Sample Paper-04 (Solved)
Mathematics
Class – XII

Time allowed: 3 hours | **ANSWERS** | Maximum Marks: 100

Section A

1. Solution:

$$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$$
$$\Rightarrow 2x^2 - 24 = -18$$
$$\Rightarrow x = \pm\sqrt{3}$$

2. Solution:

$$(A - A')' = A' - (A')' = A' - A = -(A - A')$$
Thus, A-A' is skew symmetric.

3. Solution:

$$\begin{vmatrix} -2 & -1 \\ 4 & 2 \end{vmatrix} = -4 - (-4) = 0$$

4. Solution:

Let A={1,2,3,4}

Let R={(1,2),(2,1)}

5. Solution:

x-axis makes angles, 0,90,90 with the x,y,z axis respectively.

Thus, direction cosines are cos0, cos90, cos 90, i.e 1,0,0.

6. Solution:

[-1,1]

Section B

7. Solution:

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = \begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ 2x+2y+2z & x & z+x+2y \end{vmatrix} (C_1 \to C_1 + C_2 + C_3)$$

$$= (2x+2y+2z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix} (R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$$

$$= 2(x+y+z)^3$$

8. Solution:

$$When\ t = \frac{\pi}{4}, x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}$$

$$\therefore point\ of\ contact \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$y = \sin t \Rightarrow \frac{dy}{dt} = \cos t, x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$$

$$\therefore slope\ at\ t = \frac{\pi}{4} = -\cot\frac{\pi}{4} = -1$$

$$\therefore equation\ of\ \tan gent = (y - \frac{1}{\sqrt{2}}) = -1(x - \frac{1}{\sqrt{2}}) \Rightarrow x + y - \sqrt{2} = 0$$

$$Slope\ of\ normal = m, m(-1) = -1 \Rightarrow m = 1$$

$$\therefore equation\ of\ normal = (y - \frac{1}{\sqrt{2}}) = 1(x - \frac{1}{\sqrt{2}}) \Rightarrow x - y = 0$$

9. Solution:

$$f \circ g(x) = f(g(x)) = f(x^2 - 1) = \sqrt{x^2 - 1}$$

$$g \circ f(x) = g(f(x)) = g(\sqrt{x}) = \left(\sqrt{x}\right)^2 - 1 = x - 1$$

$$\therefore f \circ g \neq g \circ f$$

10. Solution:

$$y=(x\log x)^{\log(\log x)} \Rightarrow \log y=\log(\log x)\log(x\log x)=\log(\log x)[\log x+\log(\log x)]$$
$$\frac{1}{y}\frac{dy}{dx}=\left(\frac{1}{\log x}\right)\frac{1}{x}[\log x+\log(\log x)]+\log(\log x)\left[\frac{1}{x}+\left(\frac{1}{\log x}\right)\frac{1}{x}\right]$$
$$=\left(\frac{1}{x\log x}\right)[\log x+\log(\log x)]+\log(\log x)\left[\frac{1}{x}+\left(\frac{1}{x\log x}\right)\right]$$
$$\therefore \frac{dy}{dx}=(x\log x)^{\log(\log x)}\left\{\left(\frac{1}{x\log x}\right)[\log x+\log(\log x)]+\log(\log x)\left[\frac{1}{x}+\left(\frac{1}{x\log x}\right)\right]\right\}$$

Differentiating both sides w.r.t x, we get

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=2x+2y\frac{dy}{dx}$$
$$\frac{dy}{dx}\left(\frac{1}{y}-2y\right)=2x-\frac{1}{x}$$
$$\therefore \frac{dy}{dx}=\frac{y(2x^2-1)}{x(1-2y^2)}$$

11. Solution:

$$\tan^{-1}2x+\tan^{-1}3x=\tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right)=\tan^{-1}\left(\frac{5x}{1-6x^2}\right)=\frac{\pi}{4}$$
$$\Rightarrow \frac{5x}{1-6x^2}=\tan\frac{\pi}{4}=1$$
$$\Rightarrow 6x^2+5x-1=0 \Rightarrow (6x-1)(x+1)=0$$
$$\Rightarrow x=1/6,-1$$
$$If\ \ x=-1, \tan^{-1}-2+\tan^{-1}-3<0, but\ \frac{\pi}{4}>0$$
$$Hence, x\neq -1.$$
$$\therefore x=1/6$$

12. Solution:

The position vectors of A, B,C are 2i, j, 2k respectively.

$\therefore$ $\overrightarrow{AB}$ = p.v of $\vec{B}$ - p.v of $\vec{A}$ =j-2i

$\therefore$ $\overrightarrow{BC}$ = p.v of $\vec{C}$ - p.v of $\vec{B}$ =2k-j

$\therefore$ $\overrightarrow{CA}$ = p.v of $\vec{A}$ - p.v of $\vec{C}$ =2i-2k

$$|\overrightarrow{AB}|^2 = (2)^2 + (1)^2 = 5 \qquad |\overrightarrow{AB}| \quad \sqrt{5}$$
$$|\overrightarrow{BC}|^2 = (1)^2 + (2)^2 = 5 \qquad |\overrightarrow{BC}| \quad \sqrt{5}$$
$$|\overrightarrow{CA}|^2 = (2)^2 + (2)^2 = 8 \qquad |\overrightarrow{CA}| \quad \sqrt{8}$$
$$|\overrightarrow{AB}| = |\overrightarrow{BC}| \neq |\overrightarrow{CA}|$$

Thus, A, B,C form the vertices of an isosceles triangle.

13. Solution:

Let A denote the event of getting a doublet.

Let B denote the event of getting a total of 10.

For A favorable cases are: {(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)}

Thus, P(A)=6/36

For B favorable cases are: {(4,6),(5,5),(6,4)}

Thus, P(B)=3/36

$$A \cap B = \{(5,5)\}$$
$$\therefore P(A \cap B) = 1/36$$
$$P(A^C \cap B^C) = P(A \cup B)^C = 1 - P(A \cup B) = 1 - \left[\frac{6}{36} + \frac{3}{36} - \frac{1}{36}\right] = \frac{7}{9}$$

14. Solution:

R(x)=x³-eˣ-1/x

$$M\arg inal\ \mathrm{Re}\, venue = R'(x) = 3x^2 - e^x + \frac{1}{x^2}$$

$$When\ x = 5, MR = 75 - e^5 + \frac{1}{25}$$

Precautions:

He should not give any drugs without a proper prescription.

He should not sell any medicines after expiry date.

15. Solution:

The circles in the system will have centres on the y-axis. Let (0,a) be the center of a circle touching the x-axis at the origin.

Thus, radius=$|a|$.

Equation of circles is:

$$(x-0)^2+(y-a)^2=|a|^2$$
$$x^2+y^2-2ay=0$$
Differentiating both sides w.r.t x, we get
$$2x+2y\frac{dy}{dx}-2a\frac{dy}{dx}=0$$
$$2x+2yy_1-2ay_1=0 \Rightarrow a=\frac{2x+2yy_1}{2y_1}=\frac{x+yy_1}{y_1}$$
$$\Rightarrow x^2+y^2-2\left(\frac{x+yy_1}{y_1}\right)y=0$$
$$\Rightarrow x^2y_1-y^2y_1-2xy=0$$

16. Solution:

$$\left|\vec{a}+\vec{b}\right|^2=\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right)$$
$$=\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}$$
$$=\left|\vec{a}\right|^2+2\vec{a}.\vec{b}+\left|\vec{b}\right|^2\le\left|\vec{a}\right|^2+2\left|\vec{a}.\vec{b}\right|+\left|\vec{b}\right|^2\le\left|\vec{a}\right|^2+2\left|\vec{a}\right|\left|\vec{b}\right|+\left|\vec{b}\right|^2$$
$$=\left(\left|\vec{a}\right|+\left|\vec{b}\right|\right)^2$$
$$\therefore\left|\vec{a}+\vec{b}\right|\le\left|\vec{a}\right|+\left|\vec{b}\right|$$

17. Solution:

$$I=\sqrt{\frac{a-x}{a+x}}dx=\sqrt{\frac{a-x}{a+x}\ \frac{a-x}{a-x}}\ dx=\frac{a-x}{\sqrt{a^2-x^2}}dx$$
$$=a\ \frac{dx}{\sqrt{a^2-x^2}}-\frac{xdx}{\sqrt{a^2-x^2}}=aI_1-I_2$$
$$I_1=\sin^{-1}\ \frac{x}{a}\ +C$$

$$I_2 = \frac{xdx}{\sqrt{a^2 - x^2}}, put\ a^2 - x^2 = z$$

$$\therefore -2xdx = dz$$

$$I_2 = -\frac{1}{2}\ \frac{dz}{\sqrt{z}} = -\sqrt{z} + c' = -\sqrt{a^2 - x^2} + c'$$

$$I = a\sin^{-1}\ \frac{x}{a}\ + \sqrt{a^2 - x^2} + k$$

18. Solution:

$$x_1 = 1, y_1 = 2, z_1 = 3; a_1 = 2, b_1 = 3, c_1 = 4$$
$$x_2 = 2, y_2 = 3, z_2 = 4; a_2 = 3, b_2 = 4, c_2 = 5$$
$$Then, \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

Thus, the lines are co-planar.

The equation of the plane containing the given lines is:

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$
$$i.e.\ x - 2y + z = 0$$

19. Solution:

Planes passing through the intersection of the given planes are:

$$\vec{r}.(2i + 6j) + 12 = 0 + \lambda[\vec{r}.(3i - j = 4k)] = 0$$
$$\because dis\tan ce\ \ from\ origin = 0$$
$$\therefore \frac{12}{\sqrt{(2+3\lambda)^2 + (6-\lambda)^2 + 16\lambda^2}} = 1$$
$$26\lambda^2 \quad 104$$
$$\lambda \quad 2$$

Substituting value of λ, we get the required equations of the plane as:

$$\vec{r}.(2i + j + 2k) + 3 = 0\ and\ \vec{r}.(i - 2j + 2k) - 3 = 0$$

Section C

20. Solution:

Let (x,y) be the point on the curve which is nearest to (0,5).

$$\therefore Distance = D = \sqrt{(x-0)^2+(y-5)^2}$$
$$\therefore E = D^2 = (x-0)^2+(y-5)^2$$
$$D \text{ minimum} \Rightarrow D^2 \text{ minimum}$$
$$E = 4y+(y-5)^2$$
$$\frac{dE}{dy} = 4+2(y-5)$$
$$\frac{dE}{dy} = 0 \Rightarrow y = 3$$
$$\frac{d^2E}{dy^2} = 2 > 0$$
$$y = 3 \Rightarrow x = \pm 2\sqrt{3}$$

Hence, for y=3 distance is minimum. The points are $\left(\pm 2\sqrt{3}, 3\right)$

21. Solution:

Let x be the cost of 1 kg onions, y be the cost of 1 kg wheat, z be the cost of 1 kg rice.

Thus we get the following equations:

4x+3y+2z=60

2x+4y+6z=90

6x+2y+2z=70

$$Let\ A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, b = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$|A| = 50 \neq 0, A^{-1} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}, X = A^{-1}b = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$$\therefore x = 5, y = 8, z = 8$$

Thus , per kg cost of onions, wheat and rice is Rs.5,Rs.8, Rs.8 respectively.

22. Solution:

Suppose x is the number of pieces of model A and y is the number of pieces of model B.

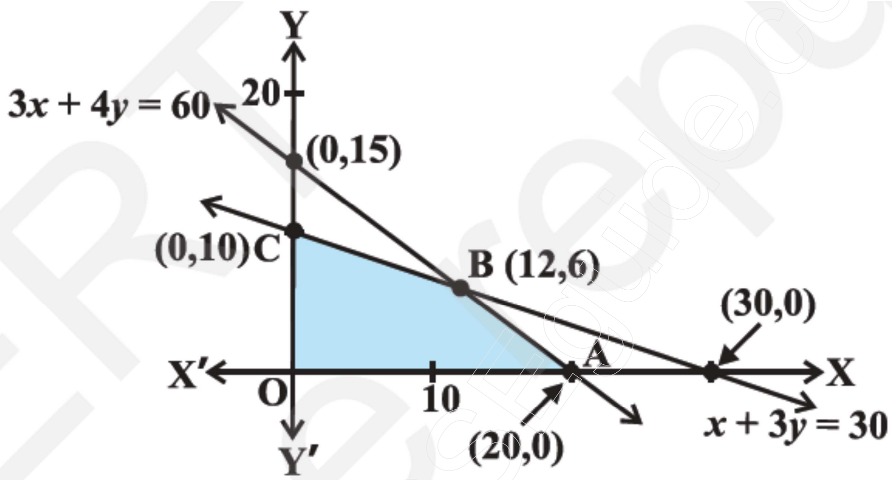
Then, Profit Z= 8000x + 12000y

The mathematical formulation of the problem is as follows:

Max Z= 8000x+12000y

s.t

$$\begin{aligned}&9x+12y\le 180(\,fabricating\ constraint)\\&3x+4y\le 60\\&x+3y\le 30(\,finishing\ constraint)\\&x\ge 0,\ y\ge 0\end{aligned}$$



We graph the above inequalities. The feasible region is as shown in the figure. The corner points are O,A,B and C. The co-ordinates of the corner points are (0,0),(20,0), (12,6),(0,10).

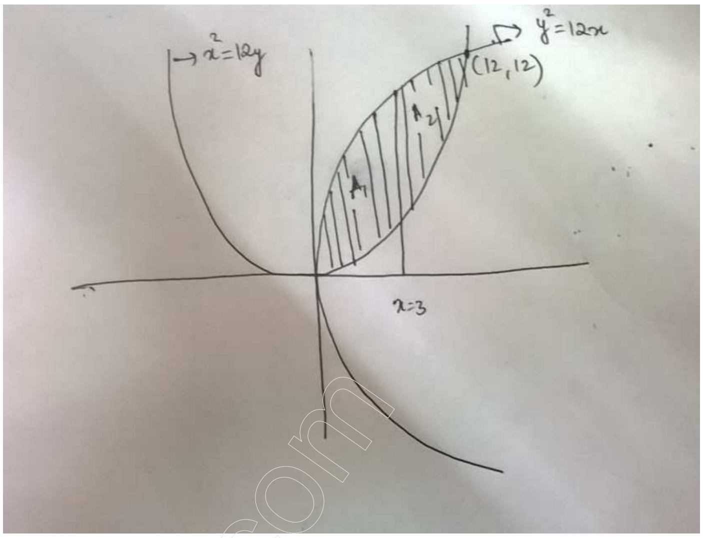
Corner Point	Z=8000x +12000y
(0,0)	0
(20,0)	16000
(12,6)	**16800**
(0,10)	12000

Thus profit is maximized by producing 12 units of A and 6 units of B and maximum profit is 16800.

23. Solution:

The point of intersection of the curves y^2=12x, x^2=12y:

$$y=\frac{x^2}{12}\Rightarrow y^2=\frac{x^4}{144}$$
$$\Rightarrow 12x=\frac{x^4}{144}$$
$$\Rightarrow x(x^3-1728)=0$$
$$\Rightarrow x=0,12$$



The shaded area is the required area.

$$A_1=\int_0^3 (y_2-y_1)\,dx$$
$$=\int_0^3\left(\sqrt{12x}-\frac{x^2}{12}\right)dx$$
$$=\sqrt{12}\left.\frac{x^{3/2}}{3/2}\right|_0^3-\left.\frac{x^3}{36}\right|_0^3=\frac{45}{4}$$
$$A_2=\int_3^{12}(y_2-y_1)\,dx$$
$$=\int_3^{12}\left(\sqrt{12x}-\frac{x^2}{12}\right)dx$$
$$=\sqrt{12}\left.\frac{x^{3/2}}{3/2}\right|_3^{12}-\left.\frac{x^3}{36}\right|_3^{12}=\frac{147}{4}$$

Thus, ratio of the areas is 45:147=15:49

24. Solution:

Let E be the event that the man reports 4.

Let S_1 denote the event that 4 occurs. Let S_2 denote the event that 4 does not occur.

$P(S_1/E)$=?

$P(E/S_1)$= Probability that he reports 4 when 4 has occurred= probability that he speaks truth=3/4

$P(E/S_2)$= Probability that he reports 4 when 4 has not occurred= probability that he does not speaks truth=1/4

Thus, by Baye's Theorem

$$P(S_1/E)=\frac{P(S_1)P(E/S_1)}{P(S_1)P(E/S_1)+P(S_2)P(E/S_2)}=\frac{(1/6)(3/4)}{(1/6)(3/4)+(5/6)(1/4)}=\frac{3}{8}$$

25. Solution:

Squaring both sides, $(1-x^2)y^2=(\sin^{-1}x)^2$

Differentiating both sides w.r.t x,

$$(1-x^2)2y\frac{dy}{dx}+y^2(-2x)=2\sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$$

$$2y(1-x^2)\frac{dy}{dx}-2xy^2=2y$$

$$y(1-x^2)\frac{dy}{dx}-xy^2=y$$

$$(1-x^2)\frac{dy}{dx}-xy=1$$

Differentiating both sides w.r.t x,

$$(1-x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}(-2x)-x\frac{dy}{dx}-y=0$$

Hence,

$$(1-x^2)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-y=0$$

26. Solution:

$$I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos 2x\log(\sin x)dx.$$

$$=\log(\sin x).\frac{\sin 2x}{2}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{\sin x}\cos x.\frac{\sin 2x}{2}dx$$

$$=0-\log\left(\frac{1}{\sqrt{2}}\right)\frac{1}{2}-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{\cancel{\sin x}}\cos x.\frac{\cancel{2}\,\cancel{\sin x}\cos x}{\cancel{2}}dx$$

$$=-\frac{1}{2}\log\left(\frac{1}{\sqrt{2}}\right)-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1+\cos 2x}{2}dx=-\frac{1}{2}\log\left(\frac{1}{\sqrt{2}}\right)-\frac{\pi}{8}+\frac{1}{4}$$