CBSE Sample Paper-03 (Solved)
Mathematics
Class – XII

Time allowed: 3 hours **Answers** Maximum Marks: 100

Section A

1. Solution:

$$2X=\begin{bmatrix}10 & 0\\ 2 & 8\end{bmatrix}\Rightarrow X=\begin{bmatrix}5 & 0\\ 1 & 4\end{bmatrix}$$
$$2Y=\begin{bmatrix}4 & 0\\ 2 & 2\end{bmatrix}\Rightarrow X=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$$

2. Solution:

$$\begin{vmatrix}102 & 18 & 36\\ 1 & 3 & 4\\ 17 & 3 & 6\end{vmatrix}=\begin{vmatrix}6(17) & 6(18) & 6(6)\\ 1 & 3 & 4\\ 17 & 3 & 6\end{vmatrix}=6\begin{vmatrix}17 & 3 & 6\\ 1 & 3 & 4\\ 17 & 3 & 6\end{vmatrix}=0(\because R_1=R_3)$$

3. Solution:

$$(AB)'=B'A'=\begin{bmatrix}1\\ 3\\ -6\end{bmatrix}\begin{bmatrix}-2 & 4 & 5\end{bmatrix}=\begin{bmatrix}-2 & 4 & 5\\ -6 & 12 & 15\\ 12 & -24 & -30\end{bmatrix}$$

4. Solution:

No. $(2,4)\in R\ but\ (4,2)\notin R$

5. Solution:

$$|\vec{a}|=\sqrt{(3)^2+(-2)^2+(-5)^2}=\sqrt{38}$$
$$\therefore l=\frac{3}{\sqrt{38}},m=\frac{-2}{\sqrt{38}},n=\frac{-5}{\sqrt{38}}$$

6. Solution:

$[0,\pi]$

Section B

7. Solution:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\begin{vmatrix} 1/a+1 & 1/a & 1/a \\ 1/b & 1/b+1 & 1/b \\ 1/c & 1/c & 1/c+1 \end{vmatrix} (taking\ a,b,c\ common\ from\ R_1,R_2,R_3)$$

$$= \begin{vmatrix} 1/a+1/b+1/c+1 & 1/a+1/b+1/c+1 & 1/a+1/b+1/c+1 \\ 1/b & 1/b+1 & 1/b \\ 1/c & 1/c & 1/c+1 \end{vmatrix} (R_1 \to R_1+R_2+R_3)$$

$$= abc(1/a+1/b+1/c+1)\begin{vmatrix} 1 & 1 & 1 \\ 1/b & 1/b+1 & 1/b \\ 1/c & 1/c & 1/c+1 \end{vmatrix}$$

$$= abc(1/a+1/b+1/c+1)\begin{vmatrix} 1 & 0 & 0 \\ 1/b & 1 & 0 \\ 1/c & 0 & 1 \end{vmatrix} = abc(1/a+1/b+1/c+1)$$

8. Solution:

$$\frac{dy}{dx} = \frac{-1}{(x^2-2x+2)^2}(2x-2)$$

$$\because slope = 0 \Rightarrow \frac{dy}{dx} = 0 \Rightarrow 2x-2=0 \Rightarrow x=1$$

$$\because x=1, y=\frac{1}{1-2+2}=1$$

$$\therefore the\ line\ is\ \tan gent\ at\ the\ po\text{int}\ (1,1)$$

$$\therefore equation\ of\ required\ line\ is\ y-1=0.(x-1) \Rightarrow y=1$$

9. Solution:

1-1:

Let f(x_1)= f(x_2)

$$\Rightarrow \frac{x_1-1}{x_1+1} = \frac{x_2-1}{x_2+1} \Rightarrow x_1x_2+x_1-x_2-1 = x_1x_2+x_2-x_1-1 \Rightarrow x_1 = x_2$$

$$\therefore f\ is\ 1-1.$$

Range:

$$Let\ f(x)=k \Rightarrow \frac{x-1}{x+1}=k \Rightarrow x=\frac{1+k}{1-k}$$

Thus, x is not defined when k=1.

Also f(1)=0

$\therefore Range = \mathbb{R} \sim \{0,1\}$

Thus f is invertible if range= $\mathbb{R} \sim \{0,1\}$

Inverse:

$Let\ x \in \mathbb{R} \sim \{0,1\}.$

$$Let\ \ f^{-1}(x) = k \Rightarrow x = f(k) \Rightarrow \frac{k-1}{k+1} = x \Rightarrow k = \frac{1+x}{1-x}$$

$$\therefore f^{-1}(x) = \frac{1+x}{1-x}, x \in \mathbb{R} \sim \{0,1\}$$

Thus,

$$f \circ f^{-1} = f\left(\frac{1+x}{1-x}\right) = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{2x}{2} = x$$

10. Solution:

$$\log(xy) = x^2 + y^2$$
$$\therefore \log(x) + \log(y) = x^2 + y^2$$

Differentiating both sides w.r.t x, we get

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$
$$\frac{dy}{dx}\left(\frac{1}{y} - 2y\right) = 2x - \frac{1}{x}$$
$$\therefore \frac{dy}{dx} = \frac{y(2x^2-1)}{x(1-2y^2)}$$

11. Solution:

$Let\ x = \tan\theta$

$$R.H.S = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right) = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1} x$$

$$= 2\tan^{-1} x + \tan^{-1} x = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$If\ \ x = \tan y, \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan y}{1-\tan^2 y}\right) = \tan^{-1}(\tan 2y) = 2y = 2\tan^{-1} x$$

12. Solution:

The position vectors of A, B,C are 2i, j, 2k respectively.

$\therefore \overrightarrow{AB}$ = p.v of $\vec{B}$ - p.v of $\vec{A}$ =j-2i

$\therefore \overrightarrow{BC}$ = p.v of $\vec{C}$ - p.v of $\vec{B}$ =2k-j

$\therefore \overrightarrow{CA}$ = p.v of $\vec{A}$ - p.v of $\vec{C}$ =2i-2k

$$\left|\overrightarrow{AB}\right|^2 = (2)^2 + (1)^2 = 5 \qquad \left|\overrightarrow{AB}\right| \quad \sqrt{5}$$
$$\left|\overrightarrow{BC}\right|^2 = (1)^2 + (2)^2 = 5 \qquad \left|\overrightarrow{BC}\right| \quad \sqrt{5}$$
$$\left|\overrightarrow{CA}\right|^2 = (2)^2 + (2)^2 = 8 \qquad \left|\overrightarrow{CA}\right| \quad \sqrt{8}$$
$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right| \neq \left|\overrightarrow{CA}\right|$$

Thus, A, B,C form the vertices of an isosceles triangle.

13. Solution:

(a)

$$P(B) = 1 - P(B^C) = 1 - 1/3 = 2/3$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\therefore \frac{5}{6} = P(A) + \frac{2}{3} - \frac{1}{3} \Rightarrow P(A) = \frac{1}{2}$$

(b)

$$P(notA\, and\, notB) = P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)] = 1 - [1/4 + 1/2 - 1/8] = 3/8$$

14. Solution:

$$f(x) = -2x^3 - 9x^2 - 12x + 1$$
$$f'(x) = -6x^2 - 18x - 12 = -6(x^2 + 3x + 2) = -6(x+1)(x+2)$$
$$f'(x) = 0 \Rightarrow x = -1, -2$$

We study the sign of f'(x) on the intervals $(-\infty, -2), (-2, -1)$ *and* $(-1, \infty)$.

If x<-2, f'(x) is negative, i.e f(x) is strictly decreasing.

If -2<x<-1, f'(x) is positive, i.e f(x) is strictly increasing.

If x>-1, f'(x) is negative, i.e f(x) is strictly decreasing.

15. Solution:

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$$

$$Let\ z = x+y \quad \frac{dz}{dx} \quad 1 \quad \frac{dy}{dx} \qquad \frac{dz}{dx} \quad 1 \quad \sin z \quad \cos z$$

$$\therefore \frac{dz}{\sin z + \cos z + 1} = dx$$

Integrating both sides, we get

$$\frac{dz}{\sin z + \cos z + 1} = \quad dx$$

$$\frac{dz}{\frac{2\tan z/2}{1+\tan^2 z/2} + \frac{1-\tan^2 z/2}{1+\tan^2 z/2} + 1} = x + c$$

$$\frac{\sec^2 z/2}{2\tan z/2 + 2} = x + c$$

$$Let\ \tan z/2 = t \qquad \frac{1}{2}\sec^2 z/2 \quad dt$$

$$\therefore \quad \frac{dt}{t+1} = x + c$$

$$\therefore \log|t+1| = x + c$$

$$\log|\tan(x+y)/2 + 1| = x + c$$

16. Solution:

$$\left(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}\right).\left(|\vec{a}|\vec{b} - |\vec{b}|\vec{a}\right) = |\vec{a}||\vec{a}|\vec{b}.\vec{b} + |\vec{b}||\vec{a}|\vec{a}.\vec{b} - |\vec{a}||\vec{b}|\vec{b}.\vec{a} - |\vec{b}||\vec{b}|\vec{a}.\vec{a}$$

$$= \cancel{|\vec{a}|^2|\vec{b}|^2} + \cancel{|\vec{b}||\vec{a}|\vec{a}.\vec{b}} - \cancel{|\vec{a}||\vec{b}|\vec{b}.\vec{a}} - \cancel{|\vec{b}|^2|\vec{a}|^2} = 0(\because \vec{a}.\vec{b} = \vec{b}.\vec{a})$$

17. Solution:

$$I = \int \log(1+x^2)dx = \int 1.\log(1+x^2)dx$$

Integrating by parts,

$$I = \log(1+x^2)\int 1.dx - \int (\frac{d}{dx}\log(1+x^2)\int 1.dx$$

$$= \log(1+x^2)x - \int \frac{2x}{1+x^2}(x)dx$$

$$= x\log(1+x^2) - 2\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= x\log(1+x^2) - 2x + 2\tan^{-1}x + c$$

18. Solution:

$$Distance = \left|\frac{(\vec{b_1}\times\vec{b_2}).(\vec{a_2}-\vec{a_1})}{|(\vec{b_1}\times\vec{b_2})|}\right|$$

$$\vec{a_1} = i+2j+k, \vec{a_2} = 2i-j-k, \vec{b_1} = i-j+k, \vec{b_2} = 2i+j+2k,$$

$$\therefore \vec{b_1}\times\vec{b_2} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -3i+3k$$

$$\therefore |\vec{b_1}\times\vec{b_2}| = \sqrt{(-3)^2+(3)^2} = 3\sqrt{2}$$

$$\vec{a_2}-\vec{a_1} = i-3j-2k$$

$$\therefore d = \left|\frac{(-3i+3k).(i-3j-2k)}{3\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$$

19. Solution:

$$\vec{a} = i+j-k, \vec{b} = 2i+6j+k, \vec{c} = i-2j+k$$

$$Equation\ of\ plane:$$

$$(\vec{r}-\vec{a}).[(\vec{b}-\vec{a})\times\vec{c}] = 0$$

$$(\vec{b}-\vec{a}) = i+5j+2k$$

$$(\vec{b}-\vec{a})\times\vec{c} = \begin{vmatrix} i & j & k \\ 1 & 5 & 2 \\ 1 & -2 & 1 \end{vmatrix} = 9i+j-7k$$

$$\therefore [\vec{r}-(i+j-k)].(9i+j-7k) = 0$$

$$\vec{r}.(9i \quad j \quad 7k) \quad 17$$

Section C

20. Solution:

Let OC=r be the radius of the cone and OA=h be its height.

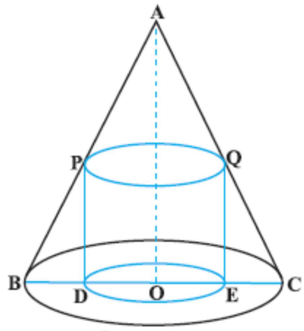
Let a cylinder with radius OE = x and height h' be inscribed in the cone.

Surface Area = 2πxh'

$\because \triangle QEC \sim \triangle AOC,$

$$\frac{QE}{AO} = \frac{CE}{CO} \Rightarrow \frac{h'}{h} = \frac{r-x}{r} \Rightarrow h' = h\left(\frac{r-x}{r}\right)$$

$$\therefore S = S(x) = 2\pi x h' = 2\pi x h\left(\frac{r-x}{r}\right) = \frac{2\pi h}{r}\left(rx - x^2\right)$$



$$S'(x) = \frac{2\pi h}{r}(r-2x)$$

$$S''(x) = \frac{2\pi h}{r}(-2)$$

$$S'(x) = 0 \Rightarrow x = r/2$$

$$Also,\ S''(r/2) = \frac{-4\pi h}{r} < 0$$

Hence, x=r/2 is a point of maxima.

Thus, the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

21. Solution:

$$Let\ A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}, b = \begin{bmatrix} 4 \\ -1 \\ 9 \end{bmatrix}$$

$$|A| = 14 \neq 0, A^{-1} = \frac{1}{14}\begin{bmatrix} 2 & 4 & 2 \\ 8 & -5 & 1 \\ -4 & 1 & -3 \end{bmatrix}, X = A^{-1}b = \frac{1}{14}\begin{bmatrix} 2 & 4 & 2 \\ 8 & -5 & 1 \\ -4 & 1 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 9 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -14 \\ 28 \\ 42 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore x = -1,\ y = 2,\ z = 3$$

22. Solution:

Suppose tailor A works for x days and tailor B works for y days.

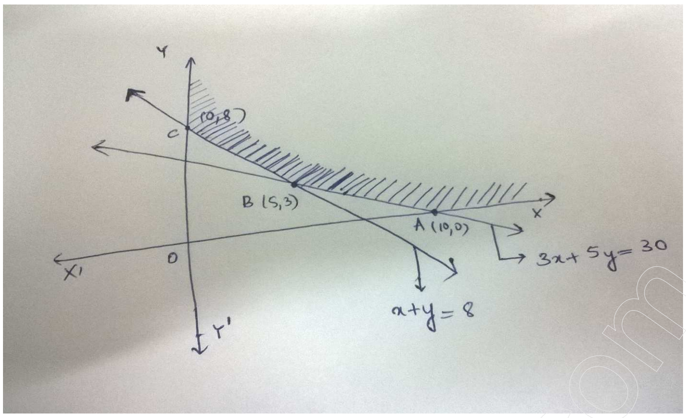
Then, Cost Z= 15x + 20y

	Tailor A	Tailor B	Min Requirement
Shirts	6	10	60
Pants	4	4	32
Cost per day	15	20	

The mathematical formulation of the problem is as follows:

Min Z= 15x+20y

$$6x+10y \ge 60 \Rightarrow 3x+5y \ge 30$$
s.t $$4x+4y \ge 32 \Rightarrow x+y \ge 8$$
$$x \ge 0, y \ge 0$$



We graph the above inequalities. The feasible region is as shown in the figure. We observe the feasible region is unbounded and the corner points are A,B and C. The co-ordinates of the corner points are (10,0), (5,3),(0,8).

Corner Point	Z=15x +20y
(10,0)	150
(5,3)	**135**
(0,8)	160

Thus cost is minimized by hiring A for 5 days and hiring B for 3 days.

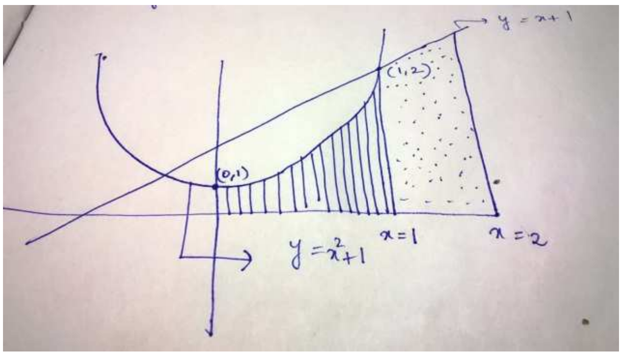
23. Solution:

The curves are x=2, y=x+1, y= x^2+1.

The point of intersection of the curves y=x+1, y=x^2+1:

$$x^2+1=x+1$$
$$\Rightarrow x(1-x)=0 \Rightarrow x=0, x=1$$

The shaded area is the required area.



$$Area=\int_0^1 y_1 dx+\int_1^2 y_2 dx$$
$$=\int_0^1 (x^2+1)dx+\int_1^2 (x+1)dx$$
$$=\left.\frac{x^3}{3}+x\right|_0^1+\left.\frac{x^2}{2}+x\right|_1^2=\frac{23}{6}$$

24. Solution:

Let A_i, B_i, C_i denote the events of winning A, B,C in their respective ith attempt.

$P(A_i)=2/3$, $P(B_i)=1/2$, $P(C_i)=1/4$

$$P(\overline{A_i})=\frac{1}{3}, P(\overline{B_i})=\frac{1}{2}, P(\overline{C_i})=\frac{3}{4}$$
$$P(A\ wins)=P(A_1\ or\ \overline{A_1B_1C_1}A_2\ or\ \overline{A_1B_1C_1A_2B_2C_2}A_3\cdots)$$
$$=P(A_1)+P(\overline{A_1})P(\overline{B_1})P(\overline{C_1})P(A_2)+P(\overline{A_1})P(\overline{B_1})P(\overline{C_1})P(\overline{A_2})P(\overline{B_2})P(\overline{C_2})P(A_3)+\cdots$$
$$=\frac{2}{3}+\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)+\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)+\cdots$$
$$=\frac{2}{3}\left[1+\frac{1}{8}+\left(\frac{1}{8}\right)^2+\cdots\right]=\frac{16}{21}$$
$$P(B\ wins)=P(\overline{A_1}B_1\ or\ \overline{A_1B_1C_1A_2}B_2\ or\ \overline{A_1B_1C_1A_2B_2C_2A_3}B_3\cdots)$$
$$=P(\overline{A_1})P(B_1)+P(\overline{A_1})P(\overline{B_1})P(\overline{C_1})P(\overline{A_2})P(B_2)+P(\overline{A_1})P(\overline{B_1})P(\overline{C_1})P(\overline{A_2})P(\overline{B_2})P(\overline{C_2})P(\overline{A_3})P(B_3)+\cdots$$
$$=\frac{1}{6}\left[1+\frac{1}{8}+\left(\frac{1}{8}\right)^2+\cdots\right]=\frac{4}{21}$$
$$P(C\ wins)=1-P(A\ wins)-P(B\ wins)=1-\frac{16}{21}-\frac{4}{21}=\frac{1}{21}$$

25. Solution:

$$x=a\sec^3\theta,\ y=a\tan^3\theta$$
$$\frac{dx}{d\theta}=3a\sec^2\theta(\sec\theta\tan\theta)=3a\sec^3\theta\tan\theta\Rightarrow\frac{d\theta}{dx}=\frac{1}{3a\sec^3\theta\tan\theta}$$
$$\frac{dy}{d\theta}=3a\tan^2\theta(\sec^2\theta)$$
$$\frac{dy}{dx}=\left(\frac{dy}{d\theta}\right)\left(\frac{d\theta}{dx}\right)=3a\tan^2\theta\sec^2\theta\frac{1}{3a\sec^3\theta\tan\theta}=\frac{\tan\theta}{\sec\theta}=\sin\theta$$
$$\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}=\frac{d}{d\theta}(\sin\theta)\frac{d\theta}{dx}=\cos\theta\frac{1}{3a\sec^3\theta\tan\theta}=\frac{1}{3a}\frac{\cos^5\theta}{\sin\theta}$$
$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{4}}=\frac{1}{12a}$$

26. Solution:

$$I=\int\frac{x^2dx}{(x+3)\sqrt{3x+4}}$$

$$Let\ z=\sqrt{3x+4}\quad \therefore x=\frac{z^2-4}{3}\Rightarrow dx=\frac{2z}{3}dz$$

$$\therefore I=\int\frac{\left(\frac{z^2-4}{3}\right)^2\frac{2z}{3}}{\left(\frac{z^2-4}{3}+3\right)z}dz$$

$$=\frac{2}{9}\int\frac{(z^2-4)^2}{z^2+5}dz=\frac{2}{9}\int\left(z^2-13+\frac{81}{z^2+5}\right)dz$$

$$=\frac{2}{9}\left[\frac{z^3}{3}-13z+\frac{81}{\sqrt{5}}\tan^{-1}\frac{z}{\sqrt{5}}\right]+c$$

$$=\frac{2}{27}(3x+4)^{3/2}-\frac{26}{9}\sqrt{3x+4}+\frac{18}{\sqrt{5}}\tan^{-1}\sqrt{\frac{3x+4}{5}}+c$$