Sample Paper-02
Mathematics
Class – XII

Time allowed: 3 hours **ANSWERS** Maximum Marks: 100

Section A

1. Solution: $f(x)=x^2$, is neither one-one nor onto.
   f(3)=f(-3)=9, hence not one-one. Also f(x) does not assume any negative values, hence it is not onto.
2. Solution:
   $$|\vec{a}|=\sqrt{(3)^2+(-2)^2+(5)^2}=\sqrt{38}$$
   $$\therefore l=\frac{3}{\sqrt{38}},m=\frac{-2}{\sqrt{38}},n=\frac{5}{\sqrt{38}}$$
3. Solution: Yes, since every line is parallel to itself, thus the above relation is reflexive.
4. Solution: Let the order of Y be nxp.
   $\because$ XY is defined $\Rightarrow$ n=3, $\because$ YZ is defined $\Rightarrow$ p=5
   $\because$ Y has order 3x5.
5. Solution:
   $$Area=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}1 & 0 & 1\\ 6 & 0 & 1\\ 4 & 3 & 1\end{vmatrix}=\frac{15}{2}sq.units$$
6. Solution:
   $$\begin{vmatrix}3 & x\\ x & 1\end{vmatrix}=\begin{vmatrix}3 & 4\\ 2 & 1\end{vmatrix}\Rightarrow 3-x^2=3-8\Rightarrow x^2=8\Rightarrow x=\pm 2\sqrt{2}$$

Section B

7. Solution:
   $$Let\ x=\sin^{-1}\left(\frac{5}{13}\right),y=\cos^{-1}\left(\frac{3}{5}\right)$$
   $$\Rightarrow \sin x=\frac{5}{13},\cos x=\sqrt{1-\sin^2 x}=\frac{12}{13},\tan x=\frac{5}{12}$$
   $$\cos y=\frac{3}{5},\sin y=\sqrt{1-\cos^2 y}=\frac{4}{5},\tan y=\frac{4}{3}$$
   $$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{5/12+4/3}{1-(5/12)(4/3)}=\frac{63}{16}$$
   $$x+y=\tan^{-1}\left(\frac{63}{16}\right)$$
8. Solution:

$\vec{a} = 3i - 4j - 4k, \vec{b} = 2i - j + k, \vec{c} = i - 3j - 5k$

$\therefore \overrightarrow{AB} = (2i - j + k) - (3i - 4j - 4k) = -i - 3j - 5k$

$\overrightarrow{BC} = (i - 3j - 5k) - (2i - j + k) = -i - 2j - 6k$

$\overrightarrow{CA} = (3i - 4j - 4k) - (i - 3j - 5k) = 2i - j + k$

$|\overrightarrow{AB}|^2 = 35, |\overrightarrow{BC}|^2 = 41, |\overrightarrow{CA}|^2 = 6$

$\therefore |\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{BC}|^2$

Thus, A, B,C form the vertices of a right angled triangle.

9. Solution:

(A) Let A denote the event that problem is solved by A and let B denote the event that problem is solved by B.

$\therefore P(A) = 1/2, P(B) = 1/3, P(\overline{A}) = 1 - 1/2 = 1/2, P(\overline{B}) = 2/3$

P(Problem is solved)= 1- P(Problem is not solved)=1- $P(\overline{A}\overline{B}) = 1 - (1/2)(2/3) = 2/3$

(B) P(exactly one of them solves the problem)= $P(\overline{A}B or A\overline{B}) = (1/2)(2/3) + (1/2)(1/3) = 1/2$

10. Solution:

The function is defined for all points of the real line.

Case I: If c < -3, f(c)=c^3-c+1,

$$\lim_{x\to c}(f(x)) = \lim_{x\to c}(x^3 - x + 1) = c^3 - c + 1 = f(c)$$

$\therefore$ *f is continuous* $\forall\ x < -3$

Case II:If c > -3

$f(c) = 3c + 2$

$$\lim_{x\to c}(f(x)) = \lim_{x\to c}(3x + 2) = 3c + 2 = f(c)$$

$\therefore$ *f is continuous* $\forall\ x > 3$

Case II:If c = -3

$$\lim_{x\to -3^-}(f(x)) = \lim_{x\to -3^-}(x^3 - x + 1) = -23$$

$$\lim_{x\to -3^+}(f(x)) = \lim_{x\to -3^+}(-2x) = 6$$

Since, L.H.L ≠ R.H.L at x=-3, f(x) is not continuous at x=-3.
Similarly if c=3, L.H.L=-6, R.H.L=11. Thus f is not continuous at x=3

11. Solution:

$$x\frac{dy}{dx} - y + x\cos ec\left(\frac{y}{x}\right) = 0 \Rightarrow \frac{dy}{dx} - \frac{y}{x} + \cos ec\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \cos ec\left(\frac{y}{x}\right)$$

$$Let\ \frac{y}{x} = v \Rightarrow y = vx$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$
$$\therefore v+x\frac{dv}{dx}=v-\operatorname{cosec}(v)$$

$$\frac{-dv}{\operatorname{cosec} v}=\frac{dx}{x}$$
$$\int\frac{-dv}{\operatorname{cosec} v}=\int\frac{dx}{x}$$
$$\int-\sin v\,dv=\int\frac{dx}{x}$$
$$\therefore \cos v=\log x+c\Rightarrow\cos\left(\frac{y}{x}\right)=\log x+c$$
$$At\ x=1, y=0\ \therefore 1=0+c\Rightarrow c=1$$
$$\therefore \cos\left(\frac{y}{x}\right)=\log x+1$$

12. Solution:

$$\begin{vmatrix}1&1&1\\a&b&c\\a^3&b^3&c^3\end{vmatrix}=\begin{vmatrix}0&0&1\\a-b&b-c&c\\a^3-b^3&b^3-c^3&c^3\end{vmatrix}(C_1\to C_1-C_2, C_2\to C_2-C_3)$$
$$=(a-b)(b^3-c^3)-(b-c)(a^3-b^3)$$
$$=(a-b)(b-c)(b^2+bc+c^2)-(b-c)(a-b)(a^2+ab+b^2)$$
$$=(a-b)(b-c)(\cancel{b^2}+bc+c^2-a^2-ab-\cancel{b^2})$$
$$=(a-b)(b-c)(c-a)(a+b+c)$$

13. Solution:

$$A=\pi r^2$$
$$\therefore\frac{dA}{dt}=\left(\frac{dA}{dr}\right)\left(\frac{dr}{dt}\right)=2\pi r\frac{dr}{dt}$$
$$\frac{dr}{dt}=2cm/\sec$$
$$\therefore if\ r=14,\frac{dA}{dt}=2\pi(14)(2)=176cm^2/\sec$$

Harmful effects of poisoning water bodies:

Spread of epidemic, death of animals drinking water from the water source.

14. Solution:

Let z$\in$ C

$\because$ g is onto there exists b$\in$ B s.t g(b)=z.

Now, $\because$ b$\in$ B and f is onto there exists a$\in$ As.t.f(a)=b.

$\because g(b) = z \Rightarrow g(f(a)) = z \Rightarrow (g \circ f)(a) = z$

$\therefore g \circ f$ *is onto.*

15. Solution: $\because$ f(x) is a polynomial $\Rightarrow$ f(x) is continuous on [-4,2].

$\because$ f(x) is a polynomial $\Rightarrow$ f(x) is differentiable on]-4,2[.

$f(-4)=(-4)^2 + 2(-4) - 8 = 0$

$f(2)=4+4-8=0$

$\therefore f(-4)=f(2)$

$\therefore$ all the conditions of Rolle's theorem are satisfied.

So, $\exists\, c \in\,]-4,2[$ s.t. $f'(c)=0$.

$f'(x)=2x+2$, $f'(x)=0 \Rightarrow 2x+2=0 \Rightarrow x=-1$

$\therefore$ for $c=-1 \in\,]-4,2[$, $f'(c)=0$.

Thus, Rolle 's Theorem is verified.

16. Solution:

$$\left(\left|\vec{a}\right|\vec{b}+\left|\vec{b}\right|\vec{a}\right).\left(\left|\vec{a}\right|\vec{b}-\left|\vec{b}\right|\vec{a}\right)=\left|\vec{a}\right|\left|\vec{a}\right|\vec{b}.\vec{b}+\left|\vec{b}\right|\left|\vec{a}\right|\vec{a}.\vec{b}-\left|\vec{a}\right|\left|\vec{b}\right|\vec{b}.\vec{a}-\left|\vec{b}\right|\left|\vec{b}\right|\vec{a}.\vec{a}$$

$$=\cancel{\left|\vec{a}\right|^2\left|\vec{b}\right|^2}+\cancel{\left|\vec{b}\right|\left|\vec{a}\right|\vec{a}.\vec{b}}-\cancel{\left|\vec{a}\right|\left|\vec{b}\right|\vec{b}.\vec{a}}-\cancel{\left|\vec{b}\right|^2\left|\vec{a}\right|^2}=0(\because \vec{a}.\vec{b}=\vec{b}.\vec{a})$$

17. Solution:

$$\int \frac{dx}{x(x^4-1)} = \int \frac{x^3 dx}{x^4(x^4-1)}$$

$$Let\ x^4 = t \Rightarrow 4x^3 dx = dt \Rightarrow x^3 dx = dt/4$$

$$\therefore \frac{1}{4}\int \frac{dt}{t(t-1)} = \frac{1}{4}\left[\int\left(\frac{1}{t-1}-\frac{1}{t}\right)dt\right]$$

$$=\frac{1}{4}[\log(t-1)-\log t]+c$$

$$=\frac{1}{4}\log\left(\frac{x^4-1}{x^4}\right)+c$$

18. Solution:

Clearly, the above lines are parallel.

$$\therefore Dis\tan ce = \left|\frac{\vec{b}\times(\overrightarrow{a_2}-\overrightarrow{a_1})}{\left|\vec{b}\right|}\right|$$

$\vec{b} = 2i - 3j + k, \overrightarrow{a_1} = i + 3j - 2k, \overrightarrow{a_2} = 2i + 4j - k$

$\therefore \overrightarrow{a_2} - \overrightarrow{a_1} = i + j + k$

$$\therefore \vec{b} \times (\overrightarrow{a_2} - \overrightarrow{a_1}) = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = i(-3-1) - j(2-1) + k(2+3) = -4i - j + 5k$$

$$\therefore dis\tan ce = \frac{\left|\sqrt{16+1+25}\right|}{\left|\sqrt{4+9+1}\right|} = \frac{\sqrt{42}}{\sqrt{14}}$$

19. Solution:

$\overrightarrow{n_1} = 2i + 2j - 3k, d_1 = 7$

$\overrightarrow{n_2} = 2i + 5j + 3k, d_2 = 9$

Equation of plane :

$\vec{r}.(\overrightarrow{n_1} + \lambda \overrightarrow{n_2}) = d_1 + \lambda d_2$

$\vec{r}.(2i + 2j - 3k + \lambda(2i + 5j + 3k)) = 7 + 9\lambda$

$Let\ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore x(2+2\lambda) + y(2+5\lambda) + z(-3+3\lambda) = 7 + 9\lambda$

$Putting\ (x, y, z) = (2,1,3)\ we\ get\ \lambda = \frac{10}{9}$

$Substituting\ the\ value\ of\ \lambda\ we\ get, \vec{r}.(38i + 68j + 3k) = 153$

Section C

20. Solution:

Suppose tailor A works for x days and tailor B works for y days.

Then, Cost Z= 15x + 20y

	Tailor A	Tailor B	Min Requirement
Shirts	6	10	60
Pants	4	4	32
Cost per day	15	20	

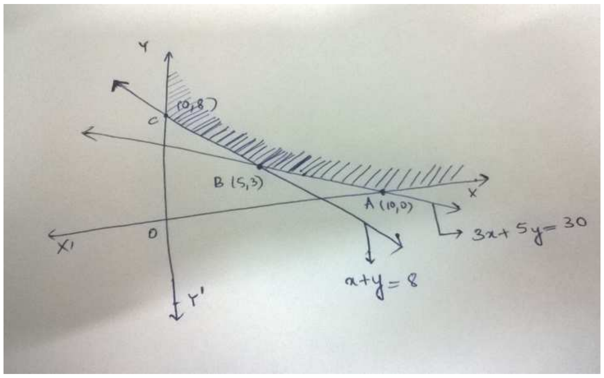
The mathematical formulation of the problem is as follows:

Min Z= 15x+20y

$6x+10y\geq 60 \Rightarrow 3x+5y\geq 30$

s.t $4x+4y\geq 32 \Rightarrow x+y\geq 8$

$x\geq 0, y\geq 0$



We graph the above inequalities. The feasible region is as shown in the figure. We observe the feasible region is unbounded and the corner points are A,Band C. The co-ordinates of the corner points are (10,0), (5,3),(0,8).

Corner Point	Z=15x +20y
(10,0)	150
(5,3)	**<u>135</u>**
(0,8)	160

Thus cost is minimized by hiring A for 5 days and hiring B for 3 days.

21. Solution:

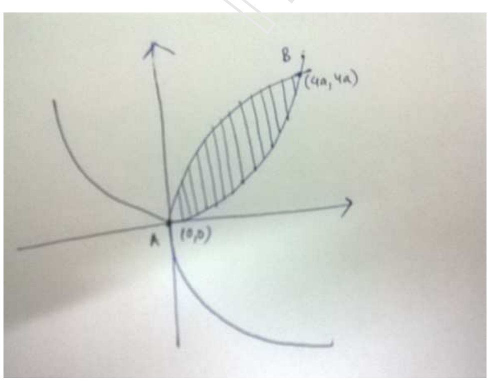
The point of intersection of the two curves:

$$x^2=\frac{y^4}{16a^2}\Rightarrow 4ay=\frac{y^4}{16a^2}$$

$$\Rightarrow y(y^3-64a^3)=0\Rightarrow y=0, y=4a$$

$$If\ \ y=0, x=0; y=4a\Rightarrow x=4a$$

$\therefore$ points of intersection are A(0,0) and B(4a,4a)



$$Area = \int_0^{4a} (y_2 - y_1)dx = \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a} \right) dx$$
$$= \sqrt{4a}\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\Bigg|_0^{4a} = \frac{16a^2}{3}$$

22. Solution:

Let E denote the event that the ball drawn is red.

Let E_1 denote the event that the ball is drawn from bag X, $P(E_1)=1/3$.

Let E_2 denote the event that the ball is drawn from bag Y, $P(E_2)=1/3$

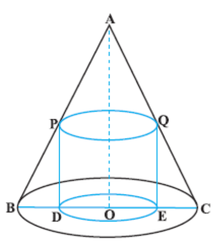
Let E_3 denote the event that the ball is drawn from bag Z, $P(E_3)=1/3$

$P(E/E_1)=3/5$,$P(E/E_2)=4/9$, $P(E/E_3)=3/5$, $P(E_2/E)=?$

By Baye's theorem,

$$P(E_2 / E) = \frac{P(E / E_2)P(E_2)}{P(E / E_1)P(E_1) + P(E / E_2)P(E_2) + P(E / E_3)P(E_3)}$$
$$= \frac{(4/9)(1/3)}{(3/5)(1/3) + (4/9)(1/3) + (3/5)(1/3)}$$
$$= \frac{10}{37}$$

23. Solution:



Let OC=r be the radius of the cone and OA=h be its height.

Let a cylinder with radius OE = x and height h' be inscribed in the cone.

Surface Area= $2\pi xh'$

$\because \triangle QEC \sim \triangle AOC,$

$$\frac{QE}{AO} = \frac{CE}{CO} \Rightarrow \frac{h'}{h} = \frac{r-x}{r} \Rightarrow h' = h\left(\frac{r-x}{r}\right)$$

$$\therefore S = S(x) = 2\pi xh' = 2\pi xh\left(\frac{r-x}{r}\right) = \frac{2\pi h}{r}\left(rx - x^2\right)$$

$$S'(x) = \frac{2\pi h}{r}\left(r - 2x\right)$$

$$S''(x)=\frac{2\pi h}{r}(-2)$$
$$S'(x)=0 \Rightarrow x=r/2$$
$$Also,\ S''(r/2)=\frac{-4\pi h}{r}<0$$

Hence, x=r/2 is a point of maxima.

Thus, the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

24. Solution:

Let $\frac{1}{x}=u, \frac{1}{y}=v, \frac{1}{z}=w$

$\therefore$ the system of equations becomes,

$$3u-2v+3w=8$$
$$2u+v-w=1$$
$$4u-3v+2w=4$$
$$Let\ A=\begin{bmatrix}3 & -2 & 3\\ 2 & 1 & -1\\ 4 & -3 & 2\end{bmatrix}, b=\begin{bmatrix}8\\1\\4\end{bmatrix}$$
$$|A|=-17\neq 0, A^{-1}=\frac{-1}{17}\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}, U=A^{-1}b=\frac{-1}{17}\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}\begin{bmatrix}8\\1\\4\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$
$$\therefore u=1 \Rightarrow x=1, v=2 \Rightarrow y=\frac{1}{2}, w=3 \Rightarrow z=\frac{1}{3}$$

25. Solution:

$$x=a(\cos t+t\sin t), y=a(\sin t-t\cos t)$$
$$\frac{dx}{dt}=a(\cancel{-\sin t}+\cancel{\sin t}+t\cos t)=at\cos t \Rightarrow \frac{dt}{dx}=\frac{1}{at\cos t}$$
$$\frac{dy}{dt}=a(\cancel{\cos t}-\cancel{\cos t}-t(-\sin t))=at\sin t$$
$$\frac{dy}{dx}=\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)=(at\sin t)\frac{1}{at\cos t}=\tan t$$
$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}=\frac{d}{dt}(\tan t)\frac{dt}{dx}=(\sec^2 t)\left(\frac{1}{at\cos t}\right)=\frac{1}{at\cos^3 t}$$

26. Solution :

$$I=\int_0^{\pi}\frac{xdx}{4\cos^2 x+9\sin^2 x}=\int_0^{\pi}\frac{(\pi-x)dx}{4\cos^2 x+9\sin^2 x}$$

$$\therefore 2I=\pi\int_0^{\pi}\frac{dx}{4\cos^2 x+9\sin^2 x}=2\pi\int_0^{\frac{\pi}{2}}\frac{dx}{4\cos^2 x+9\sin^2 x}$$

$$=2\pi\left[\int_0^{\frac{\pi}{4}}\frac{dx}{4\cos^2 x+9\sin^2 x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{4\cos^2 x+9\sin^2 x}\right]$$

$$=2\pi\left[\int_0^{\frac{\pi}{4}}\frac{\sec^2 xdx}{4+9\tan^2 x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos ec^2 xdx}{4\cot^2 x+9}\right]$$

Putting tanx=t and cotx=u, we get

$$2I=2\pi\left[\int_0^{1}\frac{dt}{4+9t^2}-\int_1^{0}\frac{du}{4u^2+9}\right]=2\pi\left[\frac{1}{9}\left(\frac{3}{2}\right)\tan^{-1}\frac{t}{2/3}\Big|_0^1-\frac{1}{4}\left(\frac{2}{3}\right)\tan^{-1}\frac{u}{3/2}\Big|_1^0\right]$$

$$=2\pi\left[\frac{1}{6}\tan^{-1}\left(\frac{3}{2}\right)+\frac{1}{6}\tan^{-1}\left(\frac{2}{3}\right)\right]=\frac{2\pi}{6}\left(\frac{\pi}{2}\right)=\frac{\pi^2}{6}$$

$$\therefore I=\frac{\pi^2}{12}$$