JEE(Advanced) – 2016 TEST PAPER WITH SOLUTION

(HELD ON SUNDAY 22nd MAY, 2016)

PHYSICS

PART-I : PHYSICS

SECTION–1 : (Maximum Marks : 15)

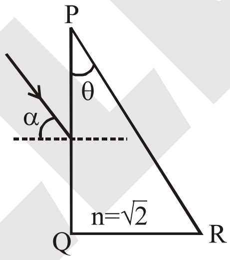
- This section contains **Five** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct
- For each question, darken the bubble corresponding to the correct option in the ORS
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

Zero Marks : 0 If none of the bubbles is darkened.

Negative Marks : –1 In all other cases

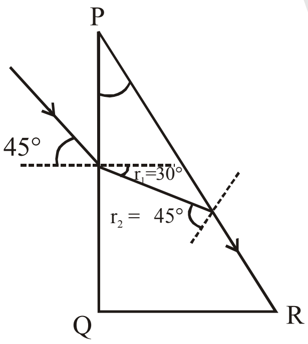
1. A parallel beam of light is incident from air at an angle α on the side PQ of a right angled triangular prism of refractive index $n = \sqrt{2}$. Light undergoes total internal reflection in the prism at the face PR when α has a minimum value of 45°. The angle θ of the prism is :



(A) 15° (B) 22.5° (C) 30° (D) 45°

Ans. (A)

Sol.



$1 \sin 45° = \sqrt{2} \sin r_1$

$r_2 - r_1 = \theta$

$\theta = 45° - 30°$

$\Rightarrow \theta = 15°$

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2. In a historical experiment to determine Planck's constant, a metal surface was irradiated with light of different wavelengths. The emitted photoelectron energies were measured by applying a stopping potential. The relevant data for the wavelength (λ) of incident light and the corresponding stopping potential (V_0) are given below:

$\lambda(\mu m)$	V_0 (Volt)
0.3	2.0
0.4	1.0
0.5	0.4

Given that $c = 3 \times 10^8\ ms^{-1}$ and $e = 1.6 \times 10^{-19}C$, Planck's constant (in units of J s) found from such an experiment is :

(A) 6.0×10^{-34} (B) 6.4×10^{-34} (C) 6.6×10^{-34} (D) 6.8×10^{-34}

Ans. (B)

Sol. $KE_{max} = \frac{hC}{\lambda} - \phi$

$$eV_s = \frac{hC}{\lambda} - \phi$$

$$1.6\times10^{-19}\times2 = \frac{h\times3\times10^8}{3000\times10^{-10}} - \phi \quad \text{... (i)}$$

$$1.6\times10^{-19}\times1 = \frac{h\times3\times10^8}{4000\times10^{-10}} - \phi \quad \text{... (ii)}$$

From (ii) $\phi = \frac{h\times3\times10^8}{4000\times10^{-10}} - 1.6\times10^{-19}$

$$1.6\times10^{-19}\times2 = \frac{h\times3\times10^8}{3000\times10^{-10}} - \frac{h\times3\times10^8}{4000\times10^{-10}} + 1.6\times10^{-19}$$

$$1.6\times10^{-19} = \frac{h\times3\times10^8}{10^{-7}}\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{h\times3\times10^8}{10^{-7}}\left[\frac{4-3}{12}\right]$$

$$1.6\times10^{-19} = \frac{h\times3\times10^8}{10^{-7}}\times\frac{1}{12}$$

$$1.6\times4\times\frac{10^{-19}\times10^{-7}}{10^8} = h$$

6.4×10^{-34} Js $= h$

3. A uniform wooden stick of mass 1.6 kg and length ℓ rests in an inclined manner on a smooth, vertical wall of height h ($<\ell$) such that a small portion of the stick extends beyond the wall. The reaction force of the wall on the stick is perpendicular to the stick. The stick makes an angle of 30° with the wall and the bottom of the stick is on a rough floor. The reaction of the wall on the stick is equal in magnitude to the reaction of the floor on the stick. The ratio h/ℓ and the frictional force f at the bottom of the stick are:

($g = 10\ ms^{-2}$)

(A) $\frac{h}{\ell} = \frac{\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3}N$

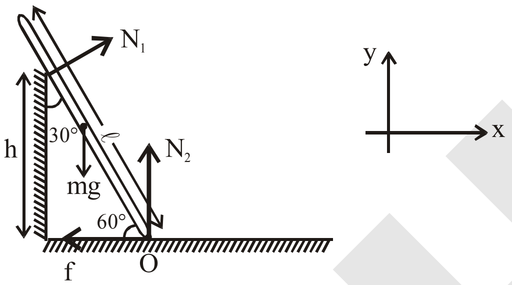
(B) $\frac{h}{\ell} = \frac{3}{16}, f = \frac{16\sqrt{3}}{3}N$

(C) $\frac{h}{\ell} = \frac{3\sqrt{3}}{16}, f = \frac{8\sqrt{3}}{3}N$

(D) $\frac{h}{\ell} = \frac{3\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3}N$

Ans. (D)

Sol.



Force equation in x-direction,

$N_1 \cos 30° - f = 0$ ... (i)

Force equation in y-direction,

$N_1 \sin 30° + N_2 - mg = 0$ ... (ii)

Torque equation about O,

$$mg\frac{\ell}{2}\cos 60° - N_1\frac{h}{\cos 30°} = 0 \quad \text{... (iii)}$$

Also, given $N_1 = N_2$ ... (iv)

[Note taking reaction from floor as normal reaction only]

solving (i), (ii), (iii) & (iv) we have

$$\frac{h}{\ell} = \frac{3\sqrt{3}}{16} \quad \& \quad f = \frac{16\sqrt{3}}{3}$$

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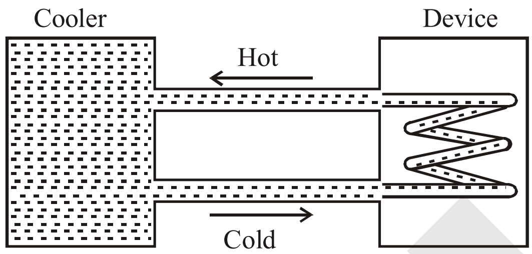
4. A water cooler of storage capacity 120 litres can cool water at constant rate of P watts. In a closed circulation system (as shown schematically in the figure), the water from the cooler is used to cool an external device that generates constantly 3 kW of heat (thermal load). The temperature of water fed into the device cannot exceed 30°C and the entire stored 120 litres of water is initially cooled to 10°C. The entire system is thermally insulated. The minimum value of P (in watts) for which the device can be operated for 3 hours is :



(Specific heat of water is 4.2 kJ kg^{-1} K^{-1} and the density of water is 1000 kg m^{-3})

(A) 1600 (B) 2067 (C) 2533 (D) 3933

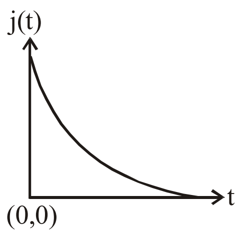
Ans. (B)

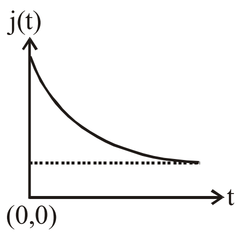
Sol. $3000 - P = (120 \times 1)(4.2 \times 10^3)\dfrac{dT}{dt}$

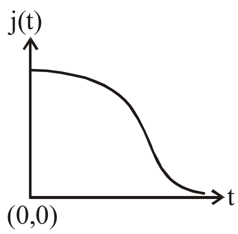
$$\frac{dT}{dt} = \frac{20}{60 \times 60 \times 3}$$

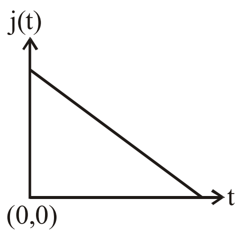
P = 2067 W

5. An infinite line charge of uniform electric charge density λ lies along the axis of an electrically conducting infinite cylindrical shell of radius R. At time t = 0, the space inside the cylinder is filled with a material of permittivity ε and electrical conductivity σ. The electrical conduction in the material follows Ohm's law. Which one of the following graphs best describes the subsequent variation of the magnitude of current density j(t) at any point in the material?

(A) 

(B) 

(C) 

(D) 

Ans. (A)

Sol. This is the problem of RC circuit where the product RC is a constant.

So due to leakage current, charge & current density will exponentially decay & will become zero at infinite time. So correct answer is (A)

for any small element

Resistance $R = \dfrac{dr}{\sigma(2\pi r\ell)}$

Capacitance $C = \dfrac{\in 2\pi r\ell}{dr}$

Product $R \times C = \dfrac{\in}{\sigma}$ = constant

$$q = q_0 e^{-\left(\frac{t\sigma}{\in}\right)}$$

$$I = \frac{dq}{dt} = \frac{q_0\sigma}{\in} e^{-\left(\frac{t\sigma}{\in}\right)}$$

$$\text{Current density} = \frac{I}{A} = \frac{q_0 \frac{\sigma}{\in} e^{-\frac{t\sigma}{\in}}}{2\pi r\ell}$$

$$j \propto e^{-\frac{t\sigma}{\in}}$$

SECTION–2 : (Maximum Marks : 32)

- This section contains **EIGHT** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct.
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS
- For each question, marks will be awarded in <u>one of the following categories</u> :

  Full Marks : +4 If only the bubble(s) corresponding to the correct option(s) is (are) darkened.

  Partial Marks : +1 For darkening a bubble corresponding **to each correct option**, Provided NO incorrect option is darkened.

  Zero Marks : 0 If none of the bubbles is darkened.

  Negative Marks : –2 In all other cases.
- for example, if (A), (C) and (D) are all the correct options for a question, darkening all these three will result in +4 marks; darkening only (A) and (D) will result in +2 marks; and darkening (A) and (B) will result in –2 marks, as a wrong option is also darkened

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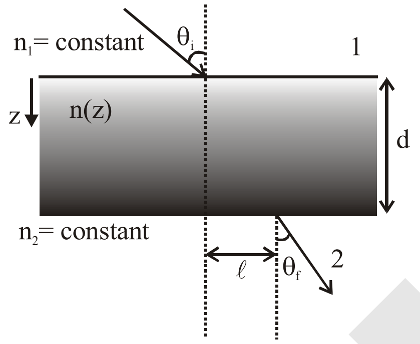
6. A transparent slab of thickness d has a refractive index n(z) that increases with z. Here z is the vertical distance inside the slab, measured from the top. The slab is placed between two media with uniform refractive indices n_1 and n_2 ($> n_1$), as shown in the figure. A ray of light is incident with angle θ_i from medium 1 and emerges in medium 2 with refraction angle θ_f with a lateral displacement ℓ.

Which of the following statement(s) is(are) true ?



(A) ℓ is independent of n_2 (B) ℓ is dependent on n(z)

(C) $n_1\sin\theta_i = (n_2 - n_1)\sin\theta_f$ (D) $n_1\sin\theta_i = n_2\sin\theta_f$

Ans. (A,B,D)

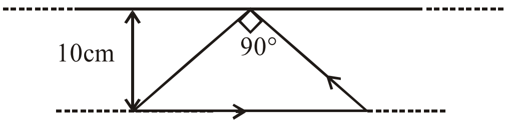
Sol. For parallel slab

$n_1 \sin\theta_i = n_2\sin\theta_f$

And ℓ depends on refractive angle in slab

$\therefore$ ℓ depends on refractive index of slab and independent of n_2

7. A conducting loop in the shape of right angled isosceles triangle of height 10 cm is kept such that the 90° vertex is very close to an infinitely long conducting wire (see the figure). The wire is electrically insulated from the loop. The hypotenuse of the triangle is parallel to the wire. The current in the triangular loop is in counterclockwise direction and increased at constant rate of 10 A s^{-1}. Which of the following statement(s) is(are) true?



(A) The induced current in the wire is in opposite direction to the current along the hypotenuse.

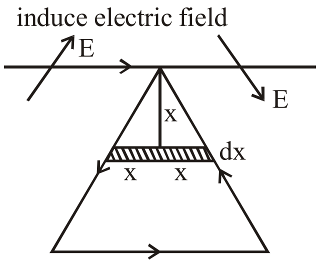
(B) There is a repulsive force between the wire and the loop

(C) If the loop is rotated at a constant angular speed about the wire, an additional emf of $\left(\frac{\mu_0}{\pi}\right)$ volt is induced in the wire

(D) The magnitude of induced emf in the wire is $\left(\frac{\mu_0}{\pi}\right)$ volt.

Ans. (B,D)

Sol.



by direction of induced electric field, current in wire is in same direction of current along the hypotenuse.

Flux through triangle if wire have current i $= \int_0^{0.1} \left(\frac{\mu_0 i}{2\pi x}\right)(2x\,dx) = \frac{\mu_0 i}{10\pi}$

$\Rightarrow$ Mutual inductance $= \frac{\mu_0}{10\pi}$

Induced emf in wire $= \frac{\mu_0}{10\pi}\frac{di}{dt} = \frac{\mu_0}{10\pi} \times 10 = \frac{\mu_0}{\pi}$

8. A plano-convex lens is made of a material of refractive index n. When a small object is placed 30 cm away in front of the curved surface of the lens, an image of double the size of the object is produced. Due to reflection from the convex surface of the lens, another faint image is observed at a distance of 10 cm away from the lens. Which of the following statement(s) is(are) true?

(A) The refractive index of the lens is 2.5

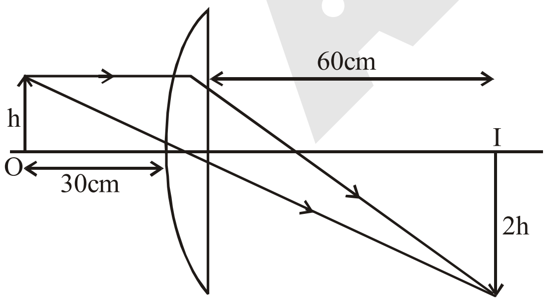
(B) The radius of curvature of the convex surface is 45 cm

(C) The faint image is erect and real

(D) The focal length of the lens is 20 cm.

Ans. (A,D)

Sol. For lens



$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

$\frac{1}{60} - \frac{1}{(-30)} = \frac{1}{f} \Rightarrow f = 20\text{cm}$ ... (i)

Also $\frac{1}{f} = (n-1)\left(\frac{1}{R} - \frac{1}{\infty}\right) = \frac{(n-1)}{R}$ ... (ii)

For reflection from convex mirror (curved surface)

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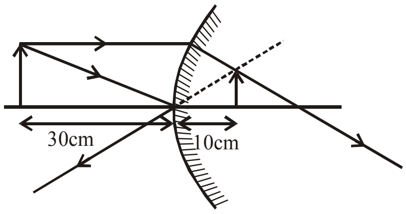

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}=\frac{2}{R}$$

$$\frac{1}{+10}+\frac{1}{-30}=\frac{1}{f}=\frac{2}{R} \quad \text{... (iii)}$$

R= 30 cm

from (i), (ii) & (iii)

n = 2.5,

faint image erect & virtual

9. An incandescent bulb has a thin filament of tungsten that is heated to high temperature by passing an electric current. The hot filament emits black-body radiation. The filament is observed to break up at random locations after a sufficiently long time of operation due to non-uniform evaporation of tungsten from the filament. If the bulb is powered at constant voltage, which of the following statement(s) is(are) true?

(A) The temperature distribution over the filament is uniform

(B) The resistance over small sections of the filament decreases with time

(C) The filament emits more light at higher band of frequencies before it breaks up

(D) The filament consumes less electrical power towards the end of the life of the bulb

Ans. **(C,D)**

Sol. Because of non-uniform evaporation at different section, area of cross-section would be different at different sections.

Region of highest evaporation rate would have rapidly reduced area and would become break up cross-section.

Resistance of the wire as whole increases with time.

Overall resistance increases hence power decreases. At break up junction temperature would be highest, thus light of highest band frequency would be emitted at those cross-section.

10. A length-scale (ℓ) depends on the permittivity (ε) of a dielectric material, Boltzmann constant k_B, the absolute temperature T, the number per unit volume (n) of certain charged particles, and the charge (q) carried by each of the particles, Which of the following expressions(s) for ℓ is(are) dimensionally correct?

(A) $\ell=\sqrt{\left(\frac{nq^2}{\varepsilon k_B T}\right)}$ (B) $\ell=\sqrt{\left(\frac{\varepsilon k_B T}{nq^2}\right)}$ (C) $\ell=\sqrt{\left(\frac{q^2}{\varepsilon n^{2/3} k_B T}\right)}$ (D) $\ell=\sqrt{\left(\frac{q^2}{\varepsilon n^{1/3} k_B T}\right)}$

Ans. **(B,D)**

Sol. We know,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2} \qquad \Rightarrow \frac{q^2}{\varepsilon_0} = (Fr^2)4\pi$$

So, dimension

$$\frac{q^2}{\varepsilon_0} = \dim(Fr^2) = MLT^{-2} \times L^2 = ML^3T^{-2}$$

Similarly; $E = \frac{3}{2}K_BT \Rightarrow \dim(K_BT) = \dim(\text{Energy}) = ML^2T^{-2}$

(A) $\sqrt{\frac{nq^2}{\varepsilon k_B T}} = \sqrt{\frac{L^{-3} \times ML^3T^{-2}}{ML^2T^{-2}}} = \frac{1}{L}$

(B) $\sqrt{\frac{(E)\times \text{vol}}{Fr^2}} = \sqrt{\frac{ML^2T^{-2} \times L^3}{MLT^{-2} \times L^2}} = L$

(C) $\sqrt{\frac{Fr^2(\text{vol})^{2/3}}{(K\varepsilon)}} = \sqrt{\frac{MLT^{-2} \times L^2 \times L^2}{ML^2T^{-2}}} = \sqrt{L^3} = L^{3/2}$

(D) $\sqrt{\frac{Fr^2(\text{vol})^{1/3}}{\text{Energy}}} = \sqrt{\frac{MLT^{-2}L^2 \times L}{ML^2T^{-2}}} = L$

[$\therefore$ dimension n = $\dim\left(\frac{1}{\text{vol}}\right) = L^{-3}$]

11. Highly excited states for hydrogen like atoms (also called Rydberg states) with nuclear charge Ze are defined by their principal quantum number n, where n >> 1. Which of the following statement(s) is (are) true?

(A) Relative change in the radii of two consecutive orbitals does not depend on Z

(B) Relative change in the radii of two consecutive oribitals varies as 1/n

(C) Relative change in the energy of two consecutive orbitals varies as $1/n^3$

(D) Relative change in the angular momenta of two consecutive orbitals varies as 1/n

Ans. (A, B, D)

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Sol. As radius $r \propto \frac{n^2}{z}$

$$\Rightarrow \frac{\Delta r}{r} = \frac{\left(\frac{n+1}{z}\right)^2 - \left(\frac{n}{z}\right)^2}{\left(\frac{n}{z}\right)^2} = \frac{2n+1}{n^2} \approx \frac{2}{n} \propto \frac{1}{n}$$

as energy $E \propto \frac{z^2}{n^2}$

$$\Rightarrow \frac{\Delta E}{E} = \frac{\frac{z^2}{n^2} - \frac{z^2}{(n+1)^2}}{\frac{z^2}{(n+1)^2}} = \frac{(n+1)^2 - n^2}{n^2.(n+1)^2}.(n+1)^2$$

$$\Rightarrow \frac{\Delta E}{E} = \frac{2n+1}{n^2} \simeq \frac{2n}{n^2} \propto \frac{1}{n}$$

as angular momentum $L = \frac{nh}{2\pi}$

$$\Rightarrow \frac{\Delta L}{L} = \frac{\frac{(n+1)h}{2\pi} - \frac{nh}{2\pi}}{\frac{nh}{2\pi}} = \frac{1}{n} \propto \frac{1}{n}$$

12. The position vector $\vec{r}$ of a particle of mass m is given by the following equation $\vec{r}(t) = \alpha t^3\hat{i} + \beta t^2\hat{j}$, where $\alpha = \frac{10}{3}\text{ms}^{-3}$, $\beta = 5\ \text{ms}^{-2}$ and m = 0.1 kg. At t = 1 s, which of the following statement(s) is(are) true about the particle?

(A) The velocity $\vec{v}$ is given by $\vec{v} = (10\hat{i} + 10\hat{j})\text{ms}^{-1}$

(B) The angular momentum $\vec{L}$ with respect to the origin is given by $\vec{L} = -\left(\frac{5}{3}\right)\hat{k}$ Nms

(C) The force $\vec{F}$ is given by $\vec{F} = (\hat{i} + 2\hat{j})$N

(D) The torque $\vec{\tau}$ with respect to the origin is given by $\vec{\tau} = -\left(\frac{20}{3}\right)\hat{k}$ Nm

Ans. (A, B, D)

Sol. $\vec{r} = \alpha t^3\hat{i} + \beta t^2\hat{j}$

$$\vec{v} = \frac{d\vec{r}}{dt} = 3\alpha t^2\hat{i} + 2\beta t\hat{j}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 6\alpha t\hat{i} + 2\beta\hat{j}$$

At t = 1

(A) $\vec{v} = 3\times\frac{10}{3}\times 1\hat{i} + 2\times J\times 1\hat{j}$

$=10\hat{i} + 10\hat{j}$

(B) $\vec{L} = \vec{r}\times\vec{p}$

$= \left(\frac{10}{3}\times 1\hat{i} + 5\times 1\hat{j}\right)\times 0.1\ \left(10\hat{i} + 10\hat{j}\right)$

$= -\frac{5}{3}\hat{k}$

(C) $\vec{F} = m\times\left(6\times\frac{10}{3}\times 1\hat{i} + 2\times 5\hat{j}\right)\ =\ 2\hat{i} + \hat{j}$

(D) $\vec{\tau} = r\times\vec{F}$

$= \left(\frac{10}{3}\hat{i} + 5\hat{j}\right)\times\left(2\hat{i} + \hat{j}\right)$

$= +\frac{10}{3}\hat{k} + 10\left(-\hat{k}\right)$

$= -\frac{20}{3}\hat{k}$

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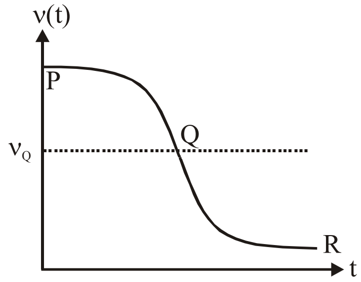
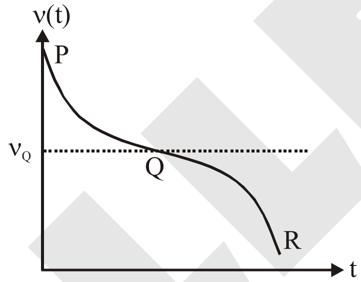
13. Two loudspeakers M and N are located 20m apart and emit sound at frequencies 118 Hz and 121 Hz, respectively. A car is initially at a point P, 1800 m away from the midpoint Q of the line MN and moves towards Q constantly at 60 km/hr along the perpendicular bisector of MN. It crosses Q and eventually reaches a point R, 1800 m away from Q. Let $\nu(t)$ represent the beat frequency measured by a person sitting in the car at time t. Let ν_P, ν_Q and ν_R be the beat frequencies measured at locations P, Q and R, respectively. The speed of sound in air is 330 ms^{-1}. Which of the following statement(s) is(are) true regarding the sound heard by the person ?

(A) The plot below represents schematically the variation of beat frequency with time



(B) The plot below represents schematically the variations of beat frequency with time

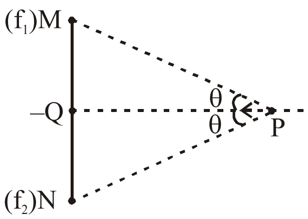


(C) The rate of change in beat frequency is maximum when the car passes through Q

(D) $\nu_P + \nu_R = 2\nu_Q$

Ans. (A, C, D)

Sol. $f_M = \dfrac{C + V\cos\theta}{C} f_1$



$f_N = \dfrac{C + V\cos\theta}{C} f_2$

$\Delta f = f_N - f_M$

$= \dfrac{C + V\cos\theta}{C}(f_2 - f_1)$

$\dfrac{d(\Delta f)}{dt} = -\dfrac{V}{C}(f_2 - f_1)\sin\theta \dfrac{d\theta}{dt}$

$\therefore$ & $\dfrac{d(\Delta f)}{dt}$ is maximum when $\theta = 90°$

[$\therefore$ C is correct]

$$\nu_P = \left(1+\frac{V}{C}\cos\theta\right)\Delta f$$

$$\nu_Q = \Delta f$$

$$\nu_R = \left(1-\frac{V}{C}\cos\theta\right)\Delta f$$

$$\therefore\ \nu_P + \nu_r = 2\nu_Q$$

SECTION–3 : (Maximum Marks : 15)

- This section contains **FIVE** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct answer is darkened.

Zero Marks : 0 In all other cases.

14. Consider two solid spheres P and Q each of density 8 gm cm^{-3} and diameters 1 cm and 0.5 cm, respectively. Sphere P is dropped into a liquid of density 0.8 gm cm^{-3} and viscosity $\eta = 3$ poiseulles. Sphere Q is dropped into a liquid of density 1.6 gm cm^{-3} and viscosity $\eta = 2$ poiseulles. The ratio of the terminal velocities of P and Q is.

Ans. 3

Sol. $V_T \propto \dfrac{r^2[d_m - d_L]}{n}$

$$\frac{V_{TP}}{V_{TQ}} = \left(\frac{r_P}{r_Q}\right)^2 \times \frac{n_{L_2}}{n_{L_1}} \times \left[\frac{d_m - d_{L_1}}{d_m - d_{L_2}}\right]$$

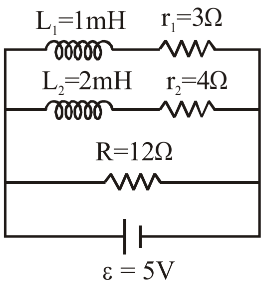
$$\frac{V_{TP}}{V_{TQ}} = \left(\frac{2}{1}\right)^2 \times \frac{2}{3} \times \left[\frac{8-0.8}{8-1.6}\right]$$

$$\frac{V_{TP}}{V_{TQ}} = 3$$

15. Two inductors L_1 (inductance 1 mH, internal resistance 3 Ω) and L_2 (inductance 2mH, internal resistance 4Ω), and a resistor R (resistance 12 Ω) are all connected in parallel across a 5V battery. The circuit is switched on at time $t = 0$. The ratio of the maximum to the minimum current (I_{max}/I_{min}) drawn from the battery is.

Ans. 8

Sol.



$$I_{max} = \frac{\varepsilon}{R} = \frac{5}{12}\text{A} \text{ (Initially at } t = 0)$$

$$I_{min} = \frac{\varepsilon}{R_{eq}} = \varepsilon\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{R}\right) \text{ (finally in steady state)}$$

$$= 5\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{12}\right)$$

$$= \frac{10}{3}\text{A}$$

$$\frac{I_{max}}{I_{min}} = 8$$

16. The isotope $^{12}_{5}B$ having a mass 12.014 u undergoes β-decay to $^{12}_{6}C$. $^{12}_{6}C$ has an excited state of the nucleus ($^{12}_{6}C^*$) at 4.041 MeV above its ground state. If $^{12}_{5}B$ decays to $^{12}_{6}C^*$, the maximum kinetic energy of the β-particle in units of MeV is (1u = 931.5 MeV/c^2, where c is the speed of light in vacuum).

Ans. 9

Sol. $^{12}_{5}B \rightarrow {}^{12}_{6}C + {}^{0}_{-1}e + \bar{v}$

Mass defect = (12.014 – 12) u

$\therefore$ Released energy = 13.041 MeV

Energy used for excitation of $^{12}_{6}C$ = 4.041 MeV

$\therefore$ Energy converted to KE of electron

= 13.041 – 4.041 = 9 MeV

17. A hydrogen atom in its ground state is irradiated by light of wavelength 970 Å. Taking hc/e = 1.237 × 10^{-6} eV m and the ground state energy of hydrogen atom as –13.6 eV, the number of lines present in the emission spectrum is

Ans. 6

Sol. $\frac{hc}{\lambda} = \frac{12370}{970}$

$-13.6 + 12.7 = -\frac{13.6}{n^2}$

$n^2 = 16$

$n = 4$

Number of lines $= {}^nC_2 = 6$

18. A metal is heated in a furnace where a sensor is kept above the metal surface to read the power radiated (P) by the metal. The sensor has a scale that displays $\log_2(P/P_0)$, where P_0 is a constant. When the metal surface is at a temperature of 487 °C, the sensor shows a value 1. Assume that the emissivity of the metallic surface remains constant. What is the value displayed by the sensor when the temperature of the metal surface is raised to 2767 °C ?

Ans. 9

Sol. $P = eA\sigma T^4$ where T is in kelvin

$$\log_2 \frac{eA\sigma(487+273)^4}{P_0} = 1 \qquad \text{...(i)}$$

$$\log_2 \frac{eA\sigma(2767+273)^4}{P_0} = x \qquad \text{...(ii)}$$

(ii) – (i)

$$\log_2\left(\frac{3040}{760}\right)^4 = x-1$$

$\therefore\ x = 9$

JEE(Advanced) – 2016 TEST PAPER WITH SOLUTION

(HELD ON SUNDAY 22nd MAY, 2016)

PART-II : CHEMISTRY

SECTION–1 : (Maximum Marks : 15)

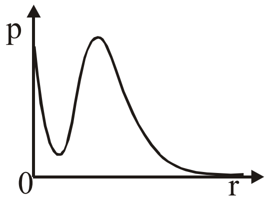
- This section contains **Five** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct
- For each question, darken the bubble corresponding to the correct option in the ORS
- For each question, marks will be awarded in one of the following categories :

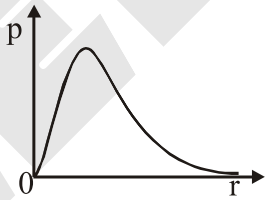
Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

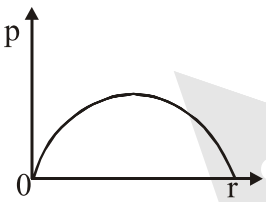
Zero Marks : 0 If none of the bubbles is darkened.

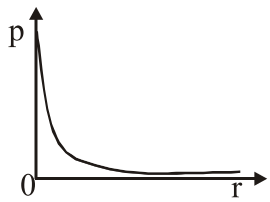
Negative Marks : –1 In all other cases

19. P is the probability of finding the 1s electron of hydrogen atom in a spherical shell of infinitesimal thickness, dr, at a distance r from the nucleus. The volume of this shell is $4\pi r^2 dr$. The qualitative sketch of the dependence of P on r is -

(A)


(B)


(C)


(D)


Ans. (B)

Sol. For 1s, radial part of wave function is

$$\psi_{(r)} = 2\left(\frac{1}{a_0}\right)^{\frac{3}{2}} e^{-\frac{r}{a_0}}$$

probability of finding an e^- in a spherical shell of thickness, 'dr' at distance 'r' from nucleus,

$P = \psi^2_{(r)} \cdot 4\pi r^2 dr$

$$= 16\pi r^2 \left(\frac{1}{a_0}\right)^3 e^{\frac{-2r}{a_0}} dr$$

So P is zero at r = 0 and r = ∞.

20. One mole of an ideal gas at 300 K in thermal contact with surroundings expands isothermally from 1.0 L to 2.0 L against a constant pressure of 3.0 atm. In this process, the change in entropy of surroundings (ΔS_{surr}) in J K^{-1} is -

(1 L atm = 101.3 J)

(A) 5.763 (B) 1.013 (C) –1.013 (D) –5.763

Ans. (C)

Sol. From 1st law of thermodynamics

$q_{sys} = \Delta U - w = 0 - [-P_{ext}.\Delta V]$

$= 3.0 \text{ atm} \times (2.0 \text{ L} - 1.0 \text{ L}) = 3.0 \text{ L-atm}$

$$\therefore \Delta S_{surr} = \frac{(q_{rev})_{surr}}{T} = -\frac{q_{sys}}{T}$$

$$= -\frac{3.0 \times 101.3 \text{J}}{300 \text{K}}$$

$= -1.013$ J/K

21. The increasing order of atomic radii of the following group 13 elements is

(A) Al < Ga < In < Tl (B) Ga < Al < In < Tl

(C) Al < In < Ga < Tl (D) Al < Ga < Tl < In

Ans. (B)

Sol. The order of radius of 13th group elements is Ga < Al < In < Tl.

Reason ⇒ Due to poor shielding effect of d-orbital, radius of Ga is smallar than Al.

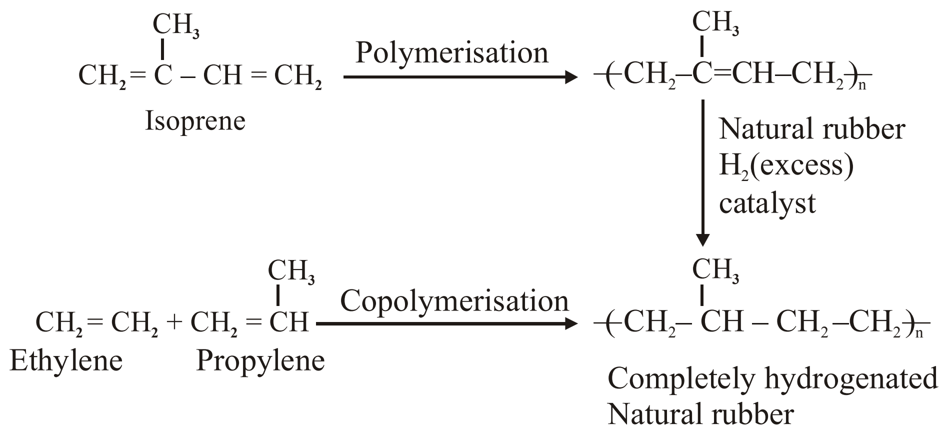
22. On complete hydrogenation, natural rubber produces

(A) ethylene-propylene copolymer (B) vulcanised rubber

(C) polypropylene (D) polybutylene

Ans. (A)

Sol.



CHEMISTRY

23. Among $[Ni(CO)_4]$, $[NiCl_4]^{2-}$, $[Co(NH_3)_4Cl_2]Cl$, $Na_3[CoF_6]$, Na_2O_2 and CsO_2, the total number of paramagnetic compounds is -

(A) 2 (B) 3 (C) 4 (D) 5

Ans. (B)

Sol.	**Compound/Ion**	**Magnetic nature of compound**
1.	$[Ni(CO)_4]$	Diamagnetic
2.	$[NiCl_4]^{2-}$	Paramagnetic
3.	$[Co(NH_3)_4Cl_2]Cl$	Diamagnetic
4.	$Na_3[CoF_6]$	Paramagnetic
5.	Na_2O_2	Diamagnetic
6.	CsO_2	Paramagnetic

So total number of paramagnetic compounds is 3.

SECTION–2 : (Maximum Marks : 32)

- This section contains **EIGHT** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct.
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS
- For each question, marks will be awarded in one of the following categories :

  Full Marks : +4 If only the bubble(s) corresponding to the correct option(s) is (are) darkened.

  Partial Marks : +1 For darkening a bubble corresponding **to each correct option**, Provided NO incorrect option is darkened.

  Zero Marks : 0 If none of the bubbles is darkened.

  Negative Marks : –2 In all other cases.
- for example, if (A), (C) and (D) are all the correct options for a question, darkening all these three will result in +4 marks; darkening only (A) and (D) will result in +2 marks; and darkening (A) and (B) will result in –2 marks, as a wrong option is also darkened

24. A plot of the number of neutrons (N) against the number of protons (P) of stable nuclei exhibits upwards deviation from linearity for atomic number, $Z > 20$. For an unstable nucleus having N/P ratio less than 1, the possible mode(s) of decay is(are) -

(A) β^- decay (β emission) (B) orbital or K-electron capture

(C) Neutron emission (D) β^+ decay (positron emission)

Ans. (B, D)

Sol. As $\frac{N}{P}$ ratio is less than 1, for possible decay mode , $\frac{N}{P}$ ratio should increase. The possible modes are α-decay, K-capture and β^+ -decay. Hence, correct option are (B), (D).

(In β^- -decay or neutron decay , $\frac{N}{P}$ ratio will decrease further)

25. The correct statements(s) about of the following reaction sequence is(are)

$$\text{Cumene}(C_9H_{12}) \xrightarrow[\text{(ii) } H_3O^+]{\text{(i) } O_2} \mathbf{P} \xrightarrow{CHCl_3/NaOH} \mathbf{Q}\ (\text{major}) + \mathbf{R}\ (\text{minor})$$

$$\mathbf{Q} \xrightarrow[PhCH_2Br]{NaOH} \mathbf{S}$$

(A) **R** is steam volatile

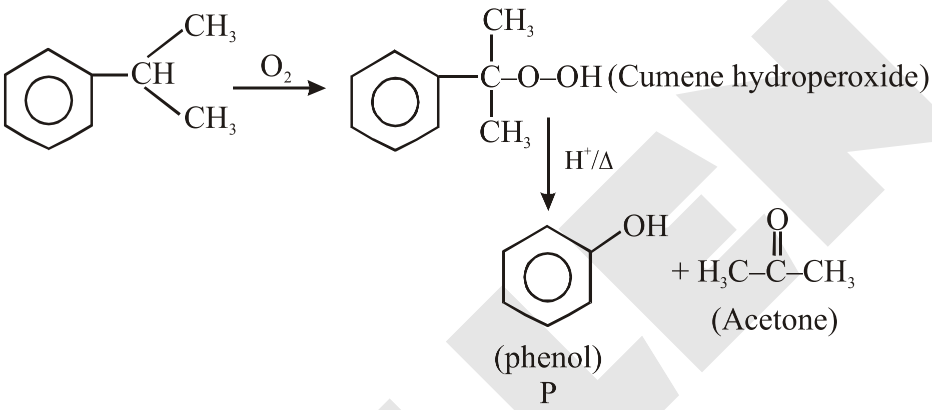
(B) **Q** gives dark violet coloration with 1% aqueous $FeCl_3$ solution

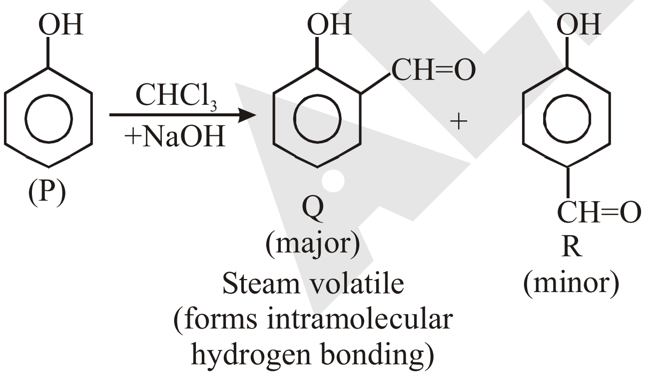
(C) **S** gives yellow precipitate with 2, 4,-dinitrophenylhydrazine

(D) **S** gives dark violet coloration with 1% aqueous $FeCl_3$ solution

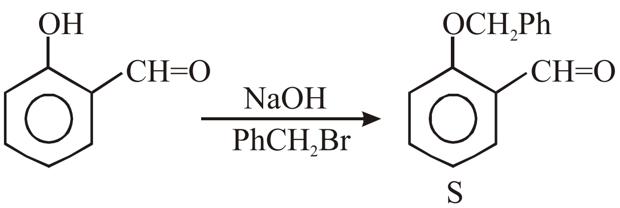
Ans. (B, C)

Sol.





High boiling point due to inter molecular H-bonding So R is not steam volatile

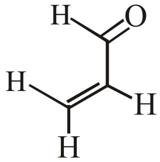
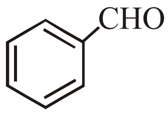
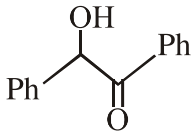
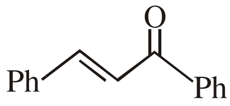


(does not give dark violet coloration with 1% $FeCl_3$ solution)

Q gives dark violet coloration with 1% aqueous $FeCl_3$ solution because it has phenolic (–OH) group.

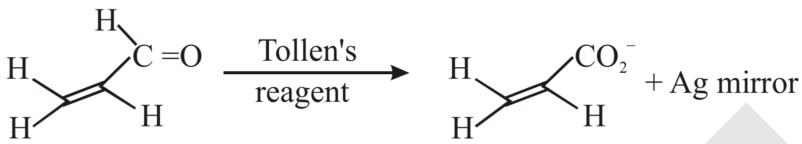
CHEMISTRY

26. Positive Tollen's test is observed for

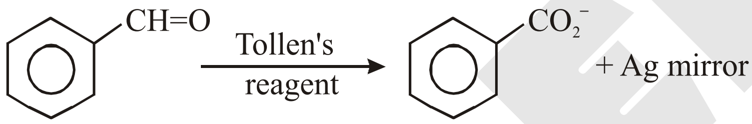
(A)  (B)  (C)  (D) 

Ans. (A,B,C)

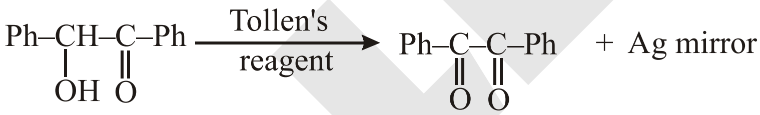
Sol. Tollens's test is given by compounds having aldehyde group. Also α-hydroxy carbonyl gives positive tollen's test.

(A)  (+ve test)

Acraldehyde

(B)  (+ve test)

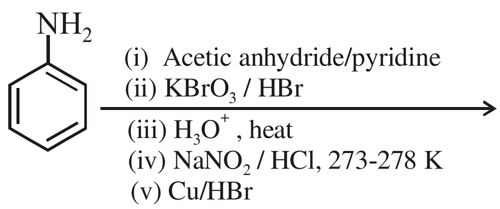
Benzaldehyde

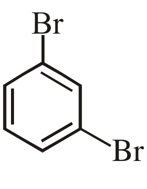
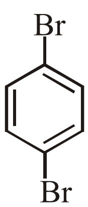
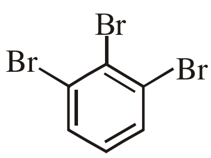
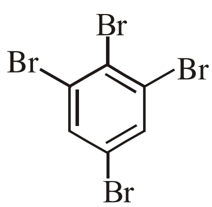
(C)  (+ve test)

Benzoin

(D)  (–ve test)

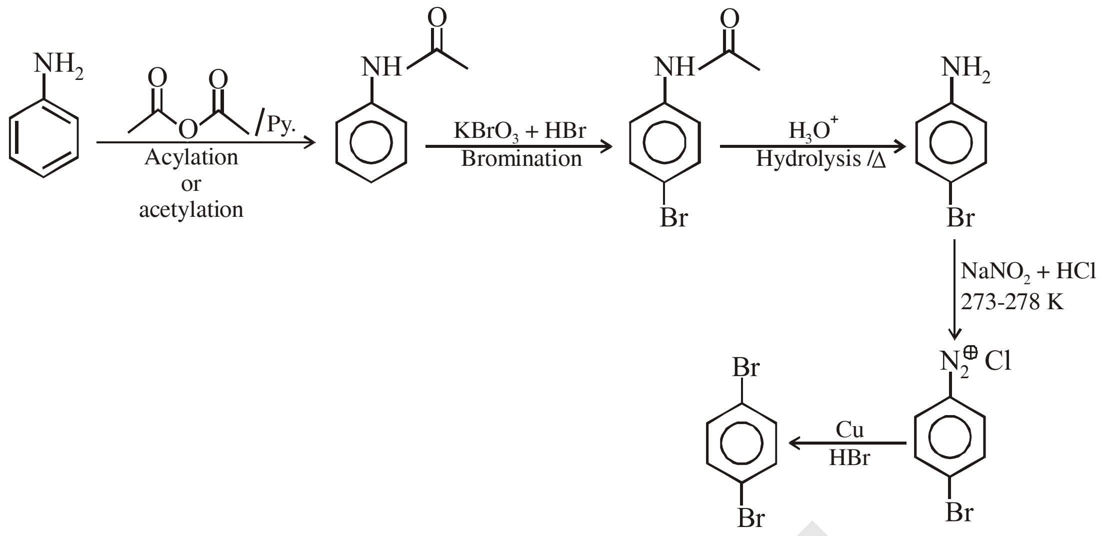
27. The product(s) of the following reaction sequence is(are)



(A)  (B)  (C)  (D) 

Ans. (B)

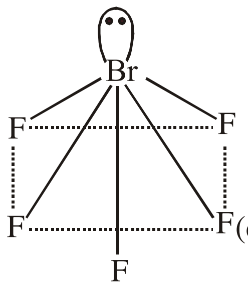
Sol.



28. The compound(s) with TWO lone pairs of electrons on the central atom is(are)

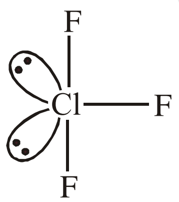
(A) BrF_5 (B) ClF_3 (C) XeF_4 (D) SF_4

Ans. (B,C)

Sol. (A) 

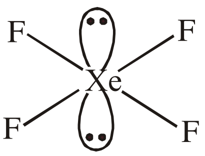
sp^3d^2 hybridisation
square pyramidal
(distorted octadehdral)

Number of lone pair on centre atom = 1

(B) 

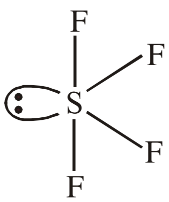
sp^3d hybridisation
T-shape

Number of lone pair on centre atom = 2

(C) 

sp^3d^2 hybridisation
square planar

Number of lone pair on centre atom = 2

(D) 

sp^3d hybridisation
see-saw

Number of lone pair on centre atom = 1

Hence **Ans. (B,C)**

29. The crystalline form of borax has

(A) Tetranuclear $[B_4O_5(OH)_4]^{2-}$ unit

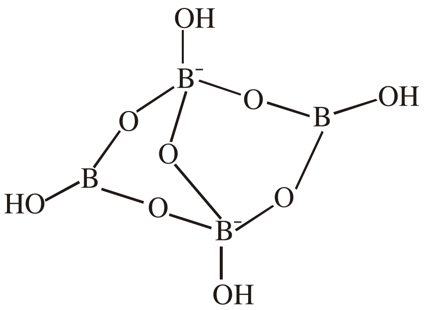
(B) All boron atoms in the same plane

(C) Equal number of sp^2 and sp^3 hybridized boron atoms

(D) One terminal hydroxide per boron atom

Ans. (A,C,D)

Sol.



(A) Having $[B_4O_5(OH)_4]^{2-}$ tetranuclear (boron) unit

(B) All boron atoms not in same plane

(C) Two boron are sp^2 hybridised and two boron are sp^3 hybridised

(D) One terminal hydroxide per boron atom is present.

30. The reagent(s) that can selectively precipitate S^{2-} from a mixture of S^{2-} and SO_4^{2-} in aqueous soltuion is(are) :

(A) $CuCl_2$ (B) $BaCl_2$ (C) $Pb(OOCCH_3)_2$ (D) $Na_2[Fe(CN)_5NO]$

Ans. (A OR A, C)

(A)
$$\underset{(Sol^n)}{CuCl_2} + \underset{(Sol^n)}{S^{2-}} \longrightarrow \underset{(Black\ ppt.)}{CuS\downarrow} + \underset{(Sol^n)}{2Cl^-}$$
$$\underset{(Sol^n)}{CuCl_2} + \underset{(Sol^n)}{SO_4^{2-}} \longrightarrow \text{No ppt.}$$

(B)
$$\underset{(Sol^n)}{BaCl_2} + \underset{(Sol^n)}{S^{2-}} \longrightarrow \underset{(No\ ppt.)}{BaS} + \underset{(Sol^n)}{2Cl^-}$$
$$\underset{(Sol^n)}{BaCl_2} + \underset{(Sol^n)}{SO_4^{2-}} \longrightarrow \underset{(White\ ppt.)}{BaSO_4\downarrow} + \underset{(Sol^n)}{2Cl^-}$$

(C)
$$\underset{(Sol^n)}{Pb(OOCCH_3)_2} + \underset{(Sol^n)}{S^{2-}} \longrightarrow \underset{(Black\ ppt.)}{PbS\downarrow} + \underset{(Sol^n)}{2CH_3COO^-}$$
$$\underset{(Sol^n)}{Pb(OOCCH_3)_2} + \underset{(Sol^n)}{SO_4^{2-}} \longrightarrow \underset{(White\ ppt.)}{PbSO_4\downarrow} + \underset{(Sol^n)}{2CH_3COO^-}$$

(D)
$$\underset{(Sol^n)}{Na_2[Fe(CN)_5NO]} + \underset{(Sol^n)}{S^{2-}} \longrightarrow \underset{(Purple\ colour\ solution)}{Na_4[Fe(CN)_5NOS]}$$
$$\underset{(Sol^n)}{Na_2[Fe(CN)_5NO]} + \underset{(Sol^n)}{SO_4^{2-}} \longrightarrow \text{No ppt.}$$

Note : $PbSO_4$ Ksp = 2.5×10^{-8}
PbS Ksp = 3×10^{-28} } Which are not given in question

As in question selective precipitation is asked PbS will be precipitate much easier than $PbSO_4$ though both are insoluble. Hence answer should be (C) also alongwith (A)

31. According to the Arrhenius equation,

(A) A high activation energy usually implies a fast reaction

(B) Rate constant increase with increase in temperature. This is due to a greater number of collisions whose energy exceeds the activation energy

(C) Higher the magnitude of activation energy, stronger is the temperature dependence of the rate constant

(D) The pre-exponential factor is a measure of the rate at which collisions occur, irrespective of their energy.

Ans. (B,C,D)

Sol. (A) $k = Ae^{-E_a/RT}$

High E_a means less k, hence slower rate.

(B) $e^{-Ea/RT}$ = fraction of molecules having kinetic energy greater than activation energy which increase as temperature increases.

(C) $\ln\frac{k_2}{k_1} = \frac{E_a}{R}\left[\frac{1}{T_1} - \frac{1}{T_2}\right]$ i.e., $\ln\frac{k_2}{k_1} \propto E_a$

(D) Rate of reaction $\propto$ Total number of collisions $\times$ Fraction of collisions which can form product

Rate of reaction $\propto Z_{AB} \times (P \times e^{-E_a/RT})$

$\propto A\ e^{-E_a/RT}$

SECTION–3 : (Maximum Marks : 15)

- This section contains **FIVE** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct answer is darkened.

Zero Marks : 0 In all other cases.

32. The mole fraction of a solute in a solution is 0.1. At 298 K, molarity of this solution is the same as its molality. Density of this solution at 298 K is 2.0 g cm^{-3}. The ratio of the molecular weights of the solute and solvent, $\left(\frac{MW_{solute}}{MW_{solvent}}\right)$, is

Ans. (9)

Sol. 1 mole solution has 0.1 mole solute and 0.9 mole solvent

Let M_1 = Molar mass solute

M_2 = Molar mass solvent

Molality, $m = \dfrac{0.1}{0.9M_2} \times 1000$ ....(1)

Molarity, $M = \dfrac{0.1}{0.1M_1 + 0.9M_2} \times 2 \times 1000$ .....(2)

$\because$ $m = M$

$\Rightarrow$ $\dfrac{0.1 \times 1000}{0.9M_2} = \dfrac{200}{0.1M_1 + 0.9M_2}$

$\Rightarrow$ $\dfrac{M_1}{M_2} = 9$

Alternate solution :

$\because$ $M = m$

$\Rightarrow$ volume of solution = mass of solvent

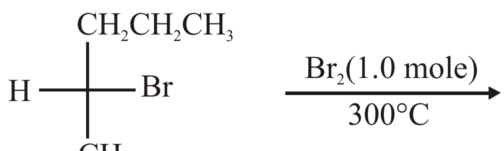
$\Rightarrow$ $\dfrac{W_{solute} + W_{solvent}}{2} = W_{solvent}$

$W_{solute} = W_{solvent}$

$0.1 \times M_{solute} = 0.9 \times M_{solvent}$

$\dfrac{M_{solute}}{M_{solvent}} = 9$

33. In the following monobromination reaction, the number of possible chiral products is

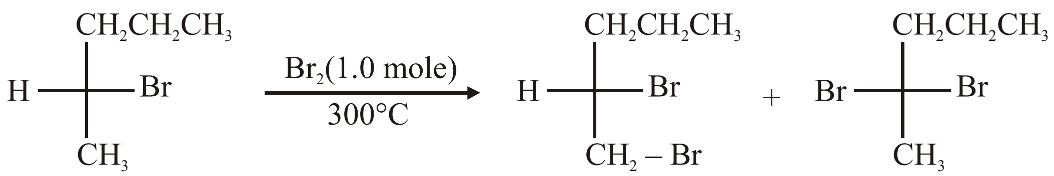


(1.0 mole)

(enantiomerically pure)

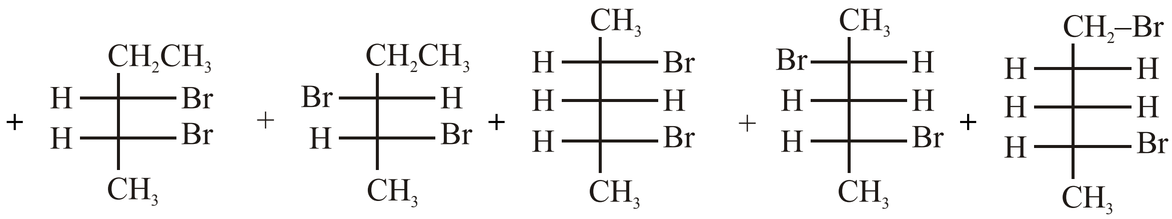
Ans. (5)

Sol.



(1.0 mole) (Chiral) (Achiral)

(enantiomerically pure)



(Chiral) (Chiral) (Achiral) (Chiral) (Chiral)

34. The diffusion coefficient of an ideal gas is proportional to its mean free path and mean speed. The absolute temperature of an ideal gas is increased 4 times and its pressure is increased 2 times. As a result, the diffusion coefficient of this gas increases x times. The value of x is

Ans. (4)

Sol. Rate of diffusion $\propto \lambda \times U_{Avg}$

$$\propto \frac{1}{\sqrt{2}\pi\sigma^2 N^*} \times U_{Avg}$$

$$\propto \frac{U_{Avg}}{\sqrt{2}\pi\sigma^2 N^*}$$

$$\propto \frac{U_{Avg}(kT)}{\sqrt{2}\pi\sigma^2 P}$$

Rate of diffusion $\propto \dfrac{T^{\frac{3}{2}}}{P}$

$$\frac{r_{final}}{r_{initial}} = \frac{(4)^{\frac{3}{2}}}{2}$$

$$\frac{r_{final}}{r_{inital}} = 4$$

35. The number of geometric isomers possible for the complex $[CoL_2Cl_2]^-$ ($L = H_2NCH_2CH_2O^-$) is

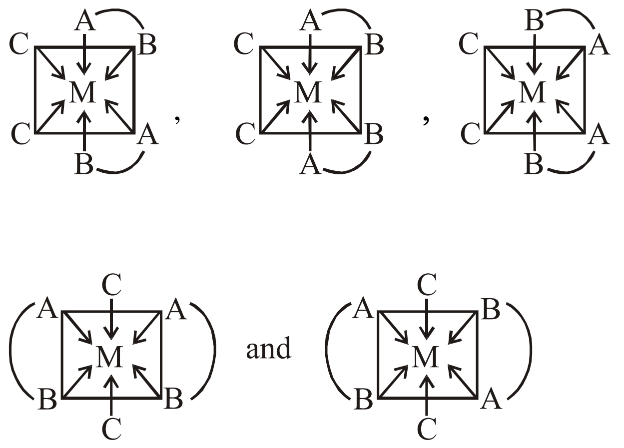
Ans. (5)

Sol. $[CoL_2Cl_2]^-$ $(L = H_2NCH_2CH_2O^-)$

$[Co(H_2NCH_2CH_2O)_2\ Cl_2]^{\ominus}$

$\downarrow$

It is $[M(AB)_2C_2]$ type of complex



Total geometrical isomers = 5

36. In neutral or faintly alkaline solution, 8 moles permanganate anion quantitatively oxidize thiosulphate anions to produce X moles of a sulphur containing product. the magnitude of X is

Ans. (6)

Sol. $\overset{+7}{Mn}O_4^- + \overset{+2}{S_2}O_3^{2-} \longrightarrow \overset{+4}{Mn}O_2 + \underset{X}{\overset{+6}{S}O_4^{2-}}$

Equivalents of MnO_4^- = equivalents of SO_4^{2-}

Moles of MnO_4^- × n-factor = moles of SO_4^{2-} × n-factor

$8 \times 3 = X \times 4$

$X = 6$

JEE(Advanced) – 2016 TEST PAPER WITH SOLUTION

(HELD ON SUNDAY 22nd MAY, 2016)

PART - III : MATHEMATICS

SECTION–1 : (Maximum Marks : 15)

- This section contains **Five** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct
- For each question, darken the bubble corresponding to the correct option in the ORS
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

Zero Marks : 0 If none of the bubbles is darkened.

Negative Marks : –1 In all other cases

37. A debate club consists of 6 girls and 4 body. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 member) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is

(A) 380 (B) 320 (C) 260 (D) 95

Ans. (A)

Sol. $({}^6C_4 + {}^6C_3.{}^4C_1).{}^4C_1 = 380$

38. The least value of $\alpha \in R$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$, is -

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

Ans. (C)

Sol. $f(x) = 4\alpha x^2 + \frac{1}{x}; x > 0$

$$f'(x) = 8\alpha x - \frac{1}{x^2}$$

$$= \frac{8\alpha x^3 - 1}{x^2}$$

$f(x)$ attains its minimum at $x = \left(\frac{1}{8\alpha}\right)^{1/3}$

$$f\left(\left(\frac{1}{8\alpha}\right)^{1/3}\right) = 1$$

$$\Rightarrow 4\alpha\left(\frac{1}{8\alpha}\right)^{2/3} + (8\alpha)^{1/3} = 1$$

$$\Rightarrow 3\alpha^{1/3} = 1 \Rightarrow \alpha = \frac{1}{27}$$

39. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

(A) $2(\sec\theta - \tan\theta)$ (B) $2\sec\theta$ (C) $-2\tan\theta$ (D) 0

Ans. (C)

Sol. $\alpha_1 = \frac{2\sec\theta + \sqrt{4\sec^2\theta - 4}}{2}$ $\qquad \beta_2 = \frac{-2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2}$ $\{\because \alpha_2 > \beta_2\}$

$\alpha_1 = \sec\theta + |\tan\theta|$ $\{\because \alpha_1 > \beta_1\}$ $\qquad \beta_2 = -\tan\theta - \sec\theta$

$\alpha_1 = \sec\theta - \tan\theta$ $\left(\because \theta \in \left(-\frac{\pi}{6}, -\frac{\pi}{12}\right)\right)$

$\alpha_1 + \beta_2 = -2\tan\theta$

40. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to -

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

Ans. (C)

Sol. $\sqrt{3}\sin x + \cos x = 2\cos 2x$

$\Rightarrow \cos 2x = \cos\left(x - \frac{\pi}{3}\right)$

$\Rightarrow 2x = 2n\pi \pm \left(x - \frac{\pi}{3}\right)$

$x = (6n-1)\frac{\pi}{3}$ or $(6n+1)\frac{\pi}{9}$

$\Rightarrow x = -\frac{\pi}{3}, \frac{\pi}{9}, \frac{7\pi}{9}$ and $-\frac{5\pi}{9}$ in $(-\pi, \pi)$

$\Rightarrow$ sum = 0

41. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that

P(computer turns out to be defective given that is produced in plant T_1)

= 10P(computer turns out to be defective given that it is produced in plant T_2)

where P(E) denotes the probability of an event E. A computer produces in the factory is randomly selected and it does not turn out to be defective. Then the probabality that it is produced in plant T_2 is

(A) $\frac{36}{73}$ (B) $\frac{47}{79}$ (C) $\frac{78}{93}$ (D) $\frac{75}{83}$

Ans. (C)

Sol. $P(T_1)=\frac{20}{100}$ $\quad P(T_2)=\frac{80}{100}$

Let $P\left(\frac{D}{T_2}\right)=x$

$$P\left(\frac{D}{T_1}\right)=10x$$

$$P(D)=\frac{7}{100} \quad \text{(given)}$$

$$P(T_1)P\left(\frac{D}{T_1}\right)+P(T_2)P\left(\frac{D}{T_2}\right)=\frac{7}{100}$$

$$\frac{20}{100}\times 10x+\frac{80}{100}\times x=\frac{7}{100}$$

$$x=\frac{1}{40}$$

$$P\left(\frac{D}{T_2}\right)=\frac{1}{40}\Rightarrow P\left(\frac{\bar{D}}{T_2}\right)=\frac{39}{40}$$

$$P\left(\frac{D}{T_1}\right)=\frac{10}{40}\Rightarrow P\left(\frac{\bar{D}}{T_1}\right)=\frac{30}{40}$$

$$P\left(\frac{T_2}{\bar{D}}\right)=\frac{\frac{80}{100}\times\frac{39}{40}}{\frac{20}{100}\times\frac{30}{40}+\frac{80}{100}\times\frac{39}{40}}=\frac{78}{93}$$

SECTION–2 : (Maximum Marks : 32)

- This section contains **EIGHT** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct.
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS
- For each question, marks will be awarded in one of the following categories :

  Full Marks : +4 If only the bubble(s) corresponding to the correct option(s) is (are) darkened.

  Partial Marks : +1 For darkening a bubble corresponding **to each correct option**, Provided NO incorrect option is darkened.

  Zero Marks : 0 If none of the bubbles is darkened.

  Negative Marks : –2 In all other cases.
- for example, if (A), (C) and (D) are all the correct options for a question, darkening all these three will result in +4 marks; darkening only (A) and (D) will result in +2 marks; and darkening (A) and (B) will result in –2 marks, as a wrong option is also darkened

42. A solution curve of the differential equation $(x^2+xy+4x+2y+4)\frac{dy}{dx}-y^2=0, x>0$, passes through the point (1,3). The the solution curve-

(A) intersects y = x + 2 exactly at one point

(B) intersects y = x + 2 exactly at two points

(C) intersects $y = (x+2)^2$

(D) does NOT intersect $y = (x+3)^2$

Ans. (A,D)

Sol. $(x^2+xy+4x+2y+4)\frac{dy}{dx}-y^2=0$

$((x+2)^2+y(x+2))\frac{dy}{dx}=y^2$

Let x + 2 = X, y = Y

$(X)(X+Y)\frac{dY}{dX}=Y^2$

$-X^2dY = XYdY - Y^2dX$

$-X^2dY = Y(XdY - YdX)$

$-\frac{dY}{Y}=\frac{XdY-YdX}{X^2}$

$-\ell n|Y|=\left(\frac{Y}{X}\right)+C$

$-\ell n|y|=\frac{y}{x+2}+C$

$\because$ it is passing through (1, 3) $\quad -\ell n3 = 1 + C$

$C = -1 - \ell n3$

$\therefore$ curve $\frac{y}{x+2}+\ell n|y|-1-\ell n3=0, x>0$ ......(i)

put y = x + 2 in equation (i)

then $\frac{x+2}{x+2}+\ell n|x+2|-1-\ell n3=0$

x = 1, –5(reject)

$\therefore$ curve intersect y = x + 2 at point (1, 3)

for option (C), put $y = (x+2)^2$, we will get

$x + 2 + 2\ell n(x+2) = 1 + \ell n3$

Clearly left hand side is an increasing function

Hence, it is always greater than $2 + 2\ell n2$

therefore no solution

for option (C) put $y = (x + 3)^2$ in equation (i)

$$\frac{(x+3)^2}{x+2}+\ell n(x+3)^2-1-\ell n3=0$$

$$\frac{(x+3)^2}{x+2}+\ell n\frac{(x+3)^2}{3}-1=0$$

$\because \quad x > 0 \quad \Rightarrow \quad x + 3 > x + 2$

and $x + 3 > 3$

So $\dfrac{(x+3)^2}{x+2}+\ell n\dfrac{(x+3)^2}{3}>1$

$\therefore \quad \dfrac{(x+3)^2}{x+2}+\ell n\dfrac{(x+3)^2}{3}-1=0$ has no solution

$\Rightarrow$ curve $y = (x + 3)^2$ does not intersect

43. Consider a pyramid OPQRS located in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP= 3. The point S is directly above the mid-point T of diagonal OQ such that TS = 3. Then-

(A) the acute angle between OQ and OS is $\dfrac{\pi}{3}$

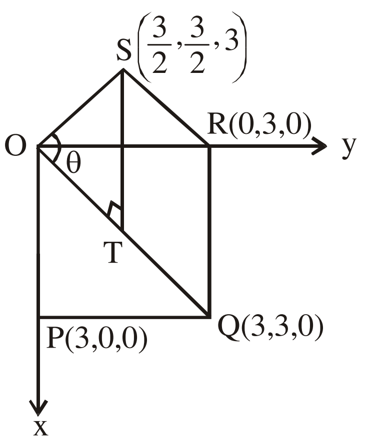
(B) the equaiton of the plane containing the triangle OQS is $x - y = 0$

(C) the length of the perpendicular from P to the plane containing the triangle OQS is $\dfrac{3}{\sqrt{2}}$

(D) the perpendicular distance from O to the straight line containing RS is $\sqrt{\dfrac{15}{2}}$

Ans. (B,C,D)

Sol.



Given OP = OR = 3 and OPQR is a square

$\Rightarrow \quad OQ = 3\sqrt{2} \quad \Rightarrow \quad OT = \frac{3}{\sqrt{2}}$ and ST = 3

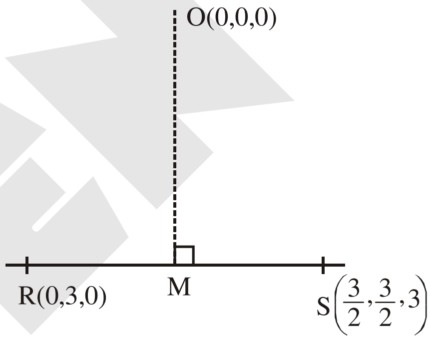
using ΔSOT, $\tan\theta = \frac{ST}{OT} = \sqrt{2} \quad \Rightarrow \quad \theta = \tan^{-1}\sqrt{2}$

clearly, equation of plane containing triangle OQS is Y – X = 0

Also, length of perpendicular from P to the plane containing the triangle OQS is PT = $\frac{3}{\sqrt{2}}$

Also equation of RS is $\bar{r} = 3\hat{j} + t\left(\frac{3}{2}\hat{i} - \frac{3}{2}\hat{j} + 3\hat{k}\right)$

$= \left(\frac{3t}{2}, 3 - \frac{3t}{2}, 3t\right)$



Let co-ordinates of $M = \left(\frac{3t}{2}, 3 - \frac{3t}{2}, 3t\right)$

$\because \quad \overrightarrow{OM}.\overrightarrow{RS} = 0$

$\Rightarrow \quad \frac{9}{4}t - \frac{3}{2}\left(3 - \frac{3t}{2}\right) + 9t = 0 \quad \Rightarrow \quad \frac{9t}{2} + 9t = \frac{9}{2} \quad \Rightarrow \quad t = \frac{1}{3}$

$\therefore \quad M = \left(\frac{1}{2}, \frac{5}{2}, 1\right) \quad \Rightarrow \quad OM = \sqrt{\frac{1}{4} + \frac{25}{4} + 1} = \sqrt{\frac{30}{4}} = \sqrt{\frac{15}{2}}$

44. In a triangle XYZ, let x,y,z be the lengths of sides opposite to the angles X,Y,Z, respectively and 2s = x + y + z. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then-

(A) area of the triangle XYZ is $6\sqrt{6}$

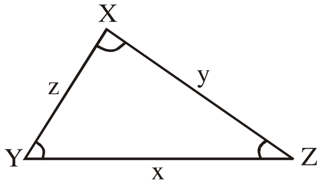
(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

Ans. (A,C,D)

Sol.



Let $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = k$

$s - x = 4k \quad \Rightarrow \quad y + z - x = 8k$

$s - y = 3k \quad \Rightarrow \quad x + z - y = 6k$

$s - z = 2k \quad \Rightarrow \quad x + y - z = 4k$

$\Rightarrow \quad x = 5k,\ y = 6k,\ z = 7k$

$\Rightarrow \quad 3s - (x + y + z) = 9k$

$s = 9k$

Let r be inradius

$$\Rightarrow \quad \pi r^2 = \frac{8\pi}{3}$$

$$\pi\left(\frac{\Delta}{s}\right)^2 = \frac{8\pi}{3}$$

$$\Delta = \sqrt{\frac{8}{3}}.s$$

$$\sqrt{s(s-x)(s-y)(s-z)} = \sqrt{\frac{8}{3}}.s$$

$$\sqrt{9k.4k.3k.2k} = \sqrt{\frac{8}{3}}9k$$

$$\sqrt{24.9k^2} = \sqrt{\frac{8}{3}}.9k$$

$k = 1$

$\Rightarrow \quad x = 5,\ y = 6,\ z = 7$

$$\Delta = \sqrt{\frac{8}{3}}.9k = \sqrt{\frac{8}{3}}.9 = 6\sqrt{6}$$

$$R = \text{circumradius} = \frac{xyz}{4\Delta} = \frac{5.6.7}{4.6\sqrt{6}} = \frac{35}{4\sqrt{6}}$$

$$\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{r}{4R} = \frac{\frac{\Delta}{s}}{\frac{xyz}{\Delta}} = \frac{\Delta^2}{s.xyz} = \frac{36.6}{9.5.6.7} = \frac{4}{35}$$

$$\cos Z = \frac{x^2+y^2-z^2}{2xy} = \frac{25+36-49}{2.5.6} = \frac{1}{5}$$

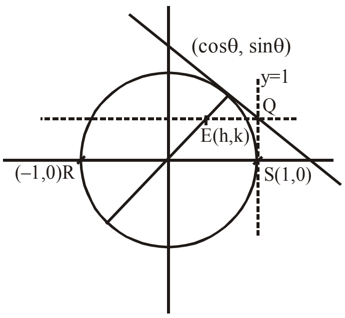
$$\sin^2\left(\frac{X+Y}{2}\right) = \cos^2\frac{Z}{2} = \frac{1+\cos Z}{2} = \frac{1+\frac{1}{5}}{2} = \frac{3}{5}$$

45. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1,0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. then the locus of E passes through the point(s)-

(A) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (C) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Ans. (A,C)

Sol.



Tangent at P : x cosθ + y sinθ = 1 ...(i)

Tangent at S : x = 1 ...(ii)

$\therefore$ By (i) & (ii) : $Q\left(1, \frac{1-\cos\theta}{\sin\theta}\right)$

Line through Q parallel to RS :

$y = \frac{1-\cos\theta}{\sin\theta} \Rightarrow y = \tan\frac{\theta}{2}$ ......(iii)

Normal at P : $y = \frac{\sin\theta}{\cos\theta}x \Rightarrow y = \tan\theta.x$ ......(iv)

Point of intersection of equation (iii) and (iv), E : $h = \frac{1-\tan^2\frac{\theta}{2}}{2}; k = \tan\frac{\theta}{2}$

eliminating θ : $h = \frac{1-k^2}{2} \Rightarrow y^2 = 1-2x$

Options (A) and (C) satisfies the locus.

46. The circle C_1: $x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then-

(A) $Q_2Q_3 = 12$

(B) $R_2R_3 = 4\sqrt{6}$

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$

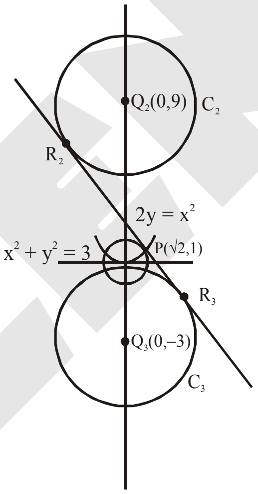
(D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

Ans. (A,B,C)

Sol. On solving $x^2 + y^2 = 3$ and $x^2 = 2y$ we get point $P(\sqrt{2},1)$

Equation of tangent at P

$\sqrt{2}.x + y = 3$



Let Q_2 be (0,k) and radius is $2\sqrt{3}$

$$\therefore \left|\frac{\sqrt{2}(0)+k-3}{\sqrt{2+1}}\right| = 2\sqrt{3}$$

$\therefore$ k = 9, – 3

$Q_2(0,9)$ and $Q_3(0, -3)$

hence $Q_2Q_3 = 12$

R_2R_3 is internal common tangent of circle C_2 and C_3

$$\therefore R_2R_3 = \sqrt{(Q_2Q_3)^2 - (2\sqrt{3}+2\sqrt{3})^2}$$

$$= \sqrt{12^2 - 48} = \sqrt{96} = 4\sqrt{6}$$

Perpendicular distance of origin O from R_2R_3 is equal to radius of circle $C_1 = \sqrt{3}$

Hence area of $\Delta OR_2R_3 = \frac{1}{2}\times(R_2R_3)\sqrt{3} = \frac{1}{2}.4\sqrt{6}.\sqrt{3} = 6\sqrt{2}$

Perpendicular Distance of P from $Q_2Q_3 = \sqrt{2}$

$\therefore$ Area of $\Delta PQ_2Q_3 = \frac{1}{2}\times 12\times\sqrt{2} = 6\sqrt{2}$

MATHEMATICS

47. Let $f : R \to R$, $g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then-

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$ (C) $h(0) = 16$ (D) $h(g(3)) = 36$

Ans. (B,C)

Sol. (A) $f'(x) = 3x^2 + 3$

so, $g'(2) = \frac{1}{f'(0)}$ (Given $g(x) = f^{-1}(x)$)

$\Rightarrow \quad g'(2) = \frac{1}{3}$

(B) $h(g(g(x))) = x$

$$h'(g(g(x)) = \frac{1}{g'(g(x)).g'(x)}$$

Now, $g(g(x)) = 1$

$g(x) = f(1) = 6$

$\therefore \; x = f(6) = 236$

so $h'(1) = \frac{1}{g'(6).g'(236)} = \frac{1}{\frac{1}{6}.\frac{1}{111}} \Rightarrow h'(1) = 666$

(C) $g(g(x)) = 0$

$\therefore \; g(x) = g^{-1}(0) \Rightarrow g(x) = f(0) \Rightarrow g(x) = 2 \Rightarrow x = g^{-1}(2) \Rightarrow x = f(2) \Rightarrow x = 16$

so $h(0) = 16$

(D) $g(x) = 3 \Rightarrow x = g^{-1}(3) \Rightarrow x = f(3) \Rightarrow x = 38$

so $h(g(3)) = 38$

48. Let $f : (0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0,\infty)$ and $f(1) \neq 1$. Then

(A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right) = 1$ (B) $\lim\limits_{x\to 0^+} x f\left(\frac{1}{x}\right) = 2$ (C) $\lim\limits_{x\to 0^+} x^2 f'(x) = 0$ (D) $|f(x)| \le 2$ for all $x \in (0,2)$

Ans. (A)

Sol. Let y = $f(x)$

$$\frac{dy}{dx}+\frac{y}{x}=2 \text{ (linear differential equation)}$$

$$\therefore\ y.e^{\int\frac{dx}{x}}=2\int e^{\int\frac{dx}{x}}=2\int e^{\int\frac{dx}{x}}dx+c$$

$$\Rightarrow yx = 2\int x dx + c$$

$$\therefore\ yx = x^2 + c$$

$$\Rightarrow f(x) = x + \frac{c}{x}\ ;\ \text{As } f(1) \neq 1 \Rightarrow c \neq 0$$

$$\Rightarrow\ f'(x)=1-\frac{c}{x^2}, c\neq 0$$

(A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right)=\lim\limits_{x\to 0^+}\left(1-cx^2\right)=1$

(B) $\lim\limits_{x\to 0^+} xf\left(\frac{1}{x}\right)=\lim\limits_{x\to 0^+} x\left(\frac{1}{x}+cx\right)=\lim\limits_{x\to 0^+}\left(1+cx^2\right)=1$

(C) $\lim\limits_{x\to 0^+} x^2 f'(x)=\lim\limits_{x\to 0^+} x^2\left(1-\frac{c}{x^2}\right)=\lim\limits_{x\to 0^+}\left(x^2-c\right)=-c$

(D) $f(x)=x+\frac{c}{x}, c\neq 0$

for c > 0

$\therefore\ \lim\limits_{x\to 0^+} f(x)=\infty \Rightarrow$ function is not bounded in (0,2).

49. Let $P=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & \alpha\\ 3 & -5 & 0\end{bmatrix}$, where $\alpha \in R$, Suppose $Q = [q_{ij}]$ is a matrix such that PQ = kI, where $k \in R$, $k\neq 0$ and I is the identity matrix of order 3. If $q_{23}=-\frac{k}{8}$ and $\det(Q)=\frac{k^2}{2}$, then-

(A) $\alpha = 0,\ k = 8$

(B) $4\alpha - k + 8 = 0$

(C) $\det(P\,adj(Q)) = 2^9$

(D) $\det(Q\,adj(P)) = 2^{13}$

Ans. (B,C)

Sol. PQ = kI

$|P|.|Q| = k^3$

$\Rightarrow |P| = 2k \neq 0 \Rightarrow$ P is an invertible matrix

$\because$ PQ = kI

$\therefore Q = kP^{-1}I$

$$\therefore Q = \frac{adj.P}{2}$$

$$\because q_{23} = -\frac{k}{8}$$

$$\therefore \frac{-(3\alpha+4)}{2} = -\frac{k}{8} \Rightarrow k = 4$$

$\therefore |P| = 2k \Rightarrow k = 10 + 6\alpha$ ...(i)

Put value of k in (i).. we get α =–1

$\therefore 4\alpha - k + 8 = 0$

$$\& \det(P(adj.Q)) = |P|\,|adj.Q| = 2k.\left(\frac{k^2}{2}\right)^2 = \frac{k^5}{2} = 2^9$$

SECTION–3 : (Maximum Marks : 15)

- This section contains **FIVE** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct answer is darkened.

Zero Marks : 0 In all other cases.

50. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots\ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is

Ans. 5

Sol. Coefficient of x^2 in the expansion of

$(1 + x)^2 + (1 + x)^3 + \ldots\ldots\ (1 + x)^{49} + (1 + mx)^{50}$ is

${}^2C_2 + {}^3C_2 + \ldots\ldots\ {}^{49}C_2 + {}^{50}C_2m^2 = (3n + 1)\ {}^{51}C_3$

${}^3C_3 + {}^3C_2 + \ldots\ldots\ {}^{49}C_2 + {}^{50}C_2m^2 = (3n + 1)\,{}^{51}C_3$

${}^{50}C_3 + {}^{50}C_2m^2 = (3n + 1)\ {}^{51}C_3$

$$\frac{50.49.48}{6} + \frac{50.49}{2}m^2 = (3n + 1)\frac{51.50.49}{6}$$

$m^2 = 51n + 1$

must be a perfect quuared

$\Rightarrow$ n = 5 and m = 16

Ans. $\Rightarrow$ 5

51. The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

Ans. 2

Sol. $x^3\begin{vmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{vmatrix} = 10$

$$\Rightarrow \quad x^3\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} + x^3.x^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 0$$

$\Rightarrow \quad x^3(25 - 23) + 6x^6.2 = 10$

$\Rightarrow \quad 6x^6 + x^3 - 5 = 0$

$\Rightarrow \quad x^3 = \frac{5}{6}, -1$

two real solutions

52. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and r, s $\in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is

Ans. 1

Sol. $z = \omega$

$P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$, $P^2 = -I$

$$\Rightarrow \quad P^2 = \begin{bmatrix} \omega^{2r} + \omega^{4s} & \omega^{r+2s}((-1)^r + 1) \\ \omega^{r+2s}((-1)^r + 1) & \omega^{4s} + \omega^{2r} \end{bmatrix} = -I$$

$\Rightarrow \quad (-1)^r + 1 = 0 \Rightarrow$ r is odd $\Rightarrow$ r = 1,3

also $\omega^{2r} + \omega^{4s} = -1 \quad \therefore \quad r \neq 3$

by r = 1 $\Rightarrow \omega^2 + \omega^{4s} = -1 \Rightarrow$ s = 1

(r, s) = (1, 1)

only 1 pair

53. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x - 1$ is

Ans. 1

Sol. Let $f(x) = \int_0^x \frac{t^2}{1+t^4}dt - 2x + 1$

$\Rightarrow \quad f'(x) = \frac{x^2}{1+x^4} - 2$

as $\frac{1+x^4}{x^2} \geq 2 \quad \Rightarrow \quad \frac{x^2}{1+x^4} \leq \frac{1}{2}$

$\Rightarrow \quad f'(x) \leq -\frac{3}{2} \quad \Rightarrow \quad f(x)$ is continuous and decreasing

$f(0) = 1$ and $f(1) = \int_0^1 \frac{t^2}{1+t^4}dt - 2 \leq -\frac{3}{2}$

by IVT $f(x) = 0$ possesses exactly one solution in $[0, 1]$

54. Let $\alpha, \beta \in R$ be such that $\lim\limits_{x\to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$. Then $6(\alpha + \beta)$ equals

Ans. 7

Sol. If $\alpha \neq 1$, then $\lim\limits_{x\to 0} \frac{x \sin\beta x}{\frac{\alpha x - \sin x}{x}} = 0$

$\therefore \quad \alpha = 1 \quad \Rightarrow \quad \lim\limits_{x\to 0} \frac{\beta x^3 \frac{\sin\beta x}{\beta x}}{x^3\left(\frac{x - \sin x}{x^3}\right)} = \frac{\beta}{1/6}$

$\Rightarrow \quad 6\beta = 1 \quad \Rightarrow \quad \beta = \frac{1}{6}$

$6(\alpha + \beta) = 7$