train station incurs a cost at a rate of nc per unit time whenever there are n customers present, find the long-run average cost.

Following a Renewal Reward Process, the long run average cost will equal to

$$\frac{E[\text{cost incurred during cycle}]}{E[\text{length of the cycle}]} \tag{27}$$

Let $X_i \;\; i \geq 1$ be independent and identically distributed random variables. For a renewal process having inter-arrival times $S_1, S_2, ...;$

$$\text{let} \;\; S_0 = 0 \;\;,\;\; S_N = \sum_{n=1}^{N} X_n \;\; n \geq 1$$

The random variable $X_n \sim \text{Exp}(\lambda)$ denotes the time between the arrival of the $(n-1)st$ and nth passenger while $S_N \sim E(N, \lambda)$ denotes the time elapsed until the nth passenger arrival. The renewal cycle will be completed at $S_N + K$ where K is a fixed value that represents the the additional time the train takes to arrive at a station.

For computing the **expected cost of a cycle** we will need to consider the cost generated by the first N passengers as well as the cost generated by the passengers that arrived between time S_N and time $(S_N + K)$.

Notice that the the expected inter-arrival time between passengers $n-1$ and n is:

$$E[S_N - S_{N-1}] = E[X_N] = 1/\lambda$$

This will mean that the expected cost generated by the Nth arrival up to time S_N will be:

$$\begin{aligned}
E[c(X_2) + 2c(X_3) + ... + (N-1)X_N] &= E\left[\sum_{n=1}^{N} c(n-1)(S_n - S_{n-1})\right] \\
&= E\left[\sum_{n=1}^{N} c(n-1)X_n\right] \\
&= \sum_{n=1}^{N} c(n-1)E[X_n] \\
&= \frac{c}{\lambda}\sum_{n=1}^{N}(n-1) \\
&= \frac{cN(N-1)}{2\lambda}
\end{aligned} \tag{28}$$

Hence, the expected the expected cost generated by the Nth arrival up to time $S_N + K$ will be:

$$\frac{cN(N-1)}{2\lambda} + cNK$$

Knowing that X_n has mean $1/\lambda$.

The number of passengers arriving in any interval of length K is distributed as a Poisson(λK). Hence, the expected cost of passengers arriving in the interval $(S_N, S_N + K)$ can be derived as:

$$\begin{aligned}
E\left[c\int_0^K N(t)dt\right] &= c\int_0^K E[N(t)]dt \\
&= c\int_0^K \lambda t dt \\
&= \frac{c\lambda K^2}{2}
\end{aligned} \tag{29}$$

And so

$$E[\text{cost incurred during cycle}] = \frac{cN(N-1)}{2\lambda} + cNK + \frac{c\lambda K^2}{2}$$

We can also compute the expected cost of passengers arriving in the interval $(S_N, S_N + K)$ by using the order statistic property. Lets consider that the joint distribution

$$(S_1, S_2, ..., S_n | N(K) = k) \sim U([S_N, S_N + K]^n)$$

meaning that $E[S_n|N(K)=k]=\frac{K}{2}$. The cost incurred during the cycle can be obtained as:

$$Cost=\sum_{n=1}^{N(K)} c(K-S_n)$$

Where S_n represents the time of arrival of the nth passenger then $c(K-S_n)$ represents the cost paid for the nth passenger. If we now take the expectation of the cost per cycle we obtain that for k customers arriving in (S_N, S_N+K) the expected cost is:

$$E[Cost|N(K)=k]=\sum_{n=1}^{N(K)} c(K-E[S_n|N(K)=k])=\sum_{n=1}^{N(K)} c\frac{K}{2}=\frac{cKk}{2}$$

While the **expected length of the cycle** will be :

$$E[\text{length of the cycle}]=E[S_N+K]=E[S_N]+K=\frac{N}{\lambda}+K \tag{30}$$

Using the result obtained in (29) and (30) we can derive that the long run cost per time unit converges to:

$$\frac{\frac{cN(N-1)}{2\lambda}+cNK+\frac{c\lambda K^2}{2}}{\frac{N}{\lambda}+K}$$

□

5 Session 5: Regenerative and Semi-Markov processes and the Inspection Paradox ; Ross-Chapter 7

Exercise 7.22 J's car buying policy is to always buy a new car, repair all breakdowns that occur during the first T time units of ownership, and then junk the car and buy a new one at the first breakdown that occurs after the car has reached age T . Suppose that the time until the first breakdown of a new car is exponential with rate λ, and that each time a car is repaired the time until the next breakdown is exponential with rate μ.

- (a) At what rate does J buy new cars?
- (b) Supposing that a new car costs C and that a cost r is incurred at each breakdown, what is J's long run average cost per unit time?

Let

$$\begin{cases} X_1 \sim Po(\lambda) \\ X_k \sim Po(\mu) \quad ,k>1 \end{cases}$$

Where X_1 denotes the time until the first breakdown and X_k the k-th breakdown after the first one. Then, let $\{N(t), t \geq T\}$ define a renewal process where each renewal occurs at time Y the first breakdown after T.

- (a)
Knowing that $E[Y] = E[\text{length of a cycle}]$ we should consider that the number of repairs before a new car is bought depends on the time of the first breakdown X_1, in the sense that if the first breakdown happens before time T, then the next cycle will begin **at the first breakdown epoch** after time T has passed i.e $E[Y|X_1 \leq T] = T + 1/\mu$ and if the first breakdown happens after time T then $E[Y|X_1 > T] = T + 1/\lambda$.

$$\begin{aligned} E[Y] &= E[Y|S_1 \leq T]P(X_1 \leq T) + E[Y|X_1 > T]P(X_1 > T) \\ &= (T + 1/\mu)P(X_1 \leq T) + (T + 1/\lambda)P(X_1 > T) \\ &= (T + 1/\mu)P(X_1 \leq T) + (T + 1/\lambda)(1 - P(X_1 \leq T)) \\ &= (T + 1/\mu)(1 - e^{-\lambda T}) + (T + 1/\lambda)(e^{-\lambda T}) \\ &= T + 1/\mu + (1/\lambda - 1/\mu)e^{-\lambda T} \end{aligned} \tag{31}$$

And so the rate at which J buys cars is $1/E[Y]$

- (b)
Knowing that

$$E[\text{cost incurred during a cycle}] = C + rE[\#\text{ of repairs}]$$

$$\begin{aligned} E[\#\text{ of repairs}] &= \int_0^T E[\#\text{ of repairs}|X_1 \leq t]f(t) \ \ dt + \int_T^\infty E[\#\text{ of repairs}|X_1 > t]f(t) \ \ dt \\ &= \int_0^T E[\#\text{ of repairs}|X_1 \leq t]\lambda e^{-\lambda t} \ \ dt + \int_T^\infty E[\#\text{ of repairs}|X_1 > t]\lambda e^{-\lambda t} \ \ dt \\ &\text{Notice that } E[\#\text{ of repairs})|X_1 > t] = 0\text{as the car will be junk by then.} \\ &= \int_0^T (1 + \mu(T - t))\lambda e^{-\lambda t} \ \ dt \\ &= \int_0^T \lambda e^{-\lambda t} \ \ dt + \int_0^T \mu T\lambda e^{-\lambda t} \ \ dt - \int_0^T \mu t\lambda e^{-\lambda t} \ \ dt \\ &= (1 + \mu T)(1 - e^{-\lambda T}) - \int_0^T \mu t\lambda e^{-\lambda t} \ \ dt \\ &= \text{integrating by parts } u = t \\ &= (1 + \mu T)(1 - e^{-\lambda T}) + \mu T e^{-\lambda T} - \frac{\mu}{\lambda}e^{-\lambda T} + \frac{\mu}{\lambda} = (1 - e^{-\lambda T})(1 - \mu T) + \mu T \end{aligned} \tag{32}$$

Notice that the expected number of repairs whenever the first repair X_1 happens before T will be 1 plus a Poisson number of further repairs with expectation $\mu(T - X_1)$

$$E[\text{cost incurred during a cycle}] = C + r((1 - e^{-\lambda T})(1 - \mu T) + \mu T)$$

By the **renewal reward theory** we can obtain the long run average cost per unit time by computing:

$$\frac{E[\text{cost incurred during a cycle}]}{E[\text{length of a cycle}]} = \frac{C + r((1 - e^{-\lambda T})(1 - \mu T) + \mu T)}{T + 1/\mu + (1/\lambda - 1/\mu)e^{-\lambda T}}$$

$\square$

Exercise 7.31 If $A(t)$ and $Y(t)$ are, respectively, the age and the excess at time t of a renewal process having an inter-arrival distribution F, calculate:

$$P\{Y(t) > x | A(t) = s\}$$

See **The Inspection Paradox**

We know that $A(t)$ denotes the average rate of a renewal process, while $Y(t)$ the excess of time t of a renewal. Let $\{N(t), t \geq 0\}$ denotes a renewal process where the time of the n-th renewal is given by $S_n = \sum_{i=1}^{n} X_i$ where X_i represent the inter-arrival time with distribution F. Then, we know that

$$N(t) = n - 1 \iff N(t) + 1 = n \quad n > 1$$

also

$$A(t) = t - S_{N(t)} = s, \iff t = s + S_{N(t)}$$

$$Y(t) = S_{N(t)+1} - t = S_{N(t)+1} - S_{N(t)} - s = X_{N(t)+1} - s \iff X_{N(t)+1} = Y(t) + s \iff X_{N(t)+1} > s$$

we can then rewrite $Y(t) > x$ as $X_{N(t)+1} > x + s$

$$P(Y(t) > x | A(t) = s) = P(X_{N(t)+1} > x + s | A(t) = s) = P(X_{N(t)+1} > x + s | X_{N(t)+1} > s)$$

$$P(X_{N(t)+1} > x + s | X_{N(t)+1} > s) = \frac{X_{N(t)+1} > x + s \ , \ X_{N(t)+1} > s)}{P(X_{N(t)+1} > s)}$$

$$\begin{aligned} P(X_{N(t)+1} > x + s | X_{N(t)+1} > s) &= \frac{X_{N(t)+1} > x + s \ , \ X_{N(t)+1} > s)}{P(X_{N(t)+1} > s)} \\ &= \frac{X_{N(t)+1} > x + s)}{P(X_{N(t)+1} > s)} = \frac{1 - F_{X_{N(t)+1}}(x + s)}{1 - F_{X_{N(t)+1}}(s)} \end{aligned} \tag{33}$$

$\square$

Exercise 7.38 A truck driver regularly drives round trips from A to B and then back to A. Each time he drives from A to B, he drives at a fixed speed that (in miles per hour) is uniformly distributed between 40 and 60; each time he drives from B to A, he drives at a fixed speed that is equally likely to be either 40 or 60.

1. (a) In the long run, what proportion of his driving time is spent going to B?
2. (b) In the long run, for what proportion of his driving time is he driving at a speed of 40 miles per hour?

The time that takes to move from one location to another is obtained by the ratio distance/speed. In this case when the truck driver goes from $A \to B$ the time he takes is $D/u, \quad u \sim U(40, 60)$, while for $B \to A$

$$\begin{cases} D/40 & \text{w.p} \quad 1/2 \\ D/60 & \text{w.p} \quad 1/2 \end{cases}$$

- (a)
  During a cycle, the driving time between locations is given by:

$$\begin{aligned} E[\text{time driving} \quad A \to B] &= \int_{40}^{60} f(u)\frac{D}{u} \quad du = \int_{40}^{60} \frac{1}{20}\frac{D}{u} \quad du = \frac{D}{20}\log(3/2) \\ E[\text{time driving} \quad B \to A] &= \frac{1}{2}(\frac{D}{40} + \frac{D}{60}) = \frac{5D}{240} \end{aligned} \tag{34}$$

  For a **regenerative process** the amount of time spent driving from A to B during a cycle is:

$$\frac{E[\text{time driving} \quad A \to B]}{E[\text{time of a cycle}]} = \frac{\frac{d}{20}\log(3/2)}{\frac{D}{20}\log(3/2) + \frac{5D}{240}}$$

- (b)
  Similarly, the long run fraction driving 40 mph is:

$$\frac{\frac{D}{40}\frac{1}{2}}{\frac{D}{20}\log(3/2) + \frac{5D}{240}}$$

$\square$

For 46 assume that jumps are independent of waiting times, while for 47 allow for dependence.

Exercise 7.46 Consider a semi-Markov process in which the amount of time that the process spends in each state before making a transition into a different state is exponentially distributed. What kind of process is this?

A **continuous-time Markov Process** which says that the probability of a transition from state i to state j does not depend on the global time and only depends on the time interval available for the transition.
A semi-Markov process is equivalent to a Markov renewal process in many aspects, except that a state is defined for every given time in the semi-Markov process, not just at the jump times. Therefore, the semi-Markov process is an actual stochastic process that evolves over time.

Exercise 7.47 In a semi-Markov process, let t_{ij} denote the conditional expected time that the process spends in state i given that the next state is j.

- (a) Present an equation relating μ_i to the t_{ij} .
- (b) Show that the proportion of time the process is in i and will next enter j is equal to $P_i P_{ij} t_{ij}/\mu_i$.

Hint: Say that a cycle begins each time state i is entered. Imagine that you receive a reward at a rate of 1 per unit time whenever the process is in i and heading for j. What is the average reward per unit time?

- (a)
  Let

$$\begin{aligned} t_{ij} &= E[\text{time per cycle in state } i|\text{next state is } j] \\ \mu_i &= E[\text{time per cycle in state } i] = E[E[\text{time per cycle in state } i|\text{next state is } j]] \\ &= \sum_j P_{ij} E[\text{time per cycle in state } i|\text{next state is } j] = \sum_j P_{ij} t_{ij} \end{aligned} \tag{35}$$

  Where P_{ij} = proportion of time that the process stays at i and transitions to j.

- (b)
  The hint tells us that we will only be once in state i per cycle. We know that P_i is the long tun fraction of time we spend in state i, hence

$$P_i = \frac{\mu_i}{E[\text{time per cycle}]} = \frac{\mu_i}{\sum_{i=1}^n \mu_i} = \frac{\mu_i}{\frac{\mu_i}{P_i}} = \frac{\mu_i P_i}{\mu_i} = \frac{P_{ij} t_{ij} P_i}{\mu_i}$$

  Where The expected time per cycle that the process is in state i on its way to state j is $P_{ij} t_{ij}$.

□

6 Session 6: Queueing Theory: Exponential models; Ross-Chapter 8

Exercise 8.1 For the $M/M/1$ queue, compute:

- (a) the expected number of arrivals during a service period and
- (b) the probability that no customers arrive during a service period.

Hint: "Condition."

A Single-Server Exponential Queuing System, where the two M's in $M/M/1$ refer to the fact that both the inter-arrival and the service distributions are exponential (and thus memoryless, or Markovian), and the 1 to the fact that there is a single server.

- $$\begin{aligned} E[\text{ \# of arrivals }] &= E[E[\text{ \# of arrivals }|\text{service period is } T]] \\ &= \int_0^\infty E[\text{ \# of arrivals }|\text{service period is t}]f(t,\mu) \;\; dt \\ &= \int_0^\infty E[\text{ \# of arrivals }|\text{service period is t}]\mu e^{-\mu t} \;\; dt \end{aligned} \tag{36}$$

  Where $f(t,\mu)$ refers to the service distribution, exponentially distributed, while the conditional expectation is just the expected value of the Poisson distribution;

  $$\begin{aligned} E[\text{ \# of arrivals }|\text{service period is t}] &= \sum_{n\geq 0} nP(\text{ \# of arrivals }|\text{service period} = n) \\ &= \sum_{n\geq 0} n\frac{(\lambda t)^n}{n!}e^{-\lambda t} = \sum_{n\geq 1}\frac{(\lambda t)^{n-1}}{(n-1)!}\lambda t e^{-\lambda t} \\ &= \lambda t e^{-\lambda t}\sum_{n\geq 1}\frac{(\lambda t)^{n-1}}{(n-1)!} = \lambda t e^{-\lambda t}\sum_{n\geq 0}\frac{(\lambda t)^n}{n!} \\ &= \lambda t e^{-\lambda t}e^{\lambda t} = \lambda t \end{aligned} \tag{37}$$

  Hence:

  $$\begin{aligned} E[\text{ \# of arrivals }] &= \lambda E[\text{service period}] = \lambda\int_0^\infty t\mu e^{-\mu t} \;\; dt \\ &= \lambda\mu\int_0^\infty t e^{-\mu t} \;\; dt = \lambda\Big[-te^{-\mu t} - \frac{e^{-\mu t}}{\mu}\Big]_0^\infty = \frac{\lambda}{\mu} \end{aligned} \tag{38}$$

- (b)
  We know want to find:

  $$\begin{aligned} E[\text{ 0 arrivals }] &= E[E[\text{ 0 arrivals }|\text{service period is } T]] \\ &= \int_0^\infty E[\text{ 0 arrivals }|\text{service period is t}]f(t,\mu) \;\; dt \\ &= \int_0^\infty E[\text{ 0 arrivals }|\text{service period is t}]\mu e^{-\mu t} \;\; dt \end{aligned} \tag{39}$$

  Where:

  $$\begin{aligned} E[\text{ 0 arrivals }|\text{service period is t}] &= P(\text{0 events in service period } t) \\ &= \frac{(\mu t)^0 e^{-\mu t}}{0!} = e^{-\mu t} \end{aligned} \tag{40}$$

  and so:

  $$E[\text{ 0 arrivals }] = \int_0^\infty e^{-\mu t}\mu e^{-\mu t} \;\; dt = \mu\int_0^\infty e^{-t(\mu+\lambda)} \;\; dt = \frac{\mu}{\mu+\lambda} \tag{41}$$

$\square$

Exercise 8.6 Show that W is smaller in an $M/M/1$ model having arrivals at rate λ and service at rate 2μ than it is in a two-server $M/M/2$ model with arrivals at rate λ and with each server at rate μ. Can you give an intuitive explanation for this result? Would it also be true for W_Q?

Recall that W denotes the average amount of time a customer spends in the system and W_Q denotes the average amount of time a customer spends waiting in queue.

- (i)
  In the **Single-Server Exponential Queueing System** $M/M/1$ where customers arrive in accordance to a Poisson Process having rate λ and a service time exponentially distributed with rate 2μ:

$$W = \frac{L}{\lambda} = \frac{\sum_{n=0}^{\infty} nP_n}{\lambda}$$

  Where the long-run proportion of time the system spends in state n and 0 is given by:

$$1 = \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \left(\frac{\lambda}{2\mu}\right)^n P_0 = \frac{1}{1-\frac{\lambda}{2\mu}} P_0 \iff P_0 = 1 - \frac{\lambda}{2\mu} \iff P_n = \left(\frac{\lambda}{2\mu}\right)^n \left(1 - \frac{\lambda}{2\mu}\right)$$

  Which gives us

$$W = \frac{\left(1-\frac{\lambda}{2\mu}\right)\sum_{n=0}^{\infty} n\left(\frac{\lambda}{2\mu}\right)^n}{\lambda} = \frac{\left(1-\frac{\lambda}{2\mu}\right)\frac{\lambda 2\mu}{(2\mu-\lambda)^2}}{\lambda} = \frac{1}{2\mu-\lambda}$$

  Knowing that by differentiating the geometric series and for $|\lambda/2\mu| < 1$

$$\sum_{n=0}^{\infty} n\left(\frac{\lambda}{2\mu}\right)^n = \sum_{n=1}^{\infty} n\left(\frac{\lambda}{2\mu}\right)^n = \frac{\lambda}{2\mu}\sum_{n=0}^{\infty} n\left(\frac{\lambda}{2\mu}\right)^{n-1} = \frac{\lambda}{2\mu}\frac{1}{\left(1-\frac{\lambda}{2\mu}\right)^2} = \frac{\lambda 2\mu}{(2\mu-\lambda)^2}$$

  Now, the average amount of time a customer spends waiting in queue is obtained as:

$$W_Q^1 = W_{M/M/1} - E[S] = \frac{1}{2\mu-\lambda} - \frac{1}{2\mu} = \frac{\lambda}{2\mu(2\mu-\lambda)}$$

  Where $E[S]$ represents the expected service time per customer.

- (ii)
  In the **Two-Server Exponential Queueing System** $M/M/2$ Consider a 2 server system in which customers arrive according to a Poisson process with rate λ. An arriving customer immediately enters service if any of the 2 servers are free. If all 2 servers

are busy, then the arrival joins the queue. All service times are exponential random variables with rate μ. Hence, the $M/M/2$ is a birth and death queueing model with arrival rates $\lambda_n = \lambda, \quad n \geq 0$ and $\mu_n = n\mu$ if $n \leq 2$ and $\mu_n = k\mu$ if $n \geq 2$.

Following Example 8.6

$$
\begin{aligned}
P_0 &= \frac{1}{1 + \frac{\lambda}{\mu}\frac{\lambda^2}{2\mu^2} + 2\sum\limits_{n=3}^{\infty}\left(\frac{\lambda}{2\mu}\right)^n} = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + 2\Big(\sum_{n=0}^{\infty}\left(\frac{\lambda}{2\mu}\right)^n - 1 - \frac{\lambda}{2\mu} - \left(\frac{\lambda}{2\mu}\right)^2\Big)} \\
&= \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + 2\Big(\frac{1}{1-\frac{\lambda}{2\mu}} - 1 - \frac{\lambda}{2\mu} - \left(\frac{\lambda}{2\mu}\right)^2\Big)} = \frac{2\mu - \lambda}{2\mu + \lambda}
\end{aligned}
\tag{42}
$$

Then, the average arrival rate of customers is

$$
\begin{aligned}
\lambda_a &= \sum_{n=0}^{\infty} \lambda P_n = \sum_{n=0}^{\infty} 2\lambda\Big(\frac{\lambda}{2\mu}\Big)^n P_0 = \lambda \sum_{n=1}^{\infty} 2\Big(\frac{\lambda}{2\mu}\Big)^n P_0 + \lambda P_0 \\
&= 2\lambda P_0 \sum_{n=1}^{\infty}\left(\frac{\lambda}{2\mu}\right)^n + \lambda P_0 = 2\lambda P_0\Big(\sum_{n=0}^{\infty}\left(\frac{\lambda}{2\mu}\right)^n - 1\Big) + \lambda P_0 \\
&= \lambda P_0\Big[2\sum_{n=0}^{\infty}\left(\frac{\lambda}{2\mu}\right)^n - 1\Big] = \lambda P_0\Big[\frac{2}{1-\frac{\lambda}{2\mu}} - 1\Big] = \lambda P_0\Big(\frac{2\mu+\lambda}{2\mu-\lambda}\Big) = \lambda
\end{aligned}
\tag{43}
$$

Now, the average number of customer seen by an arrival in the $M/M/2$ system is obtained a follows:

$$
\begin{aligned}
L &= \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} 2n\Big(\frac{\lambda}{2\mu}\Big)^n P_0 = 2P_0 \sum_{n=0}^{\infty} n\Big(\frac{\lambda}{2\mu}\Big)^n \\
&= \frac{\lambda}{\mu} P_0 \sum_{n=0}^{\infty} n\Big(\frac{\lambda}{2\mu}\Big)^{n-1} = \frac{\lambda}{\mu}\frac{2\mu-\lambda}{2\mu+\lambda}\left(\frac{1}{\Big(1-\frac{\lambda}{2\mu}\Big)^2}\right) = \frac{4\lambda\mu}{(2\mu+\lambda)(2\mu-\lambda)}
\end{aligned}
\tag{44}
$$

We obtain that

$$
W = \frac{L}{\lambda_a} = \frac{\frac{4\lambda\mu}{(2\mu+\lambda)(2\mu-\lambda)}}{\lambda} = \frac{4\mu}{(2\mu+\lambda)(2\mu-\lambda)}
$$

Furthermore, the average amount of time a customer spends waiting in queue is obtained as:

$$
W_Q^2 = W_{M/M/2} - E[S] = \frac{4\mu}{(2\mu+\lambda)(2\mu-\lambda)} - \frac{1}{\mu} = \frac{\lambda^2}{\mu(2\mu+\lambda)(2\mu-\lambda)}
$$

We then obtain that since $\lambda/2\mu < 1 \iff \lambda < 2\mu$

$$\frac{W_{M/M/1}}{W_{M/M/2}} = \frac{1}{2\mu-\lambda}\frac{(2\mu+\lambda)(2\mu-\lambda)}{4\mu} = \frac{2\mu+\lambda}{4\mu} = \frac{1}{2}\Big(1+\frac{\lambda}{2\mu}\Big) < 1 \iff W_{M/M/1} < W_{M/M/2}$$

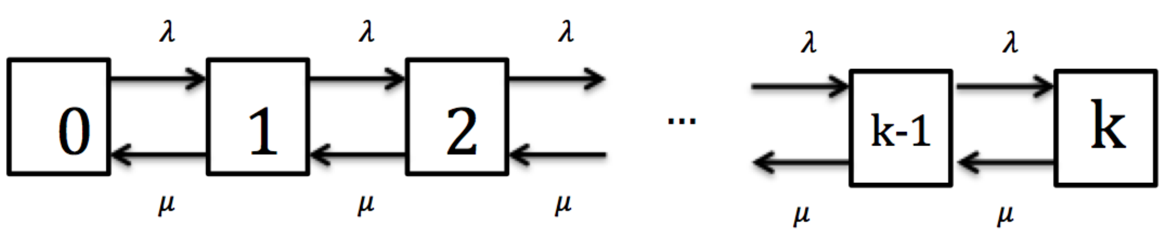
$$\frac{W_Q^1}{W_Q^2} = \frac{\mu(2\mu+\lambda)(2\mu-\lambda)}{\lambda^2}\frac{\lambda}{2\mu(2\mu-\lambda)} = \frac{2\mu+\lambda}{2\lambda} = \frac{\mu}{\lambda}+\frac{1}{2} > \frac{2\lambda}{2\lambda} = 1 \iff W_Q^1 > W_Q^2$$

Given $W_{M/M/1} < W_{M/M/2}$ we can say that the one server system is faster that the two server system (as the customer spends less time on the system). However, given $W_Q^1 > W_Q^2$ the customer has to wait longer to be served on the one server system (waiting in queue). This implies that the queue length is usually longer in the 1 sever queue, meaning that if one finds the queue empty in the $M/M/2$ case, it would do no good to have two servers. One would be better off with one faster server. □

Exercise 8.8 A facility produces items according to a Poisson process with rate λ. However, it has shelf space for only k items and so it shuts down production whenever k items are present. Customers arrive at the facility according to a Poisson process with rate μ. Each customer wants one item and will immediately depart either with the item or empty handed if there is no item available.

- (a) Find the proportion of customers that go away empty handed.
- (c) Find the average number of items on the shelf.
- (b) Find the average time that an item is on the shelf.

The $M/M/1$ **queueing system** can be represented as following: We know that items produced $\sim$



Po. process (λ) with capacity k and customer arrive $\sim$ Po. process (λ).
The number of items available in the shelf can also be described as a $M/M/1$ **Queuing System with Balking** with finite capacity k. The produced items constitute the arrivals to the queue, and the arriving customers constitute the services.

Let $P_j, 0 \leq j \leq k$, denote the limiting probability that there are j items in the system. And so

$$P_j = \begin{cases} P_0(\lambda/\mu)^j & \text{for} \quad 0 \leq j \leq k \\ 0 & \text{o.w} \end{cases}$$

- (a)
  The customer that will leave empty handed are the ones present whenever there are no items in the system, i.e, whenever $k = 0$ and so, the proportion of customers going home empty handed is P_0,

$$\begin{aligned} 1 &= P_0 \sum_{j=1}^{k} (\lambda/\mu)^j \\ &= P_0 \Big[\frac{1 - (\lambda/\mu)^{k+1}}{1 - (\lambda/\mu)} \Big] \\ \iff \quad P_0 &= \frac{1 - (\lambda/\mu)}{1 - (\lambda/\mu)^{k+1}} \end{aligned} \tag{45}$$

- (c)
  Let L be the average number of items in the system.

$$L = \sum_{j=1}^{k} j P_j = P_0 \sum_{j=1}^{k} j(\lambda/\mu)^j = \frac{1 - (\lambda/\mu)}{1 - (\lambda/\mu)^{k+1}} \sum_{j=1}^{k} j(\lambda/\mu)^j \tag{46}$$

- (b)
  The average time that an item is on the shelf is equivalent as the average time a customer is on the system. We denote it as W

$$W = \frac{L}{\lambda_a} = \frac{L}{\lambda(1 - P_k)}$$

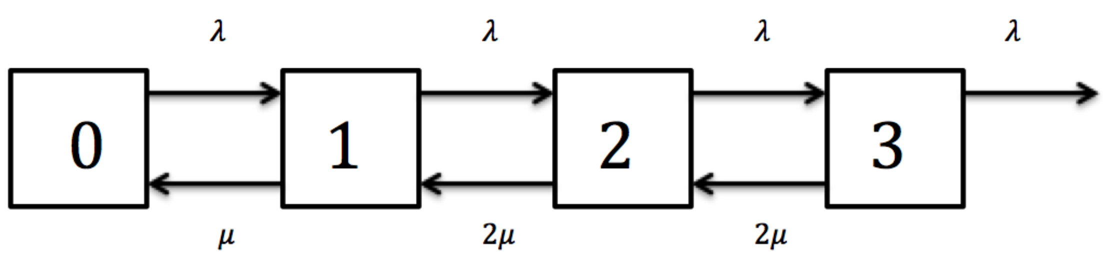
  where λ_a is defined to be average arrival rate of entering customers $\lambda_a = \lambda(1 - P_k)$, where we include only the fraction those customers that actually entered the system $(1 - P_k)$.

$\square$

Exercise 8.12 A supermarket has two exponential checkout counters, each operating at rate μ. Arrivals are Poisson at rate λ. The counters operate in the following way:

- (i) One queue feeds both counters.
- (ii) One counter is operated by a permanent checker and the other by a stock clerk who instantaneously begins checking whenever there are two or more customers in the system. The clerk returns to stocking whenever he completes a service, and there are fewer than two customers in the system.
- (a) Find P_n, proportion of time there are n in the system.
- (b) At what rate does the number in the system go from 0 to 1? From 2 to 1?

The $M/M/2$ **queueing system** can be represented as following:



Where customers arrive at a rate of λ and served at a rate μ whenever there's only one customer in the system, otherwise customers are served at a rate of 2μ (as there are two servers)

- (a)
  Following Example 8.6 we know that

$$P_0 = \frac{1}{1+\sum_{n=1}^{k}(\lambda/\mu)/n! + \sum_{n=k+1}^{\infty}(\lambda/k\mu)^n k^k/k!}$$

hence, in our case $1+\sum_{n=1}^{k}(\lambda/\mu)/n! + \sum_{n=k+1}^{\infty}(\lambda/k\mu)^n k^k/k!$ would be:

$$\begin{aligned}
&= 1+\frac{\lambda}{\mu}+\frac{1}{2}\Big(\frac{\lambda}{\mu}\Big)^2+\sum_{n=3}^{\infty}\Big(\frac{\lambda}{2\mu}\Big)^n\frac{2^2}{2!} = 1+\frac{\lambda}{\mu}+\frac{1}{2}\Big(\frac{\lambda}{\mu}\Big)^2+2\sum_{n=3}^{\infty}\Big(\frac{\lambda}{2\mu}\Big)^n \\
&= 1+\frac{\lambda}{\mu}+\frac{1}{2}\Big(\frac{\lambda}{\mu}\Big)^2+2\Big(\sum_{n=0}^{\infty}\Big(\frac{\lambda}{2\mu}\Big)^n-1-\frac{\lambda}{2\mu}-\Big(\frac{\lambda}{2\mu}\Big)^2\Big) \qquad (47)\\
&= 1+\frac{\lambda}{\mu}+\frac{1}{2}\Big(\frac{\lambda}{\mu}\Big)^2+2\Big(\frac{1}{1-\frac{\lambda}{2\mu}}-1-\frac{\lambda}{2\mu}-\Big(\frac{\lambda}{2\mu}\Big)^2\Big) = \frac{4\mu}{2\mu-\lambda}-1 = \frac{2\mu+\lambda}{2\mu-\lambda}
\end{aligned}$$

Notice that the previous result will only hold whenever $\lambda > 2\mu$

$$P_0 = \frac{2\mu - \lambda}{2\mu + \lambda}$$

Whenever $n \leq 2$

$$P_n = P_0\Big(\frac{\lambda}{2\mu}\Big)^n/n!$$

and whenever $n > 2$

$$P_n = P_0\frac{\lambda^n}{2^{n-1}\mu^n} \tag{48}$$

- (b)
Knowing customers arrive according to a Poisson process with rate λ and that the service time is exponentially distributed with rate μ ($n \leq 2$) and 2μ ($n > 2$). For this, let's define the **balance equations** for the following states:

$$\textbf{state} = 0, \quad \textbf{rate leave} = \lambda P_0, \ \textbf{rate enter} = \mu P_1$$

The process will leave state 0 at a rate λ via a customer arrival and will enter state 0 from state 1 with rate μ via customer departure.

$$\textbf{state} = 1, \quad \textbf{rate leave} = (\mu + \lambda)P_1, \ \textbf{rate enter} = \lambda P_0 + 2\mu P_2$$

The process will leave state 1 at a rate of $\lambda + \mu$ and will enter state 1 from state 2 by departure (at a rate 2μ) and state 0 by arrival (at a rate λ).

$$\textbf{state} = n, \quad n > 1 \quad \textbf{rate leave} = (2\mu + \lambda)P_n, \ \textbf{rate enter} = \lambda P_{n-1} + 2\mu P_{n+1}$$

The process will leave state n at a rate of $\lambda + 2\mu$ and will be entered by state $n - 1$ with rate λ via arrival and by state $n + 1$ with rate 2μ via departure.

$$\text{Hence, the system goes from } 0 \to 1 \text{ at a rate of} \quad \lambda P_0 = \lambda\frac{2\mu - \lambda}{2\mu + \lambda}$$

Using Eq. (48) we obtain that:

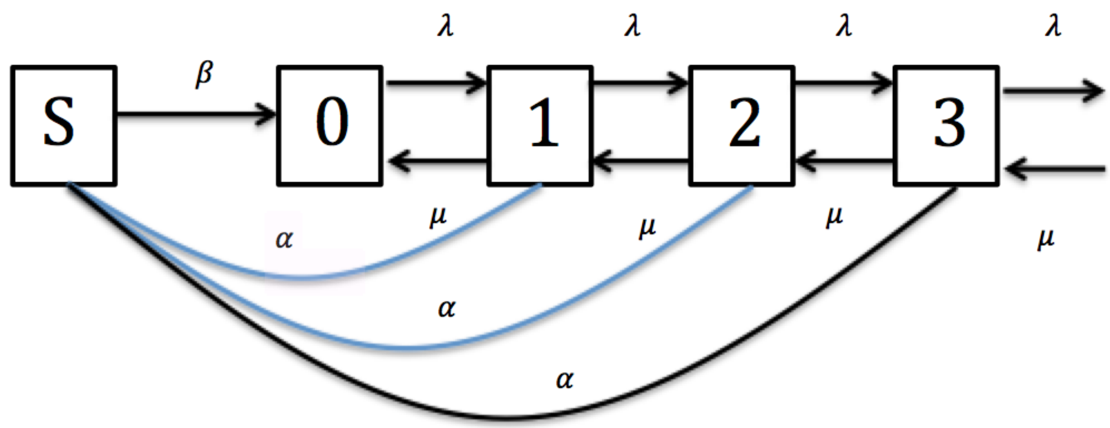
$$\text{Hence, the system goes from } 2 \to 1 \text{ at a rate of} \quad 2\mu P_2 = \frac{\lambda^2(2\mu - \lambda)}{\mu(2\mu + \lambda)}$$

□

7 Session 7: Queueing Theory: PASTA and General workloads; Ross-Chapter 8

Exercise 8.23 Consider the $M/M/1$ system in which customers arrive at rate λ and the server serves at rate μ. However, suppose that in any interval of length h in which the server is busy there is a probability $\alpha h + o(h)$ that the server will experience a breakdown, which causes the system to shut down. All customers that are in the system depart, and no additional arrivals are allowed to enter until the breakdown is fixed. The time to fix a breakdown is exponentially distributed with rate β.

The $M/M/1$ **queueing system** can be represented as following:



- (a) Define appropriate states.

  There are many states "n": infinitely many since n can be $0, 1, 2, ...$ Where state n refers to the state in which there are n customers in the system $(n \geq 0)$ and s refers the a breakdown state in which all customers that are in the system depart and no new arrivals are allowed.

- (b) Give the balance equations. In terms of the long-run probabilities.

  Knowing customers arrive according to a Poisson process with rate λ and that the service time is exponentially distributed with rate μ. The system breakdown will occur at an exponential rate of α and it will get repaired at an exponential rate of β.

  The **balance equations** are defined for the following states:

- for a system breakdown state s

$$
\begin{aligned}
\beta P_s &= \alpha \sum_{n=1}^{\infty} P_n \\
&= \alpha \left(1 - P_0 - P_s \right)
\end{aligned}
\tag{49}
$$

The process will leave state s at a rate of β and it will enter state s given the fraction of arrivals that entered the system $(1 - P_0 - P_s)$ with breakdown rate α.

- (ii) for the first state 0

$$\lambda P_0 = \beta P_s + \mu P_1 \tag{50}$$

Since state 0 can only be reached from state 1 via a departure (with rate μ), or state s via a system breakdown (with rate α).

- (iii) for the state n:

$$(\lambda + \mu + \alpha) P_n = \lambda P_{n-1} + \mu P_{n+1} \qquad 1 \leq n < \infty \tag{51}$$

The process will leave state n at a rate of $\lambda + \mu + \alpha$ and it will enter state n from state $n + 1$ by departure (with rate μ) or from state $n - 1$ via arrival (with rate λ).

- (c) What is the average amount of time that an entering customer spends in the system?

Let L be the average number of customer in the system and λ_a the average arrival rate of entering customers. Then:

$$
\begin{aligned}
L &= \sum_{n=0}^{\infty} n P_n \\
\lambda_a &= \lambda \left(1 - P_s \right) = \lambda(1 - P_s)
\end{aligned}
\tag{52}
$$

Where $1 - P_s$ represents the probability that arriving customers enter the system. The average time an entering customer will spend in the system is given by:

$$W = \frac{L}{\lambda_a} = \frac{\sum_{n=0}^{\infty} n P_n}{\lambda(1 - P_s)} \tag{53}$$

- (d) what proportion of entering customers complete their service?

Considering that the end of a service could be caused by a system breakdown or by the finishing of a service, the fraction of customers leaving is given by:

- end of a service:

$$\frac{\mu(1 - P_s - P_0)}{\lambda_a}$$

– System breakdown:

$$1 - \frac{\alpha L}{\lambda_a}$$

Following the PASTA principle (Poisson Arrivals See Time Averages: says that the fraction of Poisson arrivals finding a queue in some state, equals the fraction of time the queue is in that state), whenever a system breakdown happens, the expected number of present customers is L. Hence, in the long-run the number of customers leaving the system because of breakdowns per time unit is αL.

- (e) what proportion of customers arrive during a breakdown?

Let a_n be the proportion of customers that find n in the system when they arrive, then the proportion of entering customers that find the system in state s can be expressed as:

$$a_s = \frac{\text{the rate at which arrivals find } s}{\text{overall arrival rate}} = \frac{\lambda P_s}{\lambda} = P_s \tag{54}$$

.

□

Exercise 8.28 Let D denote the time between successive departures in a stationary $M/M/1$ queue with $\lambda < \mu$. Show, by conditioning on whether or not a departure has left the system empty, that D is exponential with rate λ.

(From wiki M/M/1 queue) The model is considered stable only if $\lambda < \mu$. If, on average, arrivals happen faster than service completions the queue will grow indefinitely long and the system will not have a stationary distribution.

The stationary distribution is the limiting distribution for large values of t. Various performance measures can be computed explicitly for the $M/M/1$ queue. We write $\rho = \lambda/\mu$ for the utilization of the buffer and require $\rho < 1$ for the queue to be stable. ρ represents the average proportion of time which the server is occupied.

$$P_k = (1 - \rho)\rho^k$$

We see that the number of customers in the system is geometrically distributed with parameter $1 - \rho$.
In this exercise we would like to prove that $D \sim \exp(\lambda)$, notice that λ represent the customer arrival rate while μ the service rate. Let

$$I = \begin{cases} 0 & \text{a departure has left the system empty} \\ 1 & \text{o.w} \end{cases}$$

If a customer leaves the **system busy**, meaning that at the moment the customer leaves there is still more people waiting to be served. The time until the next departure is

$$D = \text{time of a service}$$

However, if the customer leaves the **system empty** this means that the time until the next departure

$$D = \text{time of arrival} + \text{time of departure}$$

If we now consider the moment generating function of D, $M_D(\sigma)$ where

$$M_D(\sigma) = E[e^{\sigma D}] = E[e^{\sigma D}|I=1]P(I=1) + E[e^{\sigma D}|I=0]P(I=0)$$

Where $P(I=0) = P_0 = 1 - \frac{\lambda}{\mu}$ which is the proportion of having 0 customers on the system. Then $P(I=1) = 1 - P(I=0) = \frac{\lambda}{\mu}$.

$$\begin{aligned}
M_D(\sigma) &= \frac{\lambda}{\mu}E[e^{\sigma D}|I=1] + \Big(1-\frac{\lambda}{\mu}\Big)E[e^{\sigma D}|I=0] \\
&= \frac{\lambda}{\mu}\int_0^\infty e^{\sigma x}\mu e^{-\mu x}\ \ dx\ \ + \Big(1-\frac{\lambda}{\mu}\Big)\int_0^\infty e^{\sigma(x+y)}\mu e^{-x\mu}\lambda e^{-y\lambda}\ \ dx\ \ dy \\
&= \frac{\lambda}{\mu}\Big(\frac{\mu}{\mu-\sigma}\Big) + \Big(1-\frac{\lambda}{\mu}\Big)\Big(\int_0^\infty e^{\sigma x}\mu e^{-x\mu}\ \ dx\int_0^\infty e^{\sigma y}\lambda e^{-y\lambda}\ \ dy\Big) \\
&= \frac{\lambda}{\mu}\Big(\frac{\mu}{\mu-\sigma}\Big) + \Big(1-\frac{\lambda}{\mu}\Big)\Big[\Big(\frac{\mu}{\mu-\sigma}\Big)\Big(\frac{\lambda}{\lambda-\sigma}\Big)\Big] \\
&= \frac{\lambda\mu}{\mu(\mu-\sigma)} + \frac{\mu\lambda}{(\mu-\sigma)(\lambda-\sigma)} - \frac{\mu\lambda^2}{\mu(\mu-\sigma)(\lambda-\sigma)} = \frac{\lambda\mu(\lambda-\sigma)+m^2\lambda-\lambda^2\mu}{\mu(\mu-\sigma)(\lambda-\sigma)} \\
&= \frac{\lambda}{\lambda-\sigma}
\end{aligned} \tag{55}$$

By the uniqueness of generating functions, it follows that D has an exponential distribution with parameter λ. □

Exercise 8.36 Compare the $M/G/1$ system for first-come, first-served queue discipline with one of last-come, first-served (for instance, in which units for service are taken from the top of a stack). Would you think that the queue size, waiting time, and busy-period distribution differ? What about their means? What if the queue discipline was always to choose at random among those waiting? Intuitively, which discipline would result in the smallest variance in the waiting time distribution?

□

Exercise 8.37 In an $M/G/1$ queue,

- (a) what proportion of departures leave behind 0 work?
- (b) what is the average work in the system as seen by a departure?

A $M/G/1$ queue, arrivals (M) are determined by a Poisson Process and job service times (G) The $M/G/1$ model assumes (i) Poisson arrivals at rate λ; (ii) a general service distribution; and (iii) a single server. In addition, we will suppose that customers are served in the order of their arrival.
Consider

$$a_n = \text{proportion of customers that find } n \text{ in the system when they arrive}$$

$$d_n = \text{proportion of customers leaving behind } n \text{ in the system when they depart}$$

We have that in any system in which customers arrive and depart one at a time

$$a_n = d_n$$

and following the PASTA principle we know that $P_n = a_n$ Where the long-run probability that there will be exactly n customers in the system is the same as the rate at which arrivals find n customers. Hence, **an arrival would just see the system according to the limiting probabilities**.

- (a)
  The rate at which departures leave behind zero customers will be the same as the rate at which arrivals find zero customers which by the PASTA principle is the same as the probability of having zero customers in the system

  $$P_0 = a_0 = 1 - \lambda E[S]$$

  Where:

  $$\begin{aligned} \lambda E[S] &= \lambda E[\text{service time}] \\ &= \text{average number of busy servers} \\ &= 1 - P_0 \\ &= E[\text{service time up to time } t] \end{aligned} \tag{56}$$

- (b)
  The average amount of work as seen by a departure is equal to the average number it sees (customers in the queue) multiplied by the mean service time (since no customers seen by a departure have yet started service).

  $$\begin{aligned} &= (\text{average number it sees})\ E[\text{service time}] \\ &= (\text{average number an arrival sees}) \\ &= (\text{average number of customers in the system})\ \ E[\text{service time}] \\ &= L\ E[\text{service time}] \\ &= \lambda(W_Q + E[S])E[S] \qquad \text{(E.q 8.34)} \\ &= \frac{\lambda^2 E[S]E[S^2]}{\lambda - \lambda E[S]} + \lambda(E[S])^2 \end{aligned} \tag{57}$$

□

Exercise 8.40 Consider a $M/G/1$ system with $\lambda E[S] < 1$.

- (a) Suppose that service is about to begin at a moment when there are n customers in the system.
  - (i) Argue that the additional time until there are only $n-1$ customers in the system has the same distribution as a busy period.
  - (ii) What is the expected additional time until the system is empty?
- (b) Suppose that the work in the system at some moment is A. We are interested in the expected additional time until the system is empty—call it $E[T]$. Let N denote the number of arrivals during the first A units of time.
  - (i) Compute $E[T|N]$.
  - (ii) Compute $E[T]$.

- (a)
  First let's remember than in order for the quantities L, L_Q and W to be finite we need the condition $\lambda E[S] < 1$ to hold, as $\frac{1}{E[S]} > \lambda$ i.e departure rate > arrival rate.
  - (i)
    A busy period is when there is at least one customer in the system, and so the server is busy. Suppose we are using a **last-come, first serve** discipline, the change from n to $n-1$ customers will only occur when we have n customers on the system. Furthermore, going from n to $n-1$ is independent of the value of n.
  - (ii)
    The expected number of time until the system is empty will be $nE[B]$, where $E[B]$ is the expected length of a busy period, hence we should consider n times this value.
- (b)
  - (i)
    We would like to know the conditional expectation of the additional time until the system is empty, given that there were N arrivals during the first A units of time.
    $$E[T|N] = A + NE[B]$$
  - (ii)
    Using the law of total expectations
    $$E[T] = E[E[T|N]] = E[A] + E[NE[B]] = A + E[N]E[B] = A + A\lambda E[B]$$
    Here N is a Poisson Process
    $$E[N] = \sum_{n\geq 0} nP(N(A) = n) = \sum_{n\geq 0} n\frac{(\lambda A)^n e^{-\lambda A}}{n!} = A\lambda$$

$\square$

8 Session 8: Simulation; Ross-Chapter 11

Exercise 11.1 Suppose it is relatively easy to simulate from the distributions F_i , $i = 1, 2, ..., n$. If n is small, how can we simulate from

$$F(x) = \sum_{i=1}^{n} P_i F_i(x), \qquad P_i \geq 0, \quad \sum_i P_i = 1?$$

Give a method for simulating from:

$$F(x) = \begin{cases} \frac{1-e^{-2x}+2x}{3}, & 0 < x < 1 \\ \frac{3-e^{-2x}}{3}, & 1 < x < \infty \end{cases}$$

We know we can simulate from $F_1, F_2, ..., F_n$ and we would like to know if we can simulate from $F(x) = \sum\limits_{i=1}^{n} P_i F_i(x)$.

Simulating from Discrete Distributions: Let Y be a discrete random variable taking values in $\{1, 2, ..., n\}$ and having probability mass function

$$P\{Y = j\} = P_j, \quad j = 1, 2, ..., \quad \sum_j P_j = 1$$

To **simulate** Y for which $P\{Y = j\} = P_j$ let $U \sim U(0, 1)$

$$X = \min\{k \in R : \sum_{i=1}^{k} P_i \geq U\}$$

Now let X_i be a random variable independent of Y having distribution function $F_i(x)$. Then

$$X_y = \sum_{i=1}^{n} X_i \mathbb{1}_{\{y=i\}}$$

and;

$$\begin{aligned} F_{X_Y}(x) = P(X_Y \leq x) = \sum_{i=1}^{n} P(X_y \leq x | Y = i) P(Y = i) = \sum_{i=1}^{n} P(X_i \leq x) P(Y = i) \\ = \sum_{i=1}^{n} F_i(x) P_i \end{aligned} \tag{58}$$

For instance, having that

$$F(x) = F_1(x)P_1 + F_2(x)P_2$$

where $P_1 = 1/3, \quad P_2 = 2/3$ then $F(x) = 1/3F_1(x) + 2/3F_2(x)$

$\square$

Exercise 11.5 Suppose it is relatively easy to simulate from F_i for each $i = 1, ..., n$. How can we simulate from

- (a) $\prod_{i=1}^{n} F_i(x)$?
- (b) $1 - \prod_{i=1}^{n}(1 - F_i(x))$?
- (c) Give two methods for simulating from the distribution $F(x) = x^n,\ \ 0 < x < 1$.

- (a)Let X_i be a random variable having distribution function $F_i(x)$ and considering all X_i's to be independent, then

$$\begin{aligned}\prod_{i=1}^{n} F_i(x) &= \prod_{i=1}^{n} P(X_i \leq x) = P(X_1 \leq x, X_2 \leq x, ..., X_n \leq x) \\ &= P(\max(X_1, ..., X_n) \leq x) = P\Big(\max_{1\leq i\leq n}(X_i) \leq x\Big)\end{aligned} \tag{59}$$

- (b)

$$\begin{aligned}1 - \prod_{i=1}^{n}(1 - F_i(x)) &= 1 - \prod_{i=1}^{n} 1 - P(X_i \leq x) = 1 - \prod_{i=1}^{n} P(X_i > x) \\ &= 1 - P(X_1 > x, X_2 > x, ..., X_n > x) \\ &= 1 - P(\min(X_1, X_2, ..., X_n) > x) = 1 - P\Big(\min_{1\leq i\leq n} X_i > x\Big) \\ &= P\Big(\min_{1\leq i\leq n} X_i \leq x\Big)\end{aligned} \tag{60}$$

- (c)
  - (i) simulate n independent uniforms.
  - (ii) use the **inverse distribution method** and simulate a single uniform U

□

Exercise 11.7 Give an algorithm for simulating a random variable having density function:

$$f(x) = 30(x^2 - 2x^3 + x^4), \quad 0 < x < 1$$

Let us use the **rejection method** to generate a random variable having density function $f(x)$ Since this random variable is concentrated in the interval $(0, 1)$, let us consider the rejection method with $g(x) = 1, \quad 0 < x < 1$

To determine the constant c such that $f(x)/g(x) \leq c$, we use calculus to determine the maximum value of $f(x)/g(x)$

Differentiation of this quantity yields

$$\begin{aligned} \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= 60x(1-3x+4x^2) = 60x(4x^2 - x - 2x + 1) \\ &= 60x(x(2x-1)-(2x-1)) = 60x(2x-1)(x-1) \end{aligned} \tag{61}$$

Setting this equal to 0 we get that $x = 0, \quad x = 1/2, \quad x = 1$ where $f(0) = 0, f(1/2) = 15/8, f(1) = 0$ shows that the maximal value is attained when $x = 1/2$, and thus

$$\frac{f(x)}{g(x)} \leq \frac{15}{8} = c$$

Hence

$$\frac{f(x)}{cg(x)} = 16(x^2 - 2x^3 + x^4)$$

and thus the rejection procedure is as follows:

Step 1: Generate random numbers U_1 and U_2.

Step 2: If $U_2 \leq 16(U_1^2 - 2U_1^3 + U_1^4)$ stop and set $X = U_1$. Otherwise return to step 1.

The random variable X generated by the rejection method has density function f and the average number of times that step 1 will be performed is $c = \frac{15}{8}$.

□

Exercise 11.8 Consider the technique of simulating a gamma (n, λ) random variable by using the rejection method with g being an exponential density with rate λ/n.

- (a) Show that the average number of iterations of the algorithm needed to generate a gamma is $n^n e^{1-n}/(n-1)!$.
- (b) Use Stirling's approximation to show that for large n the answer to part (a) is approximately equal to $e[(n-1)(2\pi)]^{1/2}$.
- (c) Show that the procedure is equivalent to the following:
  Step 1: Generate Y_1 and Y_2, independent exponentials with rate 1.
  Step 2: If $Y_1 < (n-1)[Y_2 - log(Y_2) - 1]$, return to step 1.
  Step 3: Set $X = nY_2/\lambda$.
- (d)Explain how to obtain an independent exponential along with a gamma from the preceding algorithm.

- (a)
We want to generate sampling values from a target distribution X with probability density function:
$$f(x) = \frac{1}{\Gamma(n)}\lambda^n x^{n-1}e^{-\lambda x}$$
by using a proposal distribution Y with probability density
$$g(x) = \frac{\lambda}{n}e^{-\frac{\lambda}{n}x}$$
The idea is that one can generate a sample value from X by instead sampling from Y. Thus, the likelihood ratio
$$h(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{\Gamma(n)}\lambda^n x^{n-1}e^{-\lambda x}}{\frac{\lambda}{n}e^{-\frac{\lambda}{n}x}} = \frac{n}{(n-1)!}(\lambda x)^{n-1}e^{-\lambda x(1-\frac{1}{n})} \tag{62}$$
To determine the constant c such that $f(x)/g(x) \leq c$ we will look for the maximum value of $f(x)/g(x)$. Differentiation of this equation yields
$$\begin{aligned} \frac{d}{dx}\Big[h(x)\Big] &= \frac{n}{\Gamma(n)}\lambda(n-1)(\lambda x)^{n-2}e^{-\lambda x(1-\frac{1}{n})} - \frac{n}{\Gamma(n)}(\lambda x)^{n-1}\lambda\Big(1-\frac{1}{n}\Big)e^{-\lambda x(1-\frac{1}{n})} \\ &= \frac{n}{\Gamma(n)}\lambda(n-1)(\lambda x)^{n-2}e^{-\lambda x(1-\frac{1}{n})} - \frac{n}{\Gamma(n)}(\lambda x)^{n-1}\frac{\lambda}{n}(n-1)e^{-\lambda x(1-\frac{1}{n})} \\ &= \frac{n}{\Gamma(n)}\lambda(n-1)(\lambda x)^{n-2}e^{-\lambda x(1-\frac{1}{n})}\Big(1-\frac{\lambda x}{n}\Big) \end{aligned} \tag{63}$$
Setting this equal to 0 we get that
$$x = 0 \Rightarrow \min\ \ h(x) \quad , x = \frac{n}{\lambda} \Rightarrow \max \quad h(x)$$
Where the average number of iterations the algorithm needed to generate a gamma is:
$$h(x) \leq h\Big(\frac{n}{\lambda}\Big) = \frac{n}{\Gamma(n)}n^{n-1}e^{-n(1-\frac{1}{n})} = c$$

- (b)
Given the **Stirling's formula for the gamma function** we know that for all positive integers, $n! \sim \sqrt{2\pi n}(ne^{-1})^n$, hence
$$\begin{aligned} c &\approx \frac{n^n}{\sqrt{2\pi(n-1)}}\frac{e^{n-1}}{(n-1)^{n-1}e^{n-1}} = \Big(\frac{n}{n-1}\Big)^n\frac{n-1}{\sqrt{2\pi(n-1)}} \\ &= \Big(\frac{n}{n-1}\Big)^n\sqrt{\frac{n-1}{2\pi}} = \Big(1-\frac{1}{n}\Big)^{-n}\sqrt{\frac{n-1}{2\pi}} \end{aligned} \tag{64}$$
Since $(1-\frac{1}{n})^n \to e^{-1}$ as $n \to \infty$ then $(1-\frac{1}{n})^{-n} \to e$ and so
$$\frac{c}{\sqrt{(n-1)}} \to \frac{e}{\sqrt{2\pi}} \quad \text{as} \quad n \to \infty$$
$c \approx e\sqrt{\frac{n-1}{2\pi}}$

- (c)
  Since

$$
\begin{aligned}
\frac{f(x)}{cg(x)} = \frac{h(x)}{c} &= \frac{\frac{n}{(n-1)!}(\lambda x)^{n-1}e^{-\lambda x(1-\frac{1}{n})}}{\frac{n^n}{(n-1)!}e^{-(n-1)}} = n^{-(n-1)}(\lambda x)^{n-1}e^{-\lambda x(1-\frac{1}{n})}e^{n-1} \\
&= \frac{e^{n-1}}{n^{n-1}}(\lambda x)^{n-1}e^{-\lambda x(1-\frac{1}{n})}
\end{aligned}
\tag{65}
$$

and thus the rejection procedure is as follows:

Step 1: Generate Y with density function $g(x) = \frac{\lambda}{n}e^{-x\lambda/n}$; $Y \sim \exp(\lambda/n)$ and a random number U.

Step 2: If

$$U \leq \frac{h(Y)}{c} = \frac{e^{n-1}}{n^{n-1}}(\lambda Y)^{n-1}e^{-\lambda Y(1-\frac{1}{n})}$$

set $X = Y$. Otherwise return to step 1.

Notice that the probability that we accept proposed realizations from the density g is

$$P(U \leq \frac{f(Y)}{cg(Y)}) = \frac{1}{c}$$

while the probability of being rejected would be $1 - \frac{1}{c}$ meaning that the expected number of iterations before we accept a proposed random variable is c. So instead of simulating directly from the density f we on average simulate c times from the density g

$$
\begin{aligned}
&\iff \log(U) \leq \log(h(y)) - \log(c) = (n-1)\Big(1 - \frac{\lambda Y}{n} + \log\Big(\frac{\lambda Y}{n}\Big)\Big) \\
&\iff -\log(U) \geq \log(c) - \log(h(y)) = (n-1)\Big(\frac{\lambda Y}{n} - \log\Big(\frac{\lambda Y}{n}\Big) - 1\Big)
\end{aligned}
\tag{66}
$$

The previous steps will be equivalent as performing the rejection procedure as follows:

Step 1: Generate $Y_1, Y_2 \sim \exp(1)$

Consider that

$$Y_1 = -\log(U)$$

where $U \sim U(0,1)$

$$\iff F_{Y_1}(x) = P(Y_1 < x) = P(\log U > -x) = P(U > e^{-x}) = \int_0^{e^{-x}} \quad dx = e^{-x}$$

which is the density function of an exponential random variable with rate 1.

$$Y_2 = \frac{\lambda}{n} Y$$

where $Y \sim \exp(\lambda/n)$ yields to

$$F_{Y_2}(x) = P(Y_2 \le x) = P(Y \le \frac{n}{\lambda} x) = \int_0^{\frac{n}{\lambda}x} \frac{\lambda}{n} e^{-\frac{\lambda}{n}t} \ \ dt = 1 - e^{-x}$$

Which is just the cumulative density function of an exponential random variable with rate 1.

Step 2: If $Y_1 \ge (n-1)(Y_2 - \log(Y_2) - 1)$ return to step 1

Step 3: Set $X = Y = \frac{n}{y} Y_2$

- (d)
  ???

$\square$

Exercise 11.13 The *Discrete Rejection Method:* Suppose we want to simulate X having probability mass function $P\{X = i\} = P_i, \quad i = 1, ..., n$ and suppose we can easily simulate from the probability mass function Q_i, $\sum_i Q_i = 1$, $Q_i \ge 0$ Let C be such that $Pi \le CQ_i$, $i = 1, ..., n$. Show that the following algorithm generates the desired random variable

Step 1: Generate Y having mass function Q and U an independent random number.

Step 2: If $U \le P_Y / CQ_Y$, set $X = Y$. Otherwise return to step 1.

According to the ***Discrete Rejection Method*** we would like to generate a random variable X where $p(k) = P(X = k)$. We start by assuming that the F we wish to simulate from has a probability mass function $p(k)$ we would also like to find an alternative probability distribution Q, with density function $q(k)$, from which we already have an efficient algorithm for generating from but also such that the function $q(k)$ is "close" to $p(k)$.
In this case we have that $p(x)$ has mass function P_i and that $q(x)$ has mass function Q_i

Step 1: Generate a rv Y having mass function Q.

Step 2: Generate U (independent from Y).

Step 3: If

$$U \le \frac{P_Y}{CQ_Y}$$

then set $X = Y$ ("accept") ; otherwise go back to 1 ("reject").

$$
\begin{aligned}
P(X=i) = P\Big(Y=i|U \le \frac{P_Y}{CQ_y}\Big) &= \frac{P(Y=i, U \le \frac{P_Y}{CQ_y})}{P(U \le \frac{P_Y}{CQ_y})} = \frac{P(Y=i)P(U \le \frac{P_Y}{CQ_y})}{P(U \le \frac{P_Y}{CQ_y})} \\
&= \frac{Q_i P(U \le \frac{P_y}{CQ_y}|Y=i)}{\sum_{i=1}^{n}}
\end{aligned} \tag{67}
$$

□

9 Session 9: Simulation, Variance Reduction Methods; Ross-Chapter 11

The number of simulations we need depends on the variance of the estimated mean of the simulated Y's we use. If we can reduce the variance we would need less simulations. Often, computing this variance is even harder than computing the mean. But knowing that we have reduced the variance is always helpful because we know that the convergence (by the LLN) goes faster.

> **Exercise 11.30** If f is the density function of a normal random variable with mean μ and variance σ^2, show that the tilted density f_t is the density of a normal random variable with mean $\mu + \sigma^2 t$ and variance σ^2.

Let $X \sim N(\mu, \sigma^2)$ be a random variable with density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If we want to find an appropriate density function $g(x)$ (from the proposal distribution) the *tilted* density is a "good" option. Let $f_t(x)$ be a tilted density function.

From **Definition 11.22** we know that

$$f_t(x) = \frac{e^{tx} f(x)}{M(t)}, \quad M(t) = E_f[e^{tX}] = \int e^{tx} f(x) dx$$

Hence;

$$
\begin{aligned}
f_t(x) &= \frac{1}{M(t)}\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{(x-\mu)^2}{2\sigma^2}+tx\} \\
&= C\frac{1}{\sqrt{2\pi}\sigma}\exp\{\frac{1}{2\sigma^2}(2\sigma^2tx-x^2-\mu^2+2x\mu)\} = C\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(-2x(\mu-\sigma^2t)+x^2+\mu^2)\} \\
&= C\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(-2x(\mu-\sigma^2t)+x^2+(\mu-\sigma^2t)^2-(\mu-\sigma^2t)^2+\mu^2\} \\
&= C\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(x^2+(\mu-\sigma^2t)^2-2x(\mu-\sigma^2t)\}\exp\{-\frac{1}{2\sigma^2}(\mu^2-(\mu-\sigma^2t)^2)\} \\
&= C\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(x-(\mu-\sigma^2t))^2\}\exp\{-\frac{1}{2\sigma^2}(\mu^2-(\mu-\sigma^2t)^2)\}
\end{aligned}
\tag{68}
$$

where $M(t)$ is obtained as

$$
\begin{aligned}
M(t) &= \int_{-\infty}^{+\infty} e^{tx}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad dx \\
&= \exp\{-\frac{1}{2\sigma^2}(\mu^2-(\mu-\sigma^2t)^2)\}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(x-(\mu-\sigma^2t))^2\}dx
\end{aligned}
\tag{69}
$$

Which yields to:

$$
f_t(x) = \frac{\exp\{-\frac{1}{2\sigma^2}(x-(\mu-\sigma^2t))^2\}}{\int_{-\infty}^{+\infty}\exp\{-\frac{1}{2\sigma^2}(x-(\mu-\sigma^2t))^2\}dx} = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-\frac{1}{2\sigma^2}(x-(\mu-\sigma^2t))^2\} \tag{70}
$$

Which is just the density function of the normal distribution with mean $(\mu-\sigma^2t)$ and variance σ^2. □

Exercise 11.31 Consider a queueing system in which each service time, independent of the past, has mean μ. Let W_n and D_n denote, respectively, the amounts of time customer n spends in the system and in queue. Hence, $D_n = W_n - S_n$ where S_n is the service time of customer n. Therefore,

$$E[D_n] = E[W_n] - \mu$$

If we use simulation to estimate $E[D_n]$, should we

- (a) use the simulated data to determine D_n, which is then used as an estimate of $E[D_n]$; or
- (b) use the simulated data to determine W_n and then use this quantity minus μ as an estimate of $E[D_n]$? Repeat for when we want to estimate $E[W_n]$.

We would like to use the **variance reduction by conditioning** technique, we know that

$$\begin{aligned} W_n &= \text{amount of time customer } n \text{ spends in the system} \\ D_n &= \text{amount of time customer } n \text{ spends in the queue} \\ S_n &= \text{amount of time customer } n \text{ spends in service} \end{aligned} \tag{71}$$

Hence

$$D_n = W_n - S_n, \qquad E[D_n] = E[W_n] - \mu$$

as $E[S_n]$ is the mean service time.

- (a)
  We know from the conditional variance formula that

  $$\text{Var}(E[D_n|W_n]) \leq Var(D_n) = E[\text{Var}(E[D_n|W_n])] + \text{Var}(E[D_n|W_n])$$

  Implying since $E[E[D_n|W_n]] = E[D_n]$, that $E[D_n|W_n]$ is a better estimator of $E[D_n]$ than is D_n.

  By conditioning, we know that :

  $$E[D_n] = E[E[D_n|W_n]] = E[E[W_n - S_n|W_n]] = E[W_n - E[S_n|W_n]] = E[W_n] - E[S_n|W_n]$$

  For this equality to hold we would need $E[S_n|W_n] = \mu$ but this will only be the case if S_n was independent of W_n which is not. Let's imagine that we have infinite number of servers, meaning that customers will be served without having to wait in line, $D_n = 0 \iff W_n = S_n$.

  Using the estimated data to determine D_n is not a viable way for estimating $E[D_n]$

- (b)

  If we want to use D_n to simulate W_n it is possible to use conditioning for variance reduction since:

  $$E[W_n|D_n] = E[D_n + S_n|D_n] = D_n + E[S_n|D_n] = D_n + \mu \tag{72}$$

  as the service time of a customer S_n is independent of the time he spends on the queue D_n. Thus

  $$E[W_n] = E[E[W_n|D_n]] = E[E[D_n + S_n|D_n]] = E[D_n] + \mu$$

  Finally, we can then estimate $E[D_n] = E[W_n] - \mu$

□

Exercise 11.32 Show that if X and Y have the same distribution then

$$\text{Var}((X+Y)/2) \leq \text{Var}(X)$$

Hence, conclude that the use of antithetic variables can never increase variance (though it need not be as efficient as generating an independent set of random numbers).

Let's start by computing $\text{Var}(X+Y)$:

$$\begin{aligned}
\text{Var}(\frac{X+Y}{2}) &= \frac{1}{4}\Big[\text{Cov}(X+Y,X+Y)\Big] \\
&= \frac{1}{4}\Big[\text{Cov}(X,X)+\text{Cov}(X,Y)+\text{Cov}(Y,X)+\text{Cov}(Y,Y)\Big] \\
&= \frac{1}{4}\Big[\text{Var}(X)+\text{Var}(Y)+2\,\text{Cov}(X,Y)\Big] \\
&= \frac{1}{4}\Big[\text{Var}(X)+\text{Var}(Y)+2\rho_{XY}\sqrt{\text{Var}(X)\,\text{Var}(Y)}\Big] \\
&= \frac{1}{4}\Big[2\,\text{Var}(X)+2\rho_{XY}\sqrt{\text{Var}(X)^2}\Big] \\
&= \frac{1}{2}\text{Var}(X)+\frac{1}{2}\rho_{XY}\,\text{Var}(X) = \frac{1}{2}\text{Var}(X)(1+\rho_{XY})) \leq \text{Var}(X)
\end{aligned} \tag{73}$$

Knowing that the correlation coefficient ρ_{XY}

$$\rho_{XY} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} \quad \in [-1,1]$$

Where ± 1 indicates the strongest possible agreement and 0 the strongest possible disagreement. □

Exercise 11.33 If $0 \leq X \leq a$, show that

- (a) $E[X^2] \leq aE[X]$,
- (b) $\text{Var}(X) \leq E[X](a-E[X])$,
- (c) $\text{Var}(X) \leq a^2/4$.

- (a)
  We know that

  $$0 \leq aE[X] - E[X^2] = E[aX - X^2] = E[X(a-X)]$$

  as $a \geq X$ and $X \geq 0$ this makes $E[X(a-X)]$ a non-negative value.

  $$\iff aE[X] \geq E[X^2]$$

- (b)
  We know that
  $$\text{Var}(x) = E[X^2] - E[X]^2 \leq aE[X] - E[X]^2$$
  Which yields to
  $$E[X^2] \leq aE[X]$$
  which is what we have proved in (a)

- (c)
  We know that $0 \leq X \leq a$ or that $P(0 \leq X \leq a) = 1, \iff X - a \leq 0$. If we now use the result obtained in (b) where $u(a-u)$ for $u = E[X]$, in order to find it's maximal value we will differentiate this result and set to zero.
  $$\frac{\partial}{\partial u} u(a-u) = 0 \iff u = \frac{a}{2}$$
  which means that $u(a-u)$ will be at most $a^2/4$.
  $$\text{Var}(X \leq E[X](a - E[X]) \leq \frac{a^2}{4}$$

□

10 Session 10: Simulation of Stochastic Processes and MCMC; Ross Sections 11.5.0-1 + Section 4.9

Exercise 11.17 *Order Statistics:* Let $X_1, ..., X_n$ be i.i.d. from a continuous distribution F, and let $X_{(i)}$ denote the i-th smallest of $X_1, ..., X_n$, $i = 1, ..., n$. Suppose we want to simulate $X_{(1)} < X_{(2)} < ... < X_{(n)}$. One approach is to simulate n values from F, and then order these values. However, this ordering, or sorting, can be time consuming when n is large.

- (a) Suppose that $\lambda(t)$, the hazard rate function of F, is bounded. Show how the hazard rate method can be applied to generate the n variables in such a manner that no sorting is necessary.
  Suppose now that F^{-1} is easily computed.

Let $X_1, X_2, ..., X_n \overset{iid}{\sim} F$, we want to simulate $U_{(1)} < U_{(2)} < ... < U_{(n)}$.

(a)

The hazard rate function of F is defined by:

$$\lambda(t) = \frac{f(t)}{1 - F(t)} = \frac{F'(t)}{1 - F(t)}$$

Where $\lambda(t)$ represents the instantaneous probability intensity that an item having life distribution F will fail at time t given it has survived to that time.

Let's now suppose we want to simulate $X_{(1)}$ with distribution $F_{(1)}(t)$

$$\begin{aligned} F_{(1)}(t) &= P(X_{(1)} \leq t) = 1 - P(X_{(1)} > t) \\ &= 1 - P(\cup_{i=1}^{n}\{X_i > t\}) = 1 - (1 - F(t))^n \end{aligned} \tag{74}$$

Hence,

$$f_{(1)}(t) = \frac{dF_{(1)}(t)}{dt} = n(1 - F(t))^{n-1}(F'(t)) \tag{75}$$

With hazard function

$$\lambda_{(1)}(t) = \frac{f_{(1)}(t)}{1 - F_{(1)}(t)} = \frac{n(1 - F(t))^{n-1}(F'(t))}{(1 - F(t))^n} = n\frac{F'(t)}{1 - F(t)} = n\lambda(t) \tag{76}$$

□

- (b) Argue that $X_{(1)}, X_{(2)}, ..., X_{(n)}$ can be generated by simulating $U_{(1)} < U_{(2)} < ... < U_{(n)}$-the ordered values of n independent random numbers- and then setting $X_{(i)} = F^{-1}(U_{(i)})$. Explain why this means that $X_{(i)}$ can be generated from $F^{-1}(\beta_i)$ where β_i is beta with parameters $i, n+i+1$.

- (c) Argue that $U_{(1)}, U_{(2)}, ..., U_{(n)}$ can be generated, without any need for sorting, by simulating i.i.d. exponentials $Y_1, ..., Y_{n+1}$ and then setting

$$U_{(i)} = \frac{Y_1 + ... + Y_i}{Y_1 + ... + Y_{n+1}}, \qquad i = 1, ..., n$$

  Hint: Given the time of the $(n+1)$st event of a Poisson process, what can be said about the set of times of the first n events?

- (d) Show that if $U_{(n)} = y$ then $U_{(1)}, U_{(2)}, ..., U_{(n-1)}$ has the same joint distribution as the order statistics of a set of $n-1$ uniform $(0, y)$ random variables.

- (e) Use part (d) to show that $U_{(1)}, U_{(2)}, ..., U_{(n)}$ can be generated as follows:

  Step 1: Generate random numbers $U_1, ..., U_n$.
  Step 2: Set

$$U_{(n)} = U_1^{1/n}, \qquad U_{(n-1)=U_{(n)}(U_2)^{1/(n-1)}},$$

$$U_{(j-1)} = U_{(j)}(U_{n-j+2})^{1/(j-1)}, \quad j = 2, ..., n-11$$

Exercise 11.23 For a non-homogeneous Poisson process with intensity function $\lambda(t), t \geq 0$,where $\int_0^\infty \lambda(t)dt = \infty$, let $X_1, X_2, ...$ denote the sequence of times at which events occur.

- (a)Show that $\int_0^{X_1} \lambda(t)dt$ is exponential with rate 1.
- (b)Show that $\int_{X_{i-1}}^{X_i} \lambda(t)dt, \quad i \geq 1$ are independent exponentials with rate 1, where $X_0 = 0$. In words, independent of the past, the additional amount of hazard that must be experienced until an event occurs is exponential with rate 1.

□

Exercise 11.24 Give an efficient method for simulating a non-homogeneous Poisson process with intensity function

$$\lambda(t) = b + \frac{1}{t+a}, \quad t \geq 0$$

□

11 Session 11:Brownian Motion; Ross Chapter 10

11.1 Session 11: Book Examples

Example 10.1 In a bicycle race between two competitors, let $Y(t)$ denote the amount of time (in seconds) by which the racer that started in the inside position is ahead when $100t$ percent of the race has been completed, and suppose that $\{Y(t), 0 \leq t \leq 1\}$ can be effectively modeled as a Brownian motion process with variance parameter σ^2.

- (a) If the inside racer is leading by σ seconds at the midpoint of the race, what is the probability that she is the winner?
- (b) If the inside racer wins the race by a margin of σ seconds, what is the probability that she was ahead at the midpoint?

We have that $\{Y(t), 0 \leq t \leq 1\}$ is a Brownian motion process. Where $Y(0)$ indicates the distance between competitors at the beginning of the race while $Y(1)$ at the end of the race, as $(0 \leq t \leq 1)$

- (a)
  The probability that the inside racer wins given that she is leading by σ seconds at the midpoint of the race (1/2) is given by:

$$P(Y(1) > 0|Y(1/2) = \sigma)$$

Where $Y(1) > 0$ (1 is the end of the race while 0 refers to the distance of the competitors). Hence

$$\begin{aligned} P(Y(1) > 0|Y(1/2) = \sigma) &= P(Y(1) - Y(1/2) > -Y(1/2)|Y(1/2) = \sigma) \\ &= P(Y(1) - Y(1/2) > -\sigma|Y(1/2) = \sigma) \\ &= P(Y(1) - Y(1/2) > -\sigma) = P(Y(1/2) > -\sigma) \end{aligned} \tag{77}$$

As $Y(1/2)$ is a Brownian Motion, we know that $Y(1/2) \sim \mathcal{N}(0, \sigma^2 t)$ and so

$$\begin{aligned} P(Y(1/2) > -\sigma) &= \int_{-\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2}}{\sigma} \exp\Big\{\frac{-1}{2}\Big(\frac{2t^2}{\sigma^2}\Big)\Big\} dt \\ &\text{We apply chance of variable } u = \sqrt{2}t/2 \\ &= \int_{-\sqrt{2}}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2}}{\sigma} \exp\Big\{\frac{-1}{2}(u^2)\Big\} \frac{\sigma}{\sqrt{2}} du = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2}}^{\infty} \exp\Big\{\frac{-1}{2}(u^2)\Big\} du \end{aligned} \tag{78}$$

- (b) We now want to compute the probability that the inside racer was ahead of her competitors at the middle of the race ($Y(1/2) > 0$) given that she wins the race ($Y(1) = \sigma$) i.e $P(Y(1/2) > 0|Y(1) = \sigma)$. If we denote the constant σ as C, $s = 1/2$ and $t = 1$, knowing that $s < t$, and since $\{X(t), t \geq 0\}$ is standard Brownian motion when $X(t) = Y(t)/\sigma$ (we convert to a standard process and suppose $\sigma = 1$, which will make the variance be t) we can compute the conditional density as follows:

$$X(s) \sim \mathcal{N}(0, s), \quad X(t-s) \sim \mathcal{N}(0, t-s), \quad X(t) \sim \mathcal{N}(0, t)$$

$$\begin{aligned} f_{s|t}(x|C/\sigma) &= \frac{f_{s,t}(x, C/\sigma)}{f_t(C/\sigma)} = \frac{f_s(x) f_{t-s}(\frac{C}{\sigma} - x)}{f_t(\frac{C}{\sigma})} = \frac{\frac{1}{\sqrt{2\pi s}} \frac{1}{\sqrt{2\pi(t-s)}} \exp\Big\{\frac{-(C/\sigma - x)^2}{2(t-s)} \frac{-x^2}{2s}\Big\}}{\frac{1}{\sqrt{2\pi t}} \exp\{\frac{(C/\sigma)^2}{2t}\}} \\ &= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{t}{s(t-s)}} \exp\Big\{\frac{-(C/\sigma)^2 + x^2 - 2(C/\sigma)x}{2(t-s)} - \frac{x^2}{2s} + \frac{(C/\sigma)^2}{2t}\Big\} \\ &= K \exp\Big\{ - x^2\Big(\frac{1}{2(t-s)} \frac{1}{2s}\Big) - \frac{(C/\sigma)^2 + 2x(C/\sigma)}{2(t-s)} + \frac{(C/\sigma)^2}{2t}\Big\} \\ &= K \exp\Big\{ - \frac{x^2 t}{2s(t-s)} - \frac{(C/\sigma)^2 s + 2x(C/\sigma)s}{2s(t-s)} + \frac{(C/\sigma)^2}{2t}\Big\} \\ &= K \exp\Big\{ - \frac{x^2 t}{2s(t-s)} - \frac{(C/\sigma)^2 st + 2x(C/\sigma)st + (C/\sigma)^2 st - (C/\sigma)^2 s^2}{2st(t-s)}\Big\} \\ &= K \exp\Big\{ - \frac{t}{2s(t-s)}\Big(x^2 - \frac{(C/\sigma)^2 s^2}{t} - \frac{2x(C/\sigma)s}{t}\Big)\Big\} \\ &= K \exp\Big\{ - \frac{t}{2s(t-s)}\Big(x + \frac{Cs}{\sigma t}\Big)^2\Big\} = K \exp\Big\{\frac{-\Big(x - \frac{Cs}{\sigma t}\Big)^2}{2\frac{s(t-s)}{t}}\Big\} \end{aligned} \tag{79}$$

$$E\Big[X(s)|X(t) = \frac{Y(t)}{\sigma}\Big] = E\Big[X(s)|X(t) = \frac{C}{\sigma}\Big] = \frac{Cs}{\sigma t}, \quad \text{Var}\left[X(s)|X(t) = \frac{C}{\sigma}\right] = \frac{s}{t}(t-s)$$

$\square$

In the following exercises $\{B(t), t \geq 0\}$ is a standard Brownian motion process and T_a denotes the time it takes this process to hit a.

Exercise 10.1 What is the distribution of $B(s) + B(t), \quad s \leq t$?

$$\begin{aligned} B(s) + B(t) &= B(s) + B(t) + B(s) - B(s) \\ &= 2B(s) + (B(t) - B(s)) \sim 2N_1 + N_2 \end{aligned} \tag{80}$$

Where N_1 is independent of N_2, the distribution is given by

$$N_1 \sim \mathcal{N}(0, s), \iff 2N_1 \sim \mathcal{N}(0, 2^2 s) \quad N_2 \sim \mathcal{N}(0, t - s)$$

$$2N_1 + N_2 \sim \mathcal{N}(0, \quad 3s + t)$$

$\square$

Exercise 10.2 Compute the conditional distribution of $B(s)$ given that $B(t_1) = A$ and $B(t_2) = B$, where $0 < t_1 < s < t_2$.

We would like to find the distribution of $B(s)|B(t_1) = A$ and $B(t_2) = B$

$$
\begin{aligned}
&= \frac{f_{s,t_1,t_2}(x,A,B)}{f_{t_1,t_2}(A,B)} = \frac{f_{t_1}(A)f_{t_2-s}(B-x)f_{s-t_1}(x-A)}{f_{t_1}(A)f_{t_2-t_1}(B-A)} \\
&= \frac{f_{t_2-s}(B-x)f_{s-t_1}(x-A)}{f_{t_2-t_1}(B-A)} \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{t_2-t_1}{(t_2-s)(s-t_1)}}\exp\Big\{-\frac{(B-x)^2}{2(t_2-s)} - \frac{(x-A)^2}{2(s-t_1)} + \frac{(B-A)^2}{2(t_2-t_1)}\Big\} \\
&= K\exp\Big\{-\frac{(B^2+x^2-2Bx)}{2(t_2-s)} - \frac{(x^2+A^2-2xA)}{2(s-t_1)} + \frac{(B-A)^2}{2(t_2-t_1)}\Big\} \\
&= K\exp\Big\{-\frac{1}{2}\Big[\frac{x^2(t_2-t_1)}{(t_2-s)(s-t_1)} + \frac{(B^2-2Bx)}{(t_2-s)} + \frac{(A^2-2xA)}{(s-t_1)} - \frac{(B-A)^2}{(t_2-t_1)}\Big]\Big\} \\
&= K\exp\Big\{-\frac{1}{2}\Big[\frac{x^2(t_2-t_1)}{(t_2-s)(s-t_1)} - 2x\Big(\frac{B}{(t_2-s)} + \frac{A}{(s-t_1)}\Big) + \frac{B^2}{(t_2-s)} + \frac{A^2}{(s-t_1)} - \frac{(B-A)^2}{(t_2-t_1)}\Big]\Big\} \\
&= K\exp\Big\{-\frac{1}{2}\Big[\frac{x^2(t_2-t_1)}{(t_2-s)(s-t_1)} - 2x\Big(\frac{B(s-t_1)+A(t_2-s)}{(t_2-s)(s-t_1)}\Big) \\
&+ \frac{B^2(s-t_1)+A^2(t_2-s)}{(t_2-s)(s-t_1)} - \frac{(B-A)^2}{(t_2-t_1)}\Big]\Big\} \\
&= K\exp\Big\{-\frac{1}{2}\Big[\frac{x^2(t_2-t_1)}{(t_2-s)(s-t_1)} - 2x(t_2-t_1)\Big(\frac{B(s-t_1)+A(t_2-s)}{(t_2-t_1)(t_2-s)(s-t_1)}\Big) \\
&+ \frac{B^2(s-t_1)(t_2-t_1)+A^2(t_2-s)(t_2-t_1)-(B-A)^2(t_2-s)(s-t_1)}{(t_2-s)(s-t_1)(t_2-t_1)}\Big]\Big\} \\
&= K\exp\Big\{-\frac{(t_2-t_1)}{2(t_2-s)(s-t_1)}\Big[x^2 - 2x\Big(B\frac{(s-t_1)}{(t_2-t_1)} + A\frac{(t_2-s)}{(t_2-t_1)}\Big) \\
&+ B^2\frac{(s-t_1)}{(t_2-t_1)} + A^2\frac{(t_2-s)}{(t_2-t_1)} - \frac{(B-A)^2(t_2-s)(s-t_1)}{(t_2-t_1)^2}\Big]\Big\} \\
&= K\exp\Big\{-\frac{(t_2-t_1)}{2(t_2-s)(s-t_1)}\Big[x - B\frac{(s-t_1)}{(t_2-t_1)} - A\frac{(t_2-s)}{(t_2-t_1)}\Big]^2\Big\} \\
&= K\exp\left\{-\frac{\Big[x - B\frac{(s-t_1)}{(t_2-t_1)} - A\frac{(t_2-s)}{(t_2-t_1)}\Big]^2}{2\frac{(t_2-s)(s-t_1)}{(t_2-t_1)}}\right\}
\end{aligned}
$$

Which is the Normal distribution with mean and variance:

$$E[B(s)|B(t_1) = A, B(t_2) = B] = A\frac{(t_2-s)}{(t_2-t_1)} + B\frac{(s-t_1)}{(t_2-t_1)}$$

$$\text{Var}[B(s)|B(t_1) = A, B(t_2) = B] = \frac{(t_2-s)(s-t_1)}{(t_2-t_1)}$$

.

$\square$

Exercise 10.3 Compute $E[B(t_1)B(t_2)B(t_3)]$ for $t_1 < t_2 < t_3$.

$$\begin{aligned}
E[B(t_1)B(t_2)B(t_3)] &= E[B(t_1)B(t_2)\ (B(t_3)+B(t_2)-B(t_2))\ ] \\
&= E[B(t_1)B(t_2)\ (B(t_3)-B(t_2))\ ] + E[B(1)B(t_2)^2] \\
\text{by independent increments} &= E[B(t_1)B(t_2)]E[(B(t_3)-B(t_2))] + E[B(1)B(t_2)^2] \\
&= \text{Notice that}\ \ B(t_3)-B(t_2) \sim \mathcal{N}(0, t_3-t_2) \\
&= E[B(t_1)\ (\ B(t_2)-B(t_1)+B(t_1)\ )^2] \\
&= E[B(t_1)\ (\ B(t_2)-B(t_1))^2 + B(t_1)^2 - 2B(t_1)(B(t_2)-B(t_1))\ )^2] \\
&= E[B(t_1)\ (\ B(t_2)-B(t_1))^2] + E[B(t_1)^3] - 2E[B(t_1)^2(B(t_2)-B(t_1))] \\
&= E[B(t_1)]E[(\ B(t_2)-B(t_1))^2] + E[B(t_1)^3] - 2E[B(t_1)^2]E[(B(t_2)-B(t_1))] \\
&= E[B(t_1)^3] = 0
\end{aligned} \tag{81}$$

Where we have used that the odd moments of the normal distribution are 0. □

Exercise 10.4 Show that

- $P\{T_a < \infty\} = 1$,
- $E[T_a] = \infty, \quad a \neq 0$

Let T_a denote the first time the Brownian motion process hits a. When $a > 0$ we will compute $P\{T_a \leq t\}$ by considering $P\{X(t) \geq a\}$ and conditioning on whether or not $T_a \leq t$. This gives:

$$\begin{aligned}
P\{X(t) \geq a\} &= P\{X(t) \geq a | T_a \leq t\}P\{T_a \leq t\} + P\{X(t) \geq a | T_a > t\}P\{T_a > t\} \\
&= \frac{P\{T_a \leq t\}}{2}
\end{aligned} \tag{82}$$

Now if $Ta \leq t$, then the process hits a at some point in $[0, t]$ and, by symmetry, it is just as likely to be above a or below a at time t. That is, $P\{X(t) \geq a | Ta \leq t\} = 1/2$ while $P\{X(t) \geq a | T_a > t\}P\{T_a > t\} = 0$ since, by continuity, the process value cannot be greater than a without having yet hit a. Considering the standard process we get

$$\begin{aligned}
P\{T_a \leq t\} &= 2P\{B(t) \geq a\} \\
&= 2P\{\mathcal{N}(0, t) \geq a\} \\
&= 2P\{\mathcal{N}(0, 1) \geq a/\sqrt{t}\}
\end{aligned} \tag{83}$$

$$
\begin{aligned}
\lim_{t\to\infty} P\{T_a < t\} &= 2\lim_{t\to\infty} P\{\mathcal{N}(0,1) \geq a/\sqrt{t}\} \\
&= 2P\{\mathcal{N}(0,1) \geq 0\} = 2\frac{1}{\sqrt{2\pi}}\int\limits_0^\infty e^{-t^2/2} \ \ dt = 1
\end{aligned} \tag{84}
$$

□

> **Exercise 10.6** Suppose you own one share of a stock whose price changes according to a standard Brownian motion process. Suppose that you purchased the stock at a price $b+c$, $c>0$, and the present price is b. You have decided to sell the stock either when it reaches the price $b+c$ or when an additional time t goes by (whichever occurs first). What is the probability that you do not recover your purchase price?

The stock purchase price will **not be recovered** if the stock price increases its price to $b+c$ only after time t.i.e $T_{b+c} > t$, where T_{b+c} denotes the first time the stock price hits $b+c$, given that the starting ($t=0$) value was b.

$$
\begin{aligned}
P(T_{c+b} > t | B(0) = b) &= P(T_c > t|B(0) = 0) = P(T_c > t) \\
&= 1 - P(T_c \leq t) \\
&= 1 - P\Big\{\max_{0\leq s\leq t} B(s) \geq c\Big\} \ \text{Reflection Principle} \\
&= 1 - 2P(\mathcal{N}(0,1) \geq c/\sqrt{t}) = 1 - 2\frac{1}{\sqrt{2\pi}}\int\limits_{c/\sqrt{t}}^\infty e^{-y^2/2} \ \ dy \\
&= 1 - 2(1 - \Phi(c/\sqrt{t})) = 2\Phi(c/\sqrt{t}) - 1
\end{aligned} \tag{85}
$$

□

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> **Exercise 10.7** Compute an expression for
>
> $$P\Bigg\{\max_{t_1\leq s\leq t_2} B_s > x\Bigg\}$$

We want to find the probability that the maximum of a Brownian motion in an interval $[t_1, t_2]$ is $> x$. We will assume that both $t_1, t_2 > 0$. Notice that $\max_{t_1 \leq s \leq t_2} B_s$ refers to the first time the process reaches a value greater than x in the time interval $[t_1, t_2]$.

We will solve this probability by conditioning of where the position of the Brownian motion was at time t_1. We will split the integral and consider whether the position of t_1 (which we will call y) is larger than x or not.

If it is larger than x then it is clear that with probability 1 the maximum of the Brownian motions will be greater than x.

If it is smaller than x then we need the increment in the interval $[t_1, t_2]$ is at the maximum of $x - y$. The question is whether after time t_1 we go up at least $x - y$ Also recall that $B(t_1) \sim \mathcal{N}(0, t_1)$

$$
\begin{aligned}
P\left\{\max_{t_1 \leq s \leq t_2} B_s > x\right\} &= \int_{-\infty}^{+\infty} P\left\{\max_{t_1 \leq s \leq t_2} B_s > x | B(t_1) = y\right\} f_{B(t_1)}(y) \;\; dy \\
&= \int_{-\infty}^{x} P\left\{\max_{t_1 \leq s \leq t_2} B_s > x | B(t_1) = y\right\} f_{B(t_1)}(y) \;\; dy + \int_{x}^{+\infty} 1 \cdot f_{B(t_1)}(y) \;\; dy \\
&= \int_{-\infty}^{x} P\left\{\max_{0 \leq s \leq t_2 - t_1} B_s > x - y\right\} \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2t_1} \;\; dy + \int_{x}^{+\infty} \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2t_1} \;\; dy
\end{aligned} \tag{86}
$$

And so, the first integral becomes:

$$
\begin{aligned}
\int_{-\infty}^{x} P\left\{\max_{0 \leq s \leq t_2 - t_1} B_s > x - y\right\} \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2t_1} \;\; dy &= \int_{-\infty}^{x} P(T_{x-y} < t_2 - t_1) \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2t_1} \;\; dy \\
&= \int_{-\infty}^{x} 2P(X(t_2 - t_1) > x - y) \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2t_1} \;\; dy \\
&= \int_{-\infty}^{x} \frac{2}{\sqrt{2\pi}} \int_{(x-y)/\sqrt{(t_2 - t_1)}}^{\infty} e^{-z^2/2} \;\; dz \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2t_1} \;\; dy \\
&= \frac{1}{\pi\sqrt{t_1}} \int_{-\infty}^{x} \int_{(x-y)/\sqrt{t_2 - t_1}}^{\infty} e^{-z^2/2} e^{-y^2/2t_1} \;\; dz \;\; dy
\end{aligned} \tag{87}
$$

$\square$

Exercise 10.9 Let $\{X(t), t \geq 0\}$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2. What is the joint density function of $X(s)$ and $X(t)$, $s < t$?

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and define

$$X(t) = \sigma B(t) + \mu t \iff B(t) = \frac{X(t) - \mu t}{\sigma}$$

$$X(s) = \sigma B(s) + \mu s \iff B(s) = \frac{X(s) - \mu s}{\sigma}$$

The joint density function of $X(s)$ and $X(t)$ is given by:

$$\begin{aligned}
f_{X(s),X(t)}(x,y) &= f_{\sigma B(s)+\mu s, \sigma B(t)+\mu t}(x,y) \\
&= \frac{1}{\sigma^2} f_{B(s),B(t)}\Big(\frac{x-\mu s}{\sigma}, \frac{y-\mu t}{\sigma}\Big) \\
&= \frac{1}{\sigma^2} f_{B(s),B(t)-B(s)}\Big(\frac{x-\mu s}{\sigma}, \frac{(y-x)-\mu(t-s)}{\sigma}\Big) \\
&= \frac{1}{\sqrt{2\pi s\sigma^2}} \frac{1}{\sqrt{2\pi(t-s)\sigma^2}} \exp\Big\{-\frac{1}{2}\Big[\frac{(x-\mu s)^2}{s\sigma^2} + \frac{((y-\mu(t-s))-x)^2}{(t-s)\sigma^2}\Big]\Big\} \\
&= K_1 \exp\Big\{-\frac{1}{2\sigma^2}\Big[\frac{x^2 t}{s(t-s)} + \frac{(\mu s)^2 - 2x\mu s}{s} + \frac{(y-\mu(t-s))^2 - 2x(y-\mu(t-s))}{(t-s)}\Big]\Big\} \\
&= K_2 \exp\Big\{-\frac{t}{2\sigma^2 s(t-s)}\Big[x^2 - \frac{(t-s)}{t}2sx\mu - \frac{2sx}{t}(y-\mu(t-s))\Big]\Big\} \\
&= K_2 \exp\Big\{-\frac{t}{2\sigma^2 s(t-s)}\Big[x^2 - \frac{2sx(t-s)}{t}\Big(\mu + \frac{y}{(t-s)} - \mu\Big)\Big]\Big\} \\
&= K_2 \exp\Big\{-\frac{t}{2\sigma^2 s(t-s)}\Big[x^2 - 2x\frac{sy}{t} - \Big(\frac{sy}{t}\Big)^2 + \Big(\frac{sy}{t}\Big)^2\Big]\Big\} \\
&= K_3 \exp\Big\{-\frac{t}{2\sigma^2 s(t-s)}\Big[\Big(x - \frac{sy}{t}\Big)^2\Big]\Big\} = K_3 \exp\left\{\frac{-\Big(x-\frac{sy}{t}\Big)^2}{2\sigma^2\frac{s(t-s)}{t}}\right\}
\end{aligned} \tag{88}$$

Where

$$X(s) \sim \mathcal{N}(\mu s, s\sigma^2), \quad X(t) - X(s) \sim \mathcal{N}(\mu(t-s), (t-s)\sigma^2)$$

And

$$B(s) \sim \mathcal{N}(0, s), \quad B(t) - B(s) \sim \mathcal{N}(0, (t-s))$$

$\square$

Exercise 10.10 Let $\{X(t), t \geq 0\}$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2. What is the conditional distribution of $X(t)$ given that $X(s) = c$ when

- (a) $s < t$?
- (b) $t < s$?

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and define

$$X(t) = \sigma B(t) + \mu t \iff B(t) = \frac{X(t) - \mu t}{\sigma}$$

$$X(s) = \sigma B(s) + \mu s \iff B(s) = \frac{X(s) - \mu s}{\sigma}$$

We will derive the conditional distribution of $X(t)$ given that $X(s) = c$ for the following cases:

- (a) $s < t$

$$\begin{aligned}
f_{X(t)|X(s)}(x|c) &= \frac{f_{X(t),X(s)}(x,c)}{f_{X(s)}(c)} \quad \text{by independent increments} \\
&= \frac{1}{\sigma}\frac{f_{B(s)}(\frac{c-\mu s}{\sigma}) f_{B(t)-B(s)}(\frac{x-c-\mu(t-s)}{\sigma})}{f_{B(s)}(\frac{c-\mu s}{\sigma})} \\
&= \frac{1}{\sigma} f_{B(t)-B(s)}\Big(\frac{x-c-\mu(t-s)}{\sigma}\Big) \\
&= \frac{1}{\sqrt{2\pi(t-s)\sigma^2}} \exp\Big\{\frac{-(x-c-\mu(t-s))^2}{2\sigma^2(t-s)}\Big\}
\end{aligned} \tag{89}$$

$$X(t)|(X(s) = c) = B(t) - B(s) \sim \mathcal{N}(c + \mu(t-s), \sigma^2(t-s))$$

- (b) $t < s$

$$\frac{f_{X(t),X(s)}(x,c)}{f_{X(s)}(c)} = \frac{1}{\sigma}\frac{f_{B(t)}(\frac{x-\mu t}{\sigma}) f_{B(s)-B(t)}(\frac{c-x-\mu(s-t)}{\sigma})}{f_{B(s)}(\frac{c-\mu s}{\sigma})}$$

Knowing that

$$X(t) \sim \mathcal{N}(\mu t, \sigma^2 t), \quad X(s) \sim \mathcal{N}(\mu s, \sigma^2 s), \quad X(s) - x(t) \sim (\mu(s-t), \sigma^2(s-t))$$

and

$$B(t) \sim \mathcal{N}(0, t), \quad B(s) \sim \mathcal{N}(0, s), \quad B(s) - B(t) \sim (0, (s-t))$$

$$
\begin{aligned}
f_{t|s}(x|c) &= K_1 \exp\left\{\frac{-1}{2\sigma^2}\left[\frac{(x-\mu t)^2}{t} + \frac{((c-x)-\mu(s-t))^2}{(s-t)}\right]\right\} \\
&= K_1 \exp\left\{\frac{-1}{2\sigma^2}\left[\frac{(x^2+(\mu t)^2-2x(\mu t)}{t} + \frac{((c-\mu(s-t))-x)^2}{(s-t)}\right]\right\} \\
&= K_1 \exp\left\{\frac{-1}{2\sigma^2}\left[\frac{x^2+(\mu t)^2-2x(\mu t)}{t} + \frac{x^2}{(s-t)} + \frac{(c-\mu(s-t))^2}{(s-t)} - \frac{2x(c-\mu(s-t))}{(s-t)}\right]\right\} \\
&= K_1 \exp\left\{\frac{-1}{2\sigma^2}\left[\frac{x^2 s}{t(s-t)} + \frac{(\mu t)^2}{t} + \frac{(c-\mu(s-t))^2}{(s-t)} - \frac{2x(\mu t)}{t} - \frac{2x(c-\mu(s-t))}{(s-t)}\right]\right\} \\
&= K_2 \exp\left\{\frac{-1}{2\sigma^2}\left[\frac{x^2 s}{t(s-t)} - \frac{2x(\mu t)}{t} - \frac{2x(c-\mu(s-t))}{(s-t)}\right]\right\} \\
&= K_2 \exp\left\{\frac{-1}{2\sigma^2}\left[\frac{x^2 s}{t(s-t)} - \frac{(s-t)}{t(s-t)}2x(\mu t) - \frac{t}{t(s-t)}2x(c-\mu(s-t))\right]\right\} \\
&= K_2 \exp\left\{\frac{-1}{2\sigma^2}\frac{s}{t(s-t)}\left[x^2 - \frac{(s-t)}{s}2x(\mu t) - \frac{t}{s}2x(c-\mu(s-t))\right]\right\} \\
&= K_2 \exp\left\{\frac{-1}{2\sigma^2}\frac{s}{t(s-t)}\left[x^2 - \frac{2tx}{s}\Big(\mu(s-t)+c-\mu(s-t)\Big)\right]\right\} \\
&= K_2 \exp\left\{\frac{-1}{2\sigma^2}\frac{s}{t(s-t)}\left[x^2 - \frac{2tx(s-t)}{s}\left(\mu + \frac{c}{s-t} - \mu\right)\right]\right\} \\
&= K_2 \exp\left\{\frac{-1}{2\sigma^2}\frac{s}{t(s-t)}\left[x^2 - 2x\frac{tc}{s} + \left(\frac{tc}{s}\right)^2 - \left(\frac{tc}{s}\right)^2\right]\right\} \\
&= K_3 \exp\left\{\frac{-1}{2\sigma^2}\frac{s}{t(s-t)}\left[\left(x-\frac{tc}{s}\right)^2\right]\right\} = K_3 \exp\left\{\frac{-\left(x-\frac{tc}{s}\right)^2}{\frac{2\sigma^2 t(s-t)}{s}}\right\}
\end{aligned}
$$

Where K_1, K_2, K_3 denote all the factors that do not depend on x. We can now verify that whenever $t < s$ the conditional density of $X(t)$ given $X(s)$ follows a normal distribution,

$$X(t)|X(s) = c \sim \mathcal{N}\left(\frac{tc}{s}, \frac{\sigma^2 t(s-t)}{s}\right)$$

Notice that $\frac{tc}{s} = \frac{(c-\mu s)t}{s} + \mu t$

$\square$

Exercise 10.32 Let $\{Z(t), t \geq 0\}$ denote a Brownian bridge process. Show that if

$$Y(t) = (t+1)Z\Big(\frac{t}{t+1}\Big)$$

then $\{Y(t), t \geq 0\}$ is a standard Brownian motion process.

First, notice that the Brownian bridge process $\{Z(t), t \geq 0\}$ is a Gaussian Process, meaning that the process $\{Y(t), t \geq 0\}$ is also a Gaussian Process. Hence, in order to show that the process is a Standard Brownian motion we will only need to verify that (i) $E[Y(t)] = 0$ and (ii) $\mathrm{Cov}(Y(s), Y(t)) = s$.

The Brownian bridge can be defined as a Gaussian process with mean value 0 and covariance function $s(1-t)$, $s \leq t$. Then $\{Z(t), t \geq 0\}$ is a Brownian bridge process whenever

$$Z(t) = X(t) - tX(1)$$

. Hence,

$$\begin{aligned}
E[Y(t)] &= (t+1)E\Big[Z\Big(\frac{t}{t+1}\Big)\Big] \\
&= (t+1)E\Big[X\Big(\frac{t}{t+1}\Big) - \Big(\frac{t}{t+1}\Big)X(1)\Big] \\
&= E\Big[X\Big(\frac{t}{t+1}\Big) - t\Big(X(1) - X\Big(\frac{t}{t+1}\Big)\Big)\Big] \quad \text{by independent increments} \\
&= E\Big[X\Big(\frac{t}{t+1}\Big)\Big] - tE\Big[X(1) - X\Big(\frac{t}{t+1}\Big)\Big] = 0
\end{aligned} \tag{90}$$

Knowing that the expected value of a standard Brownian motion is 0. Also, notice that $t/(t+1)$ is an increasing function, as $t \to \infty$ $t/(t+1) = 1$, meaning that $t/(t+1) \leq 1$.

$$\begin{aligned}
\mathrm{Cov}(Y(s), Y(t)) &= \mathrm{Cov}\left((s+1)Z\Big(\frac{s}{s+1}\Big), (t+1)Z\Big(\frac{t}{t+1}\Big)\right) \\
&= (s+1)(t+1)\,\mathrm{Cov}\left(Z\Big(\frac{s}{s+1}\Big), Z\Big(\frac{t}{t+1}\Big)\right) \\
&= (s+1)(t+1)\frac{s}{s+1}\Big(1 - \frac{t}{t+1}\Big) = s(t+1-t) = s
\end{aligned} \tag{91}$$

□

Exercise 10.11 Consider a process whose value changes every h time units; its new value being its old value multiplied either by the factor $e^{\sigma\sqrt{h}}$ with probability $p = \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{h})$, or by the factor $e^{-\sigma\sqrt{h}}$ with probability $1-p$. As h goes to zero, show that this process converges to geometric Brownian motion with drift coefficient μ and variance parameter σ^2.

$\square$

Exercise 10.33 Let $X(t) = N(t+1) - N(t)$ where $\{N(t), t \geq 0\}$ is a Poisson process with rate λ. Compute

$$\text{Cov}[X(t), X(t+s)]$$

We have that $X(t)$ represents the number of events between t and $t+1$. The stochastic process $\{X(t), t \geq 0\}$ is stationary, following from the stationary and independent increments assumption of the Poisson Process.

For computing the covariance between $X(t)$ and $X(t+s)$ in order to make use of the bilinearity of covariance property, we will add and subtract $N(t+s)$ and $N(t+1)$ in the first and second term respectively. For this to hold we will assume that $s \in [0,1]$ and so $N(t+1) \geq N(t+s)$.

$$\begin{aligned}
\text{Cov}[X(t), X(t+s)] &= \text{Cov}[N(t+1) - N(t), N(t+s+1) - N(t+s)] \\
&= \text{Cov}[(N(t+1) - N(t+s)) + (N(t+s) - N(t)) \\
&\quad , (N(t+s+1) - N(t+1)) + (N(t+1) - N(t+s))] \\
\\
&= \text{Cov}[(N(t+1) - N(t+s)), (N(t+s+1) - N(t+1))] \\
&+ \text{Cov}[(N(t+1) - N(t+s)), (N(t+1) - N(t+s))] \\
&+ \text{Cov}[(N(t+s) - N(t)), (N(t+s+1) - N(t+1))] \\
&+ \text{Cov}[(N(t+s) - N(t)), (N(t+1) - N(t+s))] \\
&= 0 + \text{Var}(N(t+1) - N(t+s)) + 0 + 0 \\
&= \lambda(t+1-t-s) = \lambda(1-s)
\end{aligned} \tag{92}$$

By **Theorem 5.1** we know that for all $s > 0,\ t > 0$, $N(s+t) - N(s)$ is a Poisson random variable with mean $\lambda(s+t-s) = \lambda t$. That is, the number of events in any interval of length t is a Poisson random variable with mean λt.

$\square$

Exercise 10.34 Let $\{N(t), t \geq 0\}$ denote a Poisson process with rate λ and define $Y(t)$ to be the time from t until the next Poisson event.

- (a) Argue that $\{Y(t), t \geq 0\}$ is a stationary process.
- (b) Compute $\text{Cov}[Y(t), Y(t+s)]$.

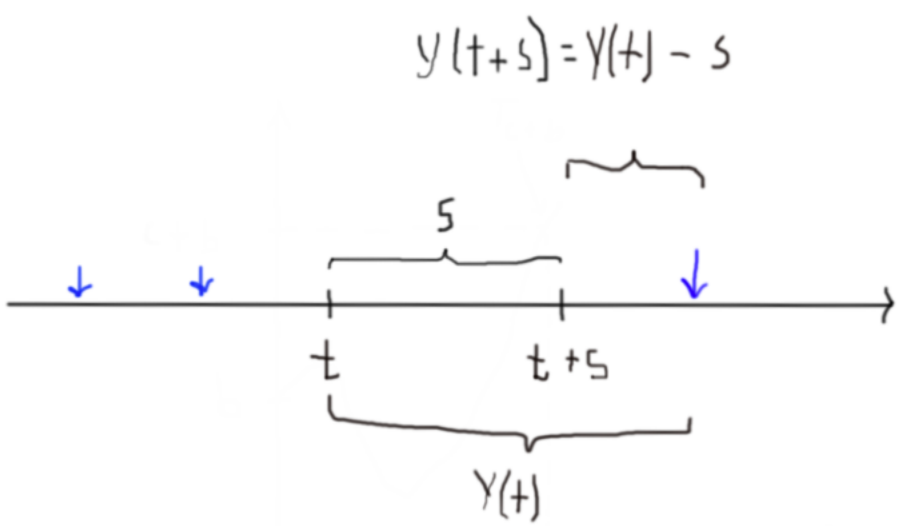
- (a)

$\{N(t), t \geq 0\}$ is an homogeneous Poisson process distributed as $\{N(t+s)-N(s); t \geq 0\}$ with rate λ and we know that $Y(t)$ is the time t until the next arrival i.e $Y(t) = \min\{u \geq 0; N(t+u)-N(t) \geq 1\}$. We know that a Poisson Process has independent and exponentially distributed arrival times, hence, memoryless. This will mean that the time until the next arrival is also exponentially distributed and it does not depend on the past or current time, i.e the continuation of a Poisson process, remains a Poisson process with rate λ. Meaning that $\{Y(t), t \geq 0\} = \{\min\{u \geq 0; N(t+u) - N(t) \geq 1\}, t \geq 0\}$ is a stationary process distributed as $\{Y(t+s); t \geq 0\}$ for all $s > 0$.

- (b)

$$\text{Cov}[Y(t), Y(t+s)] = E[Y(t)Y(t+s)] - E[Y(t)]E[Y(t+s)]$$

Where $E[Y(t)] = E[Y(t+s)] = \frac{1}{\lambda}$.



Let the blue arrows denote arrivals. Notice that $Y(t)$ and $Y(t+s)$ are very much dependent if $Y(t) > s$ as then the next arrival is after time $t+s$ meaning that $Y(t+s) = Y(t) - s$. However if $Y(t) < s$ then $Y(t+s)$ is exponentially distributed with parameter λ and independent of $Y(t)$.

Furthermore $E[Y(t)Y(t+s)]$ can be obtain applying the law of total expectations:

$$
\begin{aligned}
E[Y(t)Y(t+s)] &= \int_0^\infty E[Y(t),Y(t+s)|Y(t)=y]P(Y(t)=y) \;\; dy \\
&= \int_0^\infty E[yY(t+s)|Y(t)=y]\lambda e^{-\lambda y} \;\; dy \\
&= \int_0^s E[yY(t+s)|Y(t)=y]\lambda e^{-\lambda y} \;\; dy + \int_s^\infty E[yY(t+s)|Y(t)=y]\lambda e^{-\lambda y} \;\; dy \\
&= \int_0^s yE[Y(t+s)]\lambda e^{-\lambda y} \;\; dy + \int_s^\infty E[y(Y(t)-s)|Y(t)=y]\lambda e^{-\lambda y} \;\; dy \\
&= \int_0^s ye^{-\lambda y} \;\; dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y} \;\; dy
\end{aligned}
\tag{93}
$$

□

Exercise 10.35 Let $\{X(t), -\infty < t < \infty\}$ be a weakly stationary process having covariance function $R_X(s) = \mathrm{Cov}[X(t), X(t+s)]$.

- (a) Show that
$$Var(X(t+s) - X(t)) = 2R_X(0) - 2R_X(t)$$
- (b) If $Y(t) = X(t+1) - X(t)$ show that $\{Y(t), -\infty < t < \infty\}$ is also weakly stationary having a covariance function $R_Y(s) = \mathrm{Cov}[Y(t), Y(t+s)]$ that satisfies
$$R_Y(s) = 2R_X(s) - R_X(s-1) - R_X(s+1)$$

We know that a process $\{X(t), -\infty < t < \infty\}$ is weakly stationary if (i) the first two moments of $X(t)$ are the same for all t and (ii) the covariance between $X(s)$ and $X(t)$ depends only on $|t-s|$.

- (a)

$$
\begin{aligned}
\mathrm{Var}(X(t+s)-X(t)) &= E[(X(t+s)-X(t))^2] - (E[X(t+s)-X(t)])^2 \\
&= E[X(t+s)^2 + X(t)^2 - 2X(t+s)X(t)] - (E[X(t+s)]^2 \\
&+ E[X(t)]^2 - 2E[X(t+s)][X(t)]) \\
&= E[X(t+s)^2] + E[X(t)^2] - 2E[X(t+s)X(t)] \\
&- E[X(t+s)]^2 - E[X(t)]^2 + 2E[X(t+s)]E[X(t)] \\
&= \mathrm{Var}(X(t+s)) + \mathrm{Var}(X(t)) - 2\,\mathrm{Cov}(X(t+s), X(t)) \\
&= 2\,\mathrm{Var}(X(t+s)) - 2R_X(s) = 2\,\mathrm{Cov}(X(t+0), X(t)) - 2R_X(s) \\
&= 2R_X(0) - 2R_X(s)
\end{aligned}
\tag{94}
$$

Remember that by linearity $\mathrm{Var}(X(t)) = \mathrm{Var}(-X(t))$

- (b) For showing that the process $\{Y(t), -\infty < t < \infty\}$ is also weakly stationary we will prove:
  - (i)
  $$E[Y(t)] = E[X(t+1) - X(t)] = E[X(t+1)] - E[X(t)] = 0$$
  As $X(\cdot)$ is stationary, while
  - (ii)
  $$
  \begin{aligned}
  \mathrm{Cov}(Y(t), Y(t+s)) &= \mathrm{Cov}(X(t+1) - X(t), X(t+s+1) - X(t+s)) \\
  &= \mathrm{Cov}(X(t+1), X(t+s+1)) - \mathrm{Cov}(X(t+1), X(t+s)) \\
  &- \mathrm{Cov}(X(t+s+1), X(t)) + \mathrm{Cov}(X(t+s), X(t)) \\
  &= R_X(s) - R_X(s-1) - R_X(s+1) + R_X(s) \\
  &= 2R_X(s) - R_X(s-1) - R_X(s+1)
  \end{aligned}
  \tag{95}
  $$

□