CBSE Sample Paper-03
Mathematics
Class – XII

Time allowed: 3 hours	**Answers**	Maximum Marks: **100**

Section A

1. Solution:

    No. $(2,4) \in R \ but \ (4,2) \notin R$

2. Solution:

    $$\left|\vec{a}\right| = \sqrt{(3)^2 + (-2)^2 + (-5)^2} = \sqrt{38}$$
    $$\therefore l = \frac{3}{\sqrt{38}}, m = \frac{-2}{\sqrt{38}}, n = \frac{-5}{\sqrt{38}}$$

3. Solution: $[0, \pi]$

4. Solution:

    $$2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$
    $$2Y = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

5. Solution:

    $$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 6(17) & 6(18) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6 \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0 (\because R_1 = R_3)$$

6. Solution:

    $$(AB)' = B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \begin{bmatrix} -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$$

Section B

7. Solution:

Let $x = \tan\theta$

$R.H.S = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) = \tan^{-1}\left(\dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}x$

$= 2\tan^{-1}x + \tan^{-1}x = \tan^{-1}x + \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right)$

If $x = \tan y, \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right) = \tan^{-1}\left(\dfrac{2\tan y}{1 - \tan^2 y}\right) = \tan^{-1}(\tan 2y) = 2y = 2\tan^{-1}x$

8.   Solution:

The position vectors of A, B,C are 2i, j, 2k respectively.

$\therefore \overrightarrow{AB}$ = p.v of $\vec{B}$ - p.v of $\vec{A}$ =j-2i

$\therefore \overrightarrow{BC}$ = p.v of $\vec{C}$ - p.v of $\vec{B}$ =2k-j

$\therefore \overrightarrow{CA}$ = p.v of $\vec{A}$ - p.v of $\vec{C}$ =2i-2k

$\left|\overrightarrow{AB}\right|^2 = (2)^2 + (1)^2 = 5 \Rightarrow \left|\overrightarrow{AB}\right| = \sqrt{5}$

$\left|\overrightarrow{BC}\right|^2 = (1)^2 + (2)^2 = 5 \Rightarrow \left|\overrightarrow{BC}\right| = \sqrt{5}$

$\left|\overrightarrow{CA}\right|^2 = (2)^2 + (2)^2 = 8 \Rightarrow \left|\overrightarrow{CA}\right| = \sqrt{8}$

$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right| \neq \left|\overrightarrow{CA}\right|$

Thus, A, B,C form the vertices of an isosceles triangle.

9.   Solution:

(a)

$P(B) = 1 - P(B^C) = 1 - 1/3 = 2/3$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \dfrac{5}{6} = P(A) + \dfrac{2}{3} - \dfrac{1}{3} \Rightarrow P(A) = \dfrac{1}{2}$

(b)

$P(not A\, and\, not B) = P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)] = 1 - [1/4 + 1/2 - 1/8] = 3/8$

10.   Solution:

$$f(x) = -2x^3 - 9x^2 - 12x + 1$$
$$f'(x) = -6x^2 - 18x - 12 = -6(x^2 + 3x + 2) = -6(x+1)(x+2)$$
$$f'(x) = 0 \Rightarrow x = -1, -2$$

We study the sign of f'(x) on the intervals $(-\infty, -2), (-2, -1)$ *and* $(-1, \infty)$.

If x<-2, f'(x) is negative, i.e f(x) is strictly decreasing.

If -2<x<-1, f'(x) is positive, i.e f(x) is strictly increasing.

If x>-1, f'(x) is negative, i.e f(x) is strictly decreasing.

11. Solution:

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$$

$$Let \ z = x + y \Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx} \quad \Rightarrow \frac{dz}{dx} - 1 = \sin z + \cos z$$

$$\therefore \frac{dz}{\sin z + \cos z + 1} = dx$$

Integrating both sides, we get

$$\int \frac{dz}{\sin z + \cos z + 1} = \int dx$$

$$\int \frac{dz}{\dfrac{2\tan z/2}{1 + \tan^2 z/2} + \dfrac{1 - \tan^2 z/2}{1 + \tan^2 z/2} + 1} = x + c$$

$$\int \frac{\sec^2 z/2}{2\tan z/2 + 2} = x + c$$

$$Let \ \tan z/2 = t \Rightarrow \frac{1}{2}\sec^2 z/2 = dt$$

$$\therefore \int \frac{dt}{t+1} = x + c$$

$$\therefore \log|t+1| = x + c$$

$$\log|\tan(x+y)/2 + 1| = x + c$$

12.   Solution:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \begin{vmatrix} 1/a+1 & 1/a & 1/a \\ 1/b & 1/b+1 & 1/b \\ 1/c & 1/c & 1/c+1 \end{vmatrix} (taking \ a,b,c \ common \ from \ R_1, R_2, R_3)$$

$$= \begin{vmatrix} 1/a+1/b+1/c+1 & 1/a+1/b+1/c+1 & 1/a+1/b+1/c+1 \\ 1/b & 1/b+1 & 1/b \\ 1/c & 1/c & 1/c+1 \end{vmatrix} (R_1 \to R_1 + R_2 + R_3)$$

$$= abc(1/a + 1/b + 1/c + 1) \begin{vmatrix} 1 & 1 & 1 \\ 1/b & 1/b+1 & 1/b \\ 1/c & 1/c & 1/c+1 \end{vmatrix}$$

$$= abc(1/a + 1/b + 1/c + 1) \begin{vmatrix} 1 & 0 & 0 \\ 1/b & 1 & 0 \\ 1/c & 0 & 1 \end{vmatrix} = abc(1/a + 1/b + 1/c + 1)$$

13.   Solution:

$$\frac{dy}{dx} = \frac{-1}{(x^2 - 2x + 2)^2}(2x - 2)$$

$$\because slope = 0 \Rightarrow \frac{dy}{dx} = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$$

$$\because x = 1, y = \frac{1}{1 - 2 + 2} = 1$$

$\therefore the\ line\ is\ \tan gent\ at\ the\ po\operatorname{int}(1,1)$

$\therefore equation\ of\ required\ line\ is\ y - 1 = 0.(x - 1) \Rightarrow y = 1$

14. Solution:

1-1:

Let f(x₁)=f(x₂)

$$\Rightarrow \frac{x_1 - 1}{x_1 + 1} = \frac{x_2 - 1}{x_2 + 1} \Rightarrow x_1 x_2 + x_1 - x_2 - 1 = x_1 x_2 + x_2 - x_1 - 1 \Rightarrow x_1 = x_2$$

$\therefore f\ is\ 1 - 1.$

Range:

$$Let\ f(x) = k \Rightarrow \frac{x - 1}{x + 1} = k \Rightarrow x = \frac{1 + k}{1 - k}$$

Thus, x is not defined when k=1.

Also f(1)=0

$\therefore Range = \mathbb{R} \sim \{0,1\}$

Thus f is invertible if range= $\mathbb{R} \sim \{0,1\}$

Inverse:

$Let\ x \in \mathbb{R} \sim \{0,1\}.$

$$Let\ f^{-1}(x) = k \Rightarrow x = f(k) \Rightarrow \frac{k - 1}{k + 1} = x \Rightarrow k = \frac{1 + x}{1 - x}$$

$$\therefore f^{-1}(x) = \frac{1+x}{1-x}, x \in \mathbb{R} \sim \{0,1\}$$

Thus,

$$f \circ f^{-1} = f\left(\frac{1+x}{1-x}\right) = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{2x}{2} = x$$

15. Solution:

$$\log(xy) = x^2 + y^2$$
$$\therefore \log(x) + \log(y) = x^2 + y^2$$

Differentiating both sides w.r.t x, we get

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - 2y\right) = 2x - \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(2x^2 - 1)}{x(1 - 2y^2)}$$

16. Solution:

$$\left(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}\right).\left(|\vec{a}|\vec{b} - |\vec{b}|\vec{a}\right) = |\vec{a}||\vec{a}|\vec{b}.\vec{b} + |\vec{b}||\vec{a}|\vec{a}.\vec{b} - |\vec{a}||\vec{b}|\vec{b}.\vec{a} - |\vec{b}||\vec{b}|\vec{a}.\vec{a}$$

$$= |\vec{a}|^2|\vec{b}|^2 + |\vec{b}||\vec{a}|\vec{a}.\vec{b} - |\vec{a}||\vec{b}|\vec{b}.\vec{a} - |\vec{b}|^2|\vec{a}|^2 = 0 (\because \vec{a}.\vec{b} = \vec{b}.\vec{a})$$

17. Solution:

$$I = \int \log(1 + x^2)dx = \int 1.\log(1 + x^2)dx$$

Integrating by parts,

$$I = \log(1 + x^2)\int 1.dx - \int (\frac{d}{dx}\log(1 + x^2)\int 1.dx$$

$$= \log(1 + x^2)x - \int \frac{2x}{1 + x^2}(x)dx$$

$$= x\log(1 + x^2) - 2\int \left(1 - \frac{1}{1 + x^2}\right)dx$$

$$= x\log(1 + x^2) - 2x + 2\tan^{-1}x + c$$

18. Solution:

$$Dis\tan ce = \left| \frac{\left(\vec{b_1} \times \vec{b_2}\right).\left(\vec{a_2} - \vec{a_1}\right)}{\left|\left(\vec{b_1} \times \vec{b_2}\right)\right|} \right|$$

$$\vec{a_1} = i + 2j + k, \vec{a_2} = 2i - j - k, \vec{b_1} = i - j + k, \vec{b_2} = 2i + j + 2k,$$

$$\therefore \vec{b_1} \times \vec{b_2} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -3i + 3k$$

$$\therefore \left|\vec{b_1} \times \vec{b_2}\right| = \sqrt{(-3)^2 + (3)^2} = 3\sqrt{2}$$

$$\vec{a_2} - \vec{a_1} = i - 3j - 2k$$

$$\therefore d = \left| \frac{(-3i + 3k).(i - 3j - 2k)}{3\sqrt{2}} \right| = \frac{3}{\sqrt{2}}$$

19. Solution:

$$\vec{a} = i + j - k, \vec{b} = 2i + 6j + k, \vec{c} = i - 2j + k$$

$Equation\ of\ plane:$

$$\left(\vec{r} - \vec{a}\right).[\left(\vec{b} - \vec{a}\right) \times \vec{c}] = 0$$

$$\left(\vec{b} - \vec{a}\right) = i + 5j + 2k$$

$$\left(\vec{b} - \vec{a}\right) \times \vec{c} = \begin{vmatrix} i & j & k \\ 1 & 5 & 2 \\ 1 & -2 & 1 \end{vmatrix} = 9i + j - 7k$$

$$\therefore [\vec{r} - (i + j - k)].(9i + j - 7k) = 0$$

$$\Rightarrow \vec{r}.(9i + j - 7k) = 17$$

Section C

20. Solution:

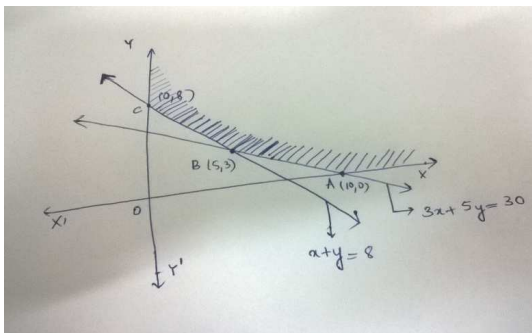Suppose tailor A works for x days and tailor B works for y days.

Then, Cost Z= 15x + 20y

	Tailor A	Tailor B	Min Requirement
Shirts	6	10	60
Pants	4	4	32
Cost per day	15	20	

The mathematical formulation of the problem is as follows:

Min Z= 15x+20y

$$6x + 10y \geq 60 \Rightarrow 3x + 5y \geq 30$$

s.t $4x + 4y \geq 32 \Rightarrow x + y \geq 8$

$$x \geq 0, y \geq 0$$

We graph the above inequalities. The feasible region is as shown in the figure. We observe the feasible region is unbounded and the corner points are A,B and C. The co-ordinates of the corner points are (10,0), (5,3),(0,8).

Corner Point	Z=15x +20y
(10,0)	150
(5,3)	**135**
(0,8)	160

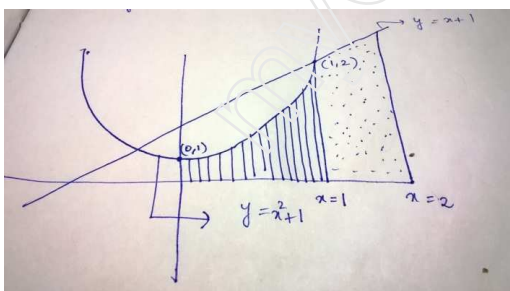Thus cost is minimized by hiring A for 5 days and hiring B for 3 days.

21. Solution:

The curves are x=2, y=x+1, y= x²+1.

The point of intersection of the curves y=x+1, y=x²+1:

$$x^2 + 1 = x + 1$$
$$\Rightarrow x(1-x) = 0 \Rightarrow x = 0, x = 1$$

The shaded area is the required area.

$$Area = \int_{0}^{1} y_1 dx + \int_{1}^{2} y_2 dx$$

$$= \int_0^1 (x^2+1)dx + \int_1^2 (x+1)dx$$

$$= \frac{x^3}{3} + x \Big|_0^1 + \frac{x^2}{2} + x \Big|_1^2 = \frac{23}{6}$$

22. Solution:

Let A_i, B_i, C_i denote the events of winning A, B, C in their respective ith attempt.

$P(A_i) = 2/3$, $P(B_i) = 1/2$, $P(C_i) = 1/4$

$$P(\overline{A_i}) = \frac{1}{3}, P(\overline{B_i}) = \frac{1}{2}, P(\overline{C_i}) = \frac{3}{4}$$

$$P(A \ wins) = P(A_1 \ or \ \overline{A_1}\overline{B_1}\overline{C_1}A_2 \ or \ \overline{A_1}\overline{B_1}\overline{C_1}\overline{A_2}\overline{B_2}\overline{C_2}A_3 \cdots)$$

$$= P(A_1) + P(\overline{A_1})P(\overline{B_1})P(\overline{C_1})P(A_2) + P(\overline{A_1})P(\overline{B_1})P(\overline{C_1})P(\overline{A_2})P(\overline{B_2})P(\overline{C_2})P(A_3) + \cdots$$

$$= \frac{2}{3} + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right) + \cdots$$

$$= \frac{2}{3}\left[1 + \frac{1}{8} + \left(\frac{1}{8}\right)^2 + \cdots\right] = \frac{16}{21}$$

$$P(B \ wins) = P(\overline{A_1}B_1 \ or \ \overline{A_1}\overline{B_1}\overline{C_1}\overline{A_2}B_2 \ or \ \overline{A_1}\overline{B_1}\overline{C_1}\overline{A_2}\overline{B_2}\overline{C_2}\overline{A_3}B_3 \cdots)$$

$$= P(\overline{A_1})P(B_1) + P(\overline{A_1})P(\overline{B_1})P(\overline{C_1})P(\overline{A_2})P(B_2) + P(\overline{A_1})P(\overline{B_1})P(\overline{C_1})P(\overline{A_2})P(\overline{B_2})P(\overline{C_2})P(\overline{A_3})P(B_3) + \cdots$$

$$= \frac{1}{6}\left[1 + \frac{1}{8} + \left(\frac{1}{8}\right)^2 + \cdots\right] = \frac{4}{21}$$

$$P(C \ wins) = 1 - P(A \ wins) - P(B \ wins) = 1 - \frac{16}{21} - \frac{4}{21} = \frac{1}{21}$$

23. Solution:

$$x = a\sec^3\theta, y = a\tan^3\theta$$

$$\frac{dx}{d\theta} = 3a\sec^2\theta(\sec\theta\tan\theta) = 3a\sec^3\theta\tan\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{3a\sec^3\theta\tan\theta}$$

$$\frac{dy}{d\theta} = 3a\tan^2\theta(\sec^2\theta)$$

$$\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right)\left(\frac{d\theta}{dx}\right) = 3a\tan^2\theta\sec^2\theta\frac{1}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = \frac{d}{d\theta}(\sin\theta)\frac{d\theta}{dx} = \cos\theta\frac{1}{3a\sec^3\theta\tan\theta} = \frac{1}{3a}\frac{\cos^5\theta}{\sin\theta}$$

$$\frac{d^2y}{dx^2}\Big|_{\theta=\frac{\pi}{4}} = \frac{1}{12a}$$

24.   Solution:

$$I = \int \frac{x^2 \, dx}{(x+3)\sqrt{3x+4}}$$

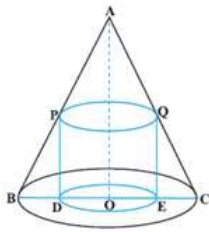Let $z = \sqrt{3x+4}$ $\therefore x = \frac{z^2-4}{3} \Rightarrow dx = \frac{2z}{3} dz$

$$\therefore I = \int \frac{\left(\frac{z^2-4}{3}\right)^2 \frac{2z}{3}}{\left(\frac{z^2-4}{3}+3\right) z} dz$$

$$= \frac{2}{9} \int \frac{(z^2-4)^2}{z^2+5} dz = \frac{2}{9} \int \left(z^2 - 13 + \frac{81}{z^2+5}\right) dz$$

$$= \frac{2}{9} \left[\frac{z^3}{3} - 13z + \frac{81}{\sqrt{5}} \tan^{-1} \frac{z}{\sqrt{5}}\right] + c$$

$$= \frac{2}{27}(3x+4)^{3/2} - \frac{26}{9}\sqrt{3x+4} + \frac{18}{\sqrt{5}} \tan^{-1} \sqrt{\frac{3x+4}{5}} + c$$

25.   Solution:

Let OC=r be the radius of the cone and OA=h be its height.

Let a cylinder with radius OE = x and height h' be inscribed in the cone.

Surface Area = $2\pi x h'$

$\because \triangle QEC \sim \triangle AOC,$

$$\frac{QE}{AO} = \frac{CE}{CO} \Rightarrow \frac{h'}{h} = \frac{r-x}{r} \Rightarrow h' = h\left(\frac{r-x}{r}\right)$$

$$\therefore S = S(x) = 2\pi x h' = 2\pi x h\left(\frac{r-x}{r}\right) = \frac{2\pi h}{r}(rx - x^2)$$

$$S'(x) = \frac{2\pi h}{r}(r - 2x)$$

$$S''(x) = \frac{2\pi h}{r}(-2)$$

$$S'(x) = 0 \Rightarrow x = r/2$$

$$Also, \ S''(r/2) = \frac{-4\pi h}{r} < 0$$

Hence, x=r/2 is a point of maxima.

Thus, the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

26.   Solution:

$$Let\ A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}, b = \begin{bmatrix} 4 \\ -1 \\ 9 \end{bmatrix}$$

$$|A| = 14 \neq 0, A^{-1} = \frac{1}{14}\begin{bmatrix} 2 & 4 & 2 \\ 8 & -5 & 1 \\ -4 & 1 & -3 \end{bmatrix}, X = A^{-1}b = \frac{1}{14}\begin{bmatrix} 2 & 4 & 2 \\ 8 & -5 & 1 \\ -4 & 1 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 9 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -14 \\ 28 \\ 42 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore x = -1, y = 2, z = 3$$