

Sample Paper-04
Mathematics
Class – XII

Time allowed: 3 hours

ANSWERS

Maximum Marks: 100

Section A

1. Solution: Let $A = \{1, 2, 3, 4\}$

Let $R = \{(1, 2), (2, 1)\}$

2. Solution:

x-axis makes angles, 0, 90, 90 with the x, y, z axis respectively.

Thus, direction cosines are $\cos 0, \cos 90, \cos 90$, i.e. 1, 0, 0.

3. Solution: $[-1, 1]$

4. Solution:

$$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$$

$$\Rightarrow 2x^2 - 24 = -18$$

$$\Rightarrow x = \pm\sqrt{3}$$

5. Solution:

$$(A - A')' = A' - (A')' = A' - A = -(A - A')$$

Thus, $A - A'$ is skew symmetric.

6. Solution:

$$\begin{vmatrix} -2 & -1 \\ 4 & 2 \end{vmatrix} = -4 - (-4) = 0$$

Section B

7. Solution:

$$\tan^{-1} 2x + \tan^{-1} 3x = \tan^{-1} \left(\frac{2x+3x}{1-6x^2} \right) = \tan^{-1} \left(\frac{5x}{1-6x^2} \right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{1-6x^2} = \tan \frac{\pi}{4} = 1$$

$$\Rightarrow 6x^2 + 5x - 1 = 0 \Rightarrow (6x-1)(x+1) = 0$$

$$\Rightarrow x = 1/6, -1$$

$$\text{If } x = -1, \tan^{-1} 2 + \tan^{-1} 3 < 0, \text{ but } \frac{\pi}{4} > 0$$

Hence, $x \neq -1$.

$$\therefore x = 1/6$$

8. Solution:

The position vectors of A, B, C are $2\mathbf{i}$, \mathbf{j} , $2\mathbf{k}$ respectively.

$$\therefore \overrightarrow{AB} = \text{p.v of } \vec{B} - \text{p.v of } \vec{A} = \mathbf{j} - 2\mathbf{i}$$

$$\therefore \overrightarrow{BC} = \text{p.v of } \vec{C} - \text{p.v of } \vec{B} = 2\mathbf{k} - \mathbf{j}$$

$$\therefore \overrightarrow{CA} = \text{p.v of } \vec{A} - \text{p.v of } \vec{C} = 2\mathbf{i} - 2\mathbf{k}$$

$$|\overrightarrow{AB}|^2 = (2)^2 + (1)^2 = 5 \Rightarrow |\overrightarrow{AB}| = \sqrt{5}$$

$$|\overrightarrow{BC}|^2 = (1)^2 + (2)^2 = 5 \Rightarrow |\overrightarrow{BC}| = \sqrt{5}$$

$$|\overrightarrow{CA}|^2 = (2)^2 + (2)^2 = 8 \Rightarrow |\overrightarrow{CA}| = \sqrt{8}$$

$$|\overrightarrow{AB}| = |\overrightarrow{BC}| \neq |\overrightarrow{CA}|$$

Thus, A, B, C form the vertices of an isosceles triangle.

9. Solution:

Let A denote the event of getting a doublet.

Let B denote the event of getting a total of 10.

For A favorable cases are: $\{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$$\text{Thus, } P(A) = 6/36$$

For B favorable cases are: $\{(4,6), (5,5), (6,4)\}$

$$\text{Thus, } P(B) = 3/36$$

$$A \cap B = \{(5, 5)\}$$

$$\therefore P(A \cap B) = 1/36$$

$$P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B) = 1 - \left[\frac{6}{36} + \frac{3}{36} - \frac{1}{36} \right] = \frac{7}{9}$$

10. Solution:

$$R(x) = x^3 - e^x - 1/x$$

$$\text{Marginal Revenue} = R'(x) = 3x^2 - e^x + \frac{1}{x^2}$$

$$\text{When } x = 5, MR = 75 - e^5 + \frac{1}{25}$$

Precautions:

He should not give any drugs without a proper prescription.

He should not sell any medicines after expiry date.

11. Solution:

The circles in the system will have centres on the y-axis. Let (0, a) be the center of a circle touching the x-axis at the origin.

Thus, radius = $|a|$.

Equation of circles is:

$$(x-0)^2 + (y-a)^2 = |a|^2$$

$$x^2 + y^2 - 2ay = 0$$

Differentiating both sides w.r.t x, we get

$$2x + 2y \frac{dy}{dx} - 2a \frac{dy}{dx} = 0$$

$$2x + 2yy_1 - 2ay_1 = 0 \Rightarrow a = \frac{2x + 2yy_1}{2y_1} = \frac{x + yy_1}{y_1}$$

$$\Rightarrow x^2 + y^2 - 2 \left(\frac{x + yy_1}{y_1} \right) y = 0$$

$$\Rightarrow x^2 y_1 - y^2 y_1 - 2xy = 0$$

12. Solution:

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = \begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ 2x+2y+2z & x & z+x+2y \end{vmatrix} (C_1 \rightarrow C_1 + C_2 + C_3)$$

$$= (2x+2y+2z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix} (R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1)$$

$$= 2(x+y+z)^3$$

13. Solution:

$$\text{When } t = \frac{\pi}{4}, x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}$$

$$\therefore \text{point of contact} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$y = \sin t \Rightarrow \frac{dy}{dt} = \cos t, x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$$

$$\therefore \text{slope at } t = \frac{\pi}{4} = -\cot \frac{\pi}{4} = -1$$

$$\therefore \text{equation of tangent} = (y - \frac{1}{\sqrt{2}}) = -1(x - \frac{1}{\sqrt{2}}) \Rightarrow x + y - \sqrt{2} = 0$$

$$\text{Slope of normal} = m, m(-1) = -1 \Rightarrow m = 1$$

$$\therefore \text{equation of normal} = (y - \frac{1}{\sqrt{2}}) = 1(x - \frac{1}{\sqrt{2}}) \Rightarrow x - y = 0$$

14. Solution:

$$f \circ g(x) = f(g(x)) = f(x^2 - 1) = \sqrt{x^2 - 1}$$

$$g \circ f(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 - 1 = x - 1$$

$$\therefore f \circ g \neq g \circ f$$

15. Solution:

$$y = (x \log x)^{\log(\log x)} \Rightarrow \log y = \log(\log x) \log(x \log x) = \log(\log x) [\log x + \log(\log x)]$$

$$\frac{1}{y} \frac{dy}{dx} = \left(\frac{1}{\log x} \right) \frac{1}{x} [\log x + \log(\log x)] + \log(\log x) \left[\frac{1}{x} + \left(\frac{1}{\log x} \right) \frac{1}{x} \right]$$

$$= \left(\frac{1}{x \log x} \right) [\log x + \log(\log x)] + \log(\log x) \left[\frac{1}{x} + \left(\frac{1}{x \log x} \right) \right]$$

$$\therefore \frac{dy}{dx} = (x \log x)^{\log(\log x)} \left\{ \left(\frac{1}{x \log x} \right) [\log x + \log(\log x)] + \log(\log x) \left[\frac{1}{x} + \left(\frac{1}{x \log x} \right) \right] \right\}$$

Differentiating both sides w.r.t x, we get

$$\frac{1}{x} + \frac{1}{y} \frac{dy}{dx} = 2x + 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} \left(\frac{1}{y} - 2y \right) = 2x - \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(2x^2 - 1)}{x(1 - 2y^2)}$$

16. Solution:

$$\begin{aligned}
 |\vec{a} + \vec{b}|^2 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \\
 &= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \\
 &= |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \leq |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 \leq |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 \\
 &= (|\vec{a}| + |\vec{b}|)^2 \\
 \therefore |\vec{a} + \vec{b}| &\leq |\vec{a}| + |\vec{b}|
 \end{aligned}$$

17. Solution:

$$I = \int \sqrt{\frac{a-x}{a+x}} dx = \int \sqrt{\left(\frac{a-x}{a+x} \right) \left(\frac{a-x}{a-x} \right)} dx = \int \frac{a-x}{\sqrt{a^2-x^2}} dx$$

$$= a \int \frac{dx}{\sqrt{a^2-x^2}} - \int \frac{xdx}{\sqrt{a^2-x^2}} = aI_1 - I_2$$

$$I_1 = \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$I_2 = \int \frac{xdx}{\sqrt{a^2-x^2}}, \text{ put } a^2 - x^2 = z$$

$$\therefore -2xdx = dz$$

$$I_2 = -\frac{1}{2} \int \frac{dz}{\sqrt{z}} = -\sqrt{z} + c' = -\sqrt{a^2-x^2} + c'$$

$$I = a \sin^{-1} \left(\frac{x}{a} \right) + \sqrt{a^2-x^2} + k$$

18. Solution:

$$x_1 = 1, y_1 = 2, z_1 = 3; a_1 = 2, b_1 = 3, c_1 = 4$$

$$x_2 = 2, y_2 = 3, z_2 = 4; a_2 = 3, b_2 = 4, c_2 = 5$$

$$\text{Then, } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

Thus, the lines are co-planar.

The equation of the plane containing the given lines is:

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$$\text{i.e. } x - 2y + z = 0$$

19. Solution:

Planes passing through the intersection of the given planes are:

$$\vec{r} \cdot (2i + 6j) + 12 = 0 + \lambda [\vec{r} \cdot (3i - j + 4k)] = 0$$

$$\therefore \text{distance from origin} = 0$$

$$\therefore \frac{12}{\sqrt{(2+3\lambda)^2 + (6-\lambda)^2 + 16\lambda^2}} = 1$$

$$\Rightarrow 26\lambda^2 = 104$$

$$\Rightarrow \lambda = \pm 2$$

Substituting value of λ , we get the required equations of the plane as:

$$\vec{r} \cdot (2i + j + 2k) + 3 = 0 \text{ and } \vec{r} \cdot (i - 2j + 2k) - 3 = 0$$

Section C

20. Solution:

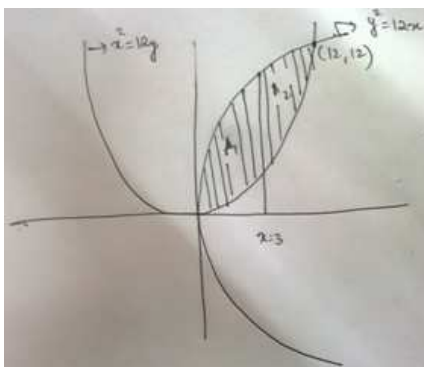
The point of intersection of the curves $y^2 = 12x$, $x^2 = 12y$:

$$y = \frac{x^2}{12} \Rightarrow y^2 = \frac{x^4}{144}$$

$$\Rightarrow 12x = \frac{x^4}{144}$$

$$\Rightarrow x(x^3 - 1728) = 0$$

$$\Rightarrow x = 0, 12$$



The shaded area is the required area.

$$\begin{aligned}
 A_1 &= \int_0^3 (y_2 - y_1) dx \\
 &= \int_0^3 \left(\sqrt{12x} - \frac{x^2}{12} \right) dx \\
 &= \sqrt{12} \frac{x^{3/2}}{3/2} \Big|_0^3 - \frac{x^3}{36} \Big|_0^3 = \frac{45}{4} \\
 A_2 &= \int_3^{12} (y_2 - y_1) dx \\
 &= \int_3^{12} \left(\sqrt{12x} - \frac{x^2}{12} \right) dx \\
 &= \sqrt{12} \frac{x^{3/2}}{3/2} \Big|_3^{12} - \frac{x^3}{36} \Big|_3^{12} = \frac{147}{4}
 \end{aligned}$$

Thus, ratio of the areas is $45:147=15:49$

21. Solution:

Let E be the event that the man reports 4.

Let S_1 denote the event that 4 occurs. Let S_2 denote the event that 4 does not occur.

$$P(S_1/E) = ?$$

$P(E/S_1)$ = Probability that he reports 4 when 4 has occurred = probability that he speaks truth = $3/4$

$P(E/S_2)$ = Probability that he reports 4 when 4 has not occurred = probability that he does not speak truth = $1/4$

Thus, by Baye's Theorem

$$P(S_1 / E) = \frac{P(S_1)P(E / S_1)}{P(S_1)P(E / S_1) + P(S_2)P(E / S_2)} = \frac{(1/6)(3/4)}{(1/6)(3/4) + (5/6)(1/4)} = \frac{3}{8}$$

22. Solution:

Squaring both sides, $(1 - x^2)y^2 = (\sin^{-1} x)^2$

Differentiating both sides w.r.t x ,

$$(1 - x^2)2y \frac{dy}{dx} + y^2(-2x) = 2 \sin^{-1} x \cdot \frac{1}{\sqrt{1 - x^2}}$$

$$2y(1 - x^2) \frac{dy}{dx} - 2xy^2 = 2y$$

$$y(1 - x^2) \frac{dy}{dx} - xy^2 = y$$

$$(1 - x^2) \frac{dy}{dx} - xy = 1$$

Differentiating both sides w.r.t x ,

$$(1 - x^2) \frac{d^2 y}{dx^2} + \frac{dy}{dx}(-2x) - x \frac{dy}{dx} - y = 0$$

Hence,

$$(1 - x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} - y = 0$$

23. Solution:

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2x \log(\sin x) dx.$$

$$= \log(\sin x) \cdot \frac{\sin 2x}{2} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin x} \cos x \cdot \frac{\sin 2x}{2} dx$$

$$= 0 - \log\left(\frac{1}{\sqrt{2}}\right) \frac{1}{2} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin x} \cos x \cdot \frac{2 \sin x \cos x}{2} dx$$

$$= -\frac{1}{2} \log\left(\frac{1}{\sqrt{2}}\right) - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} dx = -\frac{1}{2} \log\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{8} + \frac{1}{4}$$

24. Solution:

Let (x, y) be the point on the curve which is nearest to $(0, 5)$.

$$\therefore \text{Distance} = D = \sqrt{(x-0)^2 + (y-5)^2}$$

$$\therefore E = D^2 = (x-0)^2 + (y-5)^2$$

$$D \text{ minimum} \Rightarrow D^2 \text{ minimum}$$

$$E = 4y + (y-5)^2$$

$$\frac{dE}{dy} = 4 + 2(y-5)$$

$$\frac{dE}{dy} = 0 \Rightarrow y = 3$$

$$\frac{d^2E}{dy^2} = 2 > 0$$

$$y = 3 \Rightarrow x = \pm 2\sqrt{3}$$

Hence, for $y=3$ distance is minimum. The points are $(\pm 2\sqrt{3}, 3)$

25. Solution:

Let x be the cost of 1 kg onions, y be the cost of 1 kg wheat, z be the cost of 1 kg rice.

Thus we get the following equations:

$$4x + 3y + 2z = 60$$

$$2x + 4y + 6z = 90$$

$$6x + 2y + 2z = 70$$

$$\text{Let } A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, b = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$|A| = 50 \neq 0, A^{-1} = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}, X = A^{-1}b = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$$\therefore x = 5, y = 8, z = 8$$

Thus, per kg cost of onions, wheat and rice is Rs.5, Rs.8, Rs.8 respectively.

26. Solution:

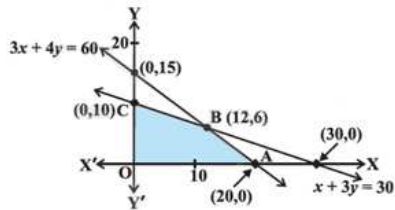
Suppose x is the number of pieces of model A and y is the number of pieces of model B.

Then, Profit $Z = 8000x + 12000y$

The mathematical formulation of the problem is as follows:

$$\text{Max } Z = 8000x + 12000y$$

$$\begin{aligned}
 &9x + 12y \leq 180 \text{ (fabricating constraint)} \\
 &3x + 4y \leq 60 \\
 \text{s.t. } &x + 3y \leq 30 \text{ (finishing constraint)} \\
 &x \geq 0, y \geq 0
 \end{aligned}$$



We graph the above inequalities. The feasible region is as shown in the figure. The corner points are O, A, B and C. The co-ordinates of the corner points are (0,0), (20,0), (12,6), (0,10).

Corner Point	$Z = 8000x + 12000y$
(0,0)	0
(20,0)	16000
(12,6)	16800
(0,10)	12000

Thus profit is maximized by producing 12 units of A and 6 units of B and maximum profit is 16800.