Development and Simulation of EOMs for a Two-Link Solar Panel Polishing Robot

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***Abstract*—The equations of motion for a two-link, haptic-interface, solar panel polishing robot are developed for the purpose of determining the joint torques to be enacted in order to constrain the polishing head's motion to the solar panel's surface.**

I. Introduction

A solar panel polishing robot with a haptic interface is to be used to train new polishers in a virtual environment. The haptic trainer needs to respond to the user as if the polishing head was constrained to move only along the panel's surface. To this end, the equations of motion for the polisher need to be developed and used in order to determine the joint torques to enforce this constraint.

II. Developing Equations of Motion

A. Lagrange's Method

I will first use the Euler-Lagrange equation to derive the equations of motion for the two-link manipulator.

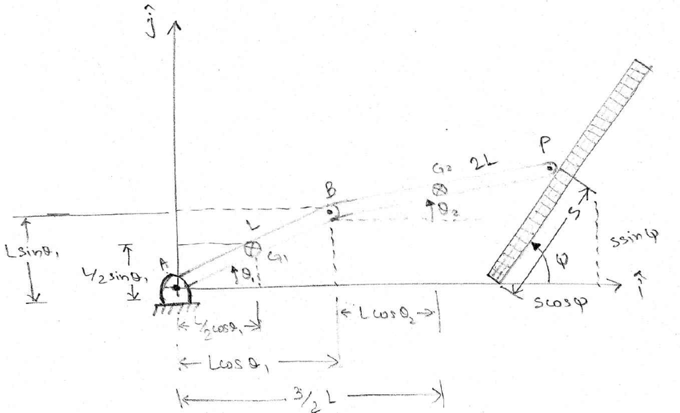

Fig. 1. Schematic showing the solar polisher

The position vectors to the centers of gravity of the two links, $\mathbf{r_{G_1}}$ and $\mathbf{r_{G_2}}$, and that to the polishing head, $\mathbf{r_P}$, are given as:

$$\mathbf{r_{G_1}} = \frac{L}{2}cos\theta_1\hat{\mathbf{i}} + \frac{L}{2}sin\theta_1\hat{\mathbf{j}}$$

$$\mathbf{r_{G_2}} = L(cos\theta_1 + cos\theta_2)\hat{\mathbf{i}} + L(sin\theta_1 + sin\theta_2)\hat{\mathbf{j}}$$

$$\mathbf{r_P} = L(cos\theta_1 + 2cos\theta_2)\hat{\mathbf{i}} + L(sin\theta_1 + 2sin\theta_2)\hat{\mathbf{j}}$$

The corresponding velocities can be obtained by differentiating the position vector expressions:

$$\mathbf{v_{G_1}} = -\frac{L}{2}\dot{\theta}_1 sin\theta_1\hat{\mathbf{i}} + \frac{L}{2}\dot{\theta}_1 cos\theta_1\hat{\mathbf{j}}$$

$$\mathbf{v_{G_2}} = -L(\dot{\theta}_1 sin\theta_1 + \dot{\theta}_2 sin\theta_2)\hat{\mathbf{i}} + L(\dot{\theta}_1 cos\theta_1 + \dot{\theta}_2 cos\theta_2)\hat{\mathbf{j}}$$

$$\mathbf{v_P} = -L(\dot{\theta}_1 sin\theta_1 + 2\dot{\theta}_2 sin\theta_2)\hat{\mathbf{i}} + L(\dot{\theta}_1 cos\theta_1 + 2\dot{\theta}_2 cos\theta_2)\hat{\mathbf{j}}$$

With the velocities known, the kinetic energy of the system can be calculated as:

$$T = \frac{1}{2}m_1 {v_{G_1}}^2 + \frac{1}{2}I_{G_1}\dot{\theta_1}^2 + \frac{1}{2}m_2 {v_{G_2}}^2 + \frac{1}{2}I_{G_2}\dot{\theta_2}^2$$

$$= (\frac{1}{3}m_1 + m_2)L^2\dot{\theta_1}^2 + \frac{2}{3}m_2 L^2\dot{\theta_2}^2 + m_2 L^2 cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2$$

The potential energy of the system can also be derived as:

$$V = (\frac{1}{2}m_1 + m_2)Lgsin\theta_1 + m_2 Lgsin\theta_2$$

Now, the constraint equation for the system can be formulated as:

$$\mathbf{v_P}.\hat{\mathbf{n}} = 0$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the planar surface, i-e,

$$\hat{\mathbf{n}} = -sin(\psi)\hat{\mathbf{i}} + cos(\psi)\hat{\mathbf{j}}$$

Substituting $\mathbf{v_P}$ and $\hat{\mathbf{n}}$ into the constraint equation gives

$$Lcos(\theta_1 - \psi)\dot{\theta}_1 + 2Lcos(\theta_2 - \psi)\dot{\theta}_2 = a_1 = 0 \qquad (1)$$

Now, the work done by the non-conservative force $\mathbf{F}$ applied by the user on the system is given by

$$\delta W_{nc} = \mathbf{F}.\delta\mathbf{r_P}$$

where

$$\mathbf{F} = Fcos\gamma\hat{\mathbf{i}} + Fsin\gamma\hat{\mathbf{j}}$$

$$\delta\mathbf{r_P} = -L(\delta\theta_1 sin\theta_1 + 2\delta\theta_2 sin\theta_2)\hat{\mathbf{i}} + L(\delta\theta_1 cos\theta_1 + 2\delta\theta_2 cos\theta_2)\hat{\mathbf{j}}$$

Substituting these into the expression for δW_{nc} gives

$$\delta W_{nc} = FLsin(\gamma - \theta_1)\delta\theta_1 + 2FLsin(\gamma - \theta_2)\delta\theta_2$$

The non-conservative forces along the generalized coordinates θ_1 and θ_2 can then be calculated to be

$$Q_{\theta_1} = \frac{\partial}{\partial \delta\theta_1}\delta W_{nc} = FLsin(\gamma - \theta_1)$$

$$Q_{\theta_2} = \frac{\partial}{\partial \delta\theta_2}\delta W_{nc} = 2FLsin(\gamma - \theta_2)$$

The Euler-Lagrange equations for the idealized two-link solar polisher robotic system can be formulated as:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta_1}}) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} + \lambda_1 \frac{\partial a_1}{\partial \dot{\theta_1}} = Q_{\theta_1}$$
$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta_2}}) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} + \lambda_1 \frac{\partial a_1}{\partial \dot{\theta_2}} = Q_{\theta_2}$$

which give the following equations of motion for the system.

$$\begin{aligned} (\frac{1}{3}m_1 + m_2)L^2\ddot{\theta_1} + m_2L^2cos(\theta_1 - \theta_2)\ddot{\theta_2} \\ +m_2L^2sin(\theta_1 - \theta_2)\dot{\theta_2}^2 + (\frac{1}{2}m_1 + m_2)Lgcos\theta_1 \\ +\lambda_1 Lcos(\theta_1 - \psi) = FLsin(\gamma - \theta_1) \end{aligned} \tag{2}$$

$$\begin{aligned} m_2L^2cos(\theta_1 - \theta_2)\ddot{\theta_1} + \frac{4}{3}m_2L^2\ddot{\theta_2} \\ -m_2L^2sin(\theta_1 - \theta_2)\dot{\theta_1}^2 + m_2Lgcos\theta_2 \\ +2\lambda_1 Lcos(\theta_2 - \psi) = 2FLsin(\gamma - \theta_2) \end{aligned} \tag{3}$$

The above equations have 3 unknowns. In order to solve the system we need a third equation, which can be obtained by differentiating the constraint equation (Eq. 1) derived earlier:

$$\begin{aligned} Lcos(\theta_1 - \psi)\ddot{\theta_1} + 2Lcos(\theta_2 - \psi)\ddot{\theta_2} \\ -Lsin(\theta_1 - \psi)\dot{\theta_1}^2 - 2Lsin(\theta_2 - \psi)\dot{\theta_2}^2 = 0 \end{aligned} \tag{4}$$

In the next section, the equations of motion will be derived using *Kane's Method*.

B. Kane's method

To apply the Kane's method, we first need the accelerations of all the bodies in our system, which can be found by differentiating the velocity relations obtained in the previous section:

$$\mathbf{a_{G_1}} = -\frac{L}{2}(\ddot{\theta_1}sin\theta_1 + \dot{\theta_1}^2cos\theta_1)\hat{\mathbf{i}} + \frac{L}{2}(\ddot{\theta_1}cos\theta_1 - \dot{\theta_1}^2sin\theta_1)\hat{\mathbf{j}}$$

$$\begin{aligned} \mathbf{a_{G_2}} = &-L(\ddot{\theta_1}sin\theta_1 + \dot{\theta_1}^2cos\theta_1 + \ddot{\theta_2}sin\theta_2 + \dot{\theta_2}^2cos\theta_2)\hat{\mathbf{i}} \\ &+L(\ddot{\theta_1}cos\theta_1 - \dot{\theta_1}^2sin\theta_1 + \ddot{\theta_2}cos\theta_2 - \dot{\theta_2}^2sin\theta_2)\hat{\mathbf{j}} \end{aligned}$$

We now need to find the projections of the velocities of each of the bodies in the system onto the generalized coordinates

$$\frac{\partial \mathbf{v_{G_1}}}{\partial \dot{\theta_1}} = \mathbf{v^1_{G_1}} = -\frac{L}{2}sin\theta_1\hat{\mathbf{i}} + \frac{L}{2}cos\theta_1\hat{\mathbf{j}}$$

$$\frac{\partial \mathbf{v_{G_1}}}{\partial \dot{\theta_2}} = \mathbf{v^2_{G_1}} = \mathbf{0}$$

$$\frac{\partial \mathbf{v_{G_2}}}{\partial \dot{\theta_1}} = \mathbf{v^1_{G_2}} = -Lsin\theta_1\hat{\mathbf{i}} + Lcos\theta_1\hat{\mathbf{j}}$$

$$\frac{\partial \mathbf{v_{G_2}}}{\partial \dot{\theta_2}} = \mathbf{v^2_{G_2}} = -Lsin\theta_2\hat{\mathbf{i}} + Lcos\theta_2\hat{\mathbf{j}}$$

Now, we need to do the same for the angular velocities

$$\omega_{\mathbf{G_1}} = \dot{\theta_1}\hat{\mathbf{k}}$$
$$\omega_{\mathbf{G_2}} = \dot{\theta_2}\hat{\mathbf{k}}$$
$$\frac{\partial \omega_{\mathbf{G_1}}}{\partial \dot{\theta_1}} = \omega^{\mathbf{1}}_{\mathbf{G_1}} = \hat{\mathbf{k}}$$
$$\frac{\partial \omega_{\mathbf{G_1}}}{\partial \dot{\theta_2}} = \omega^{\mathbf{2}}_{\mathbf{G_1}} = \mathbf{0}$$
$$\frac{\partial \omega_{\mathbf{G_2}}}{\partial \dot{\theta_1}} = \omega^{\mathbf{1}}_{\mathbf{G_2}} = \mathbf{0}$$
$$\frac{\partial \omega_{\mathbf{G_2}}}{\partial \dot{\theta_2}} = \omega^{\mathbf{2}}_{\mathbf{G_2}} = \hat{\mathbf{k}}$$

Next, we need to formulate the angular momenta and their rates of change for both the bodies

$$\mathbf{H_{G_1}} = I_{G_1}\omega_{\mathbf{G_1}} = \frac{1}{12}m_1L^2\dot{\theta_1}\hat{\mathbf{k}}$$
$$\implies \dot{\mathbf{H_{G_1}}} = \frac{1}{12}m_1L^2\ddot{\theta_1}\hat{\mathbf{k}}$$
$$\mathbf{H_{G_2}} = I_{G_2}\omega_{\mathbf{G_2}} = \frac{1}{3}m_2L^2\dot{\theta_2}\hat{\mathbf{k}}$$
$$\implies \dot{\mathbf{H_{G_2}}} = \frac{1}{3}m_2L^2\ddot{\theta_2}\hat{\mathbf{k}}$$

Using the FBD shown in Fig. 2, the forces acting on Link 1 can be found to be

$$\mathbf{F_{G_1}} = -m_1g\hat{\mathbf{j}}$$

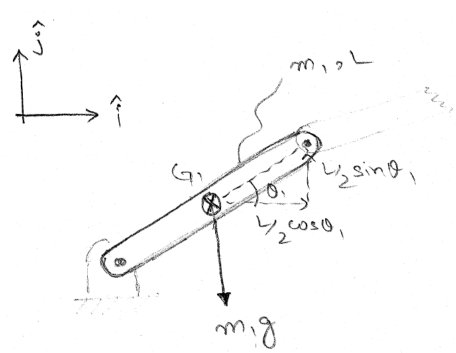

Fig. 2. Free Body Diagram (FBD) for Link 1

Similarly, for Link 2, the net force as shown in Fig. 3 is:

$$\mathbf{F_{G_2}} = Fcos\gamma\hat{\mathbf{i}} + (Fsin\gamma - m_2g)\hat{\mathbf{j}}$$

The moments acting on the bodies can then be found as:

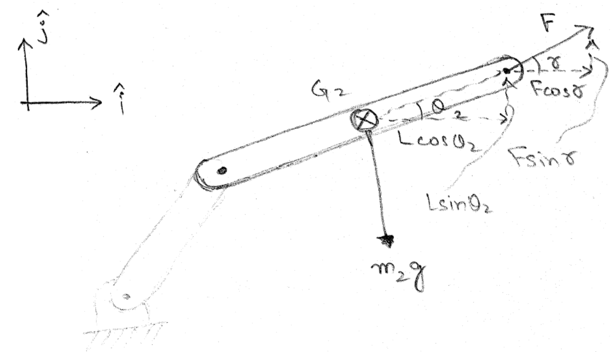

Fig. 3. Free Body Diagram (FBD) for Link 2

$$\mathbf{M_{G_1}} = \mathbf{0}$$
$$\mathbf{M_{G_2}} = FLsin(\gamma - \theta_2)\hat{\mathbf{k}}$$

Now, the Kane's equation with constraints is:

$$\sum_{i=\{G_1,G_2\}} ((\mathbf{F_i} - m_i\mathbf{a_i}).\mathbf{v_i}^k + (\mathbf{M_i} - \dot{\mathbf{H}}_\mathbf{i}).\omega_\mathbf{i}^\mathbf{k}) = R_k$$

which can be expanded as:

$$(\mathbf{F_{G_1}} - m_1\mathbf{a_{G_1}}).\mathbf{v^k_{G_1}} + (\mathbf{M_{G_1}} - \dot{\mathbf{H_{G_1}}}).\omega^\mathbf{k}_\mathbf{G_1} +$$
$$(\mathbf{F_{G_2}} - m_2\mathbf{a_{G_2}}).\mathbf{v^k_{G_2}} + (\mathbf{M_{G_2}} - \dot{\mathbf{H_{G_2}}}).\omega^\mathbf{k}_\mathbf{G_2} = \lambda_1 \frac{\partial a_1}{\partial \dot{\theta_k}}$$

For $k = \{1, 2\}$, this would give 2 equations:

$$(\frac{1}{3}m_1 + m_2)L^2\ddot{\theta_1} + m_2L^2cos(\theta_1 - \theta_2)\ddot{\theta_2}$$
$$+m_2L^2sin(\theta_1 - \theta_2)\dot{\theta_2}^2 + (\frac{1}{2}m_1 + m_2)Lgcos\theta_1$$
$$+\lambda_1Lcos(\theta_1 - \psi) = FLsin(\gamma - \theta_1)$$

$$m_2L^2cos(\theta_1 - \theta_2)\ddot{\theta_1} + \frac{4}{3}m_2L^2\ddot{\theta_2}$$
$$-m_2L^2sin(\theta_1 - \theta_2)\dot{\theta_1}^2 + m_2Lgcos\theta_2$$
$$+2\lambda_1Lcos(\theta_2 - \psi) = 2FLsin(\gamma - \theta_2)$$

These equations are exactly the same as the ones we got with the Lagrangian method.

As for the 3rd equation, it would be the same constraint equation as we had in the last section, so, the derivation for it has not been repeated here.

III. Simulation

A. Setting Up

The equations of motion and the constraint equation are converted to a matrix equation which will then be solved using ODE45 in MATLAB to simulate the system.

Eq. 2 can be written as:

$$T_{\ddot{\theta_1}}\ddot{\theta_1} + T_{\ddot{\theta_2}}\ddot{\theta_2} + T_{\lambda_1}\lambda_1 + T_0 = 0 \tag{5}$$

where

$$T_{\ddot{\theta_1}} = (\frac{1}{3}m_1 + m_2)L^2$$

$$T_{\ddot{\theta_2}} = m_2L^2cos(\theta_1 - \theta_2)$$

$$T_{\lambda_1} = Lcos(\theta_1 - \psi)$$

$$T_0 = m_2L^2sin(\theta_1 - \theta_2)\dot{\theta_2}^2 + (\frac{1}{2}m_1 + m_2)Lgcos\theta_1$$
$$-FLsin(\gamma - \theta_1)$$

Similarly, Eq. 3 can be written as:

$$Q_{\ddot{\theta_1}}\ddot{\theta_1} + Q_{\ddot{\theta_2}}\ddot{\theta_2} + Q_{\lambda_1}\lambda_1 + Q_0 = 0 \tag{6}$$

where

$$Q_{\ddot{\theta_1}} = m_2L^2cos(\theta_1 - \theta_2)$$

$$Q_{\ddot{\theta_2}} = \frac{4}{3}m_2L^2$$

$$Q_{\lambda_1} = 2Lcos(\theta_2 - \psi)$$

$$Q_0 = -m_2L^2sin(\theta_1 - \theta_2)\dot{\theta_1}^2 + m_2Lgcos\theta_2$$
$$-2FLsin(\gamma - \theta_2)$$

And Eq. 4 can be written as:

$$R_{\ddot{\theta_1}}\ddot{\theta_1} + R_{\ddot{\theta_2}}\ddot{\theta_2} + R_{\lambda_1}\lambda_1 + R_0 = 0 \tag{7}$$

where

$$R_{\ddot{\theta_1}} = Lcos(\theta_1 - \psi)$$

$$R_{\ddot{\theta_2}} = 2Lcos(\theta_2 - \psi)$$

$$R_{\lambda_1} = 0$$

$$R_0 = -Lsin(\theta_1 - \psi)\dot{\theta_1}^2 - 2Lsin(\theta_2 - \psi)\dot{\theta_2}^2$$

Eq. 5, 6 and 7 can be formulated as a matrix equation

$$\begin{bmatrix} \ddot{\theta_1} \\ \ddot{\theta_2} \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} T_{\ddot{\theta_1}} & T_{\ddot{\theta_2}} & T_{\lambda_1} \\ Q_{\ddot{\theta_1}} & Q_{\ddot{\theta_2}} & Q_{\lambda_1} \\ R_{\ddot{\theta_1}} & R_{\ddot{\theta_2}} & R_{\lambda_1} \end{bmatrix}^{-1} \begin{bmatrix} -T_0 \\ -Q_0 \\ -R_0 \end{bmatrix} \tag{8}$$

B. Results

A test case is simulated with the following system parameters:

$$m_1 = 1, m_2 = 2, L = 1, \psi = 60^o$$

The magnitude of the external force was defined as:

$$F = tsin(\frac{4}{3}\pi t)$$

And its angle, γ, was set to 60^o. Fig. 4 shows how $\mathbf{F}$ varies with time.

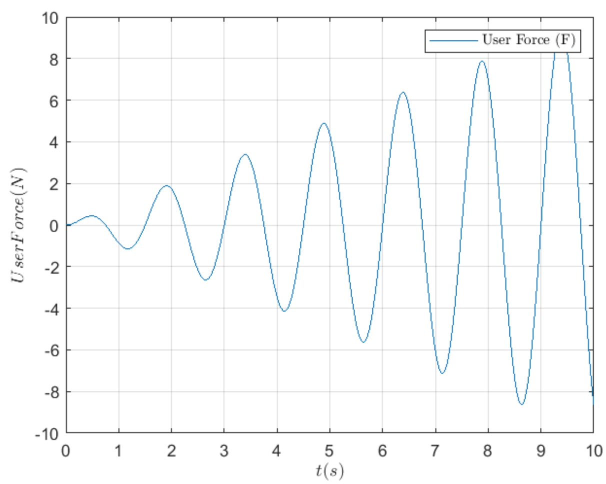

Fig. 4. Time profile of $\mathbf{F}$

The system was simulated from an initial state of x_0 (given below) using the ODE45 solver in MATLAB, and the system states were plotted against time.

$$x_0 = \begin{bmatrix} \theta_1 \\ \dot{\theta_1} \\ \theta_2 \\ \dot{\theta_2} \end{bmatrix} = \begin{bmatrix} 50^o \\ 0\ rad/s \\ 25.8^o \\ 0\ rad/s \end{bmatrix}$$

Fig. 5 shows how the angles of the two links vary over time and Fig. 6 shows how the angular velocities vary over time.

Although the ODE45 solver does not directly give angular accelerations, they can be obtained from the angle and angular velocity data. The angular acceleration profiles for the system are shown in Fig. 7.

The time profile of the constraint force ,λ_1, is shown in Fig. 8. To counter this force, the joint motors have to apply a counter torque (shown in Fig. 9).

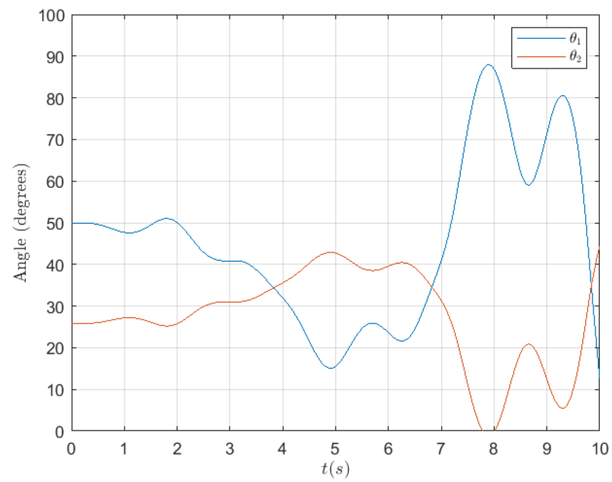

Fig. 5. Plot showing the angles, θ_1 and θ_2, of the two links

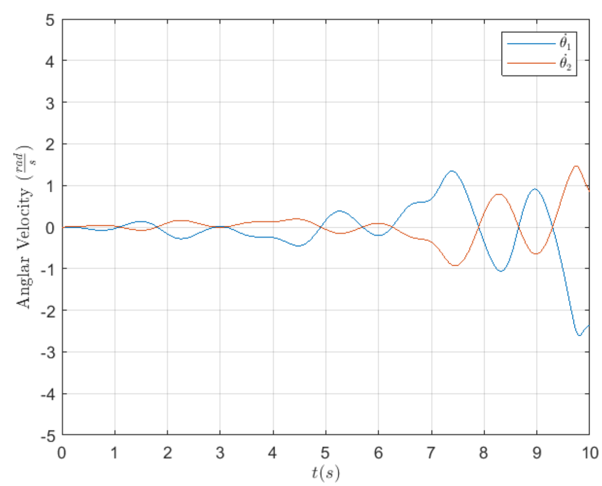

Fig. 6. Plot showing the angular velocities, $\dot{\theta_1}$ and $\dot{\theta_2}$, of the two links

IV. Some FAQs

A. How did you test your EOMs to see if they were correct?

1) Dimensional analysis was done at various points while deriving the equations to make sure the dimensions checked out.
2) The user force, $\mathbf{F}$, was set to zero and the total energy of the system was calculated at each time step in order to see if it was constant. Fig. 10 shows that the total energy indeed remains constant, which provides another verification of the equations of motion being correct.
3) As another check of the validity of the EOMs, the external force, $\mathbf{F}$, was again removed. With no force acting on the system (note: the graviational force was already zero, as described at the start of Section III-B), the links were expected to not move at all, and that is indeed what solving the equations of motion resulted into, as shown in Fig. **??**.

B. Which EOM derivation do you prefer and why?

In this problem, I found the Lagrangian method to be easy. Kane's method resulted in lengthy algebraic equations which had to be simplified very carefully to get the final forms of the equations of motion, and making an error during these algebraic manipulations is likely.

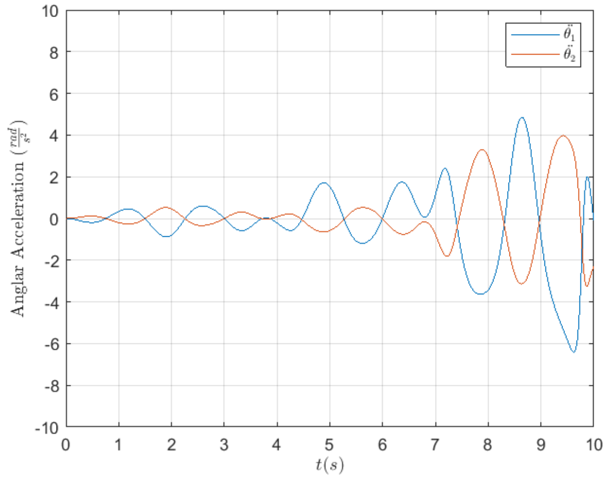

Fig. 7. Plot showing the angular accelerations, $\ddot{\theta_1}$ and $\ddot{\theta_2}$, of the two links

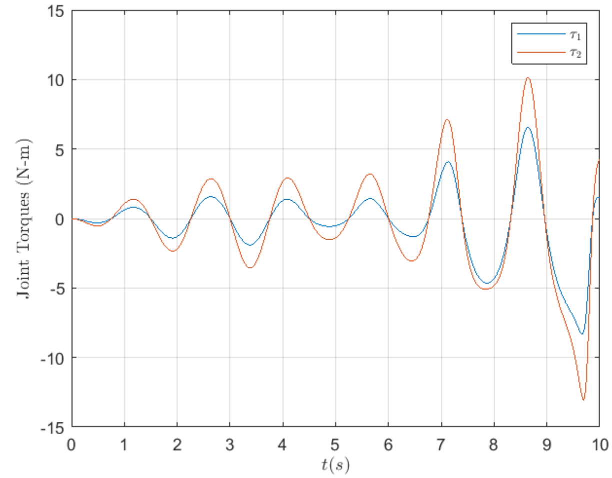

Fig. 9. Plot showing the joint torques τ_1 and τ_2

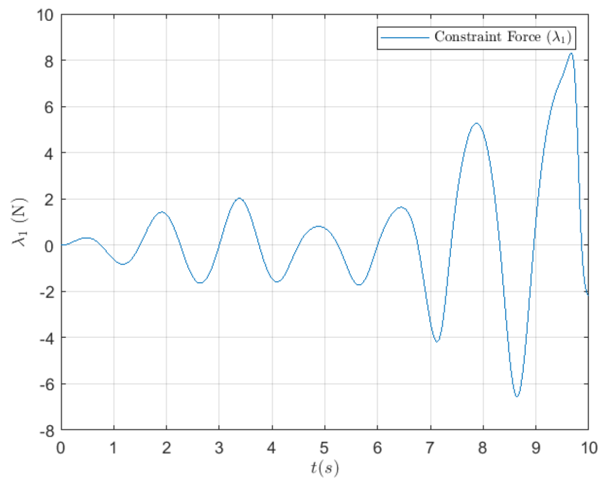

Fig. 8. Plot showing the constraint force

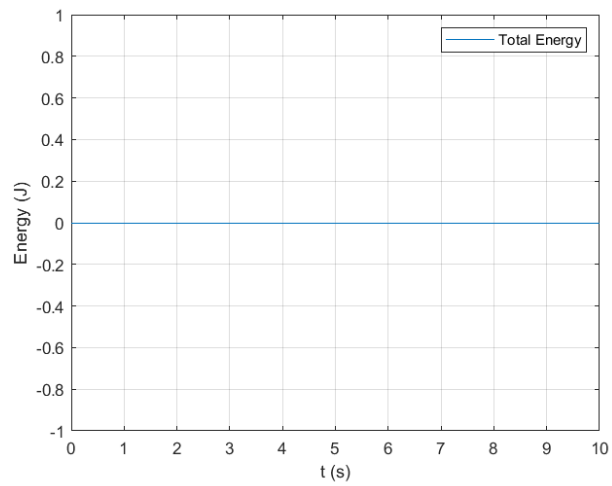

Fig. 10. Plot showing the variation of the total energy of the system over time

Also, to use the Euler-Lagrange equation, one does not have to find the accelerations, while in the Kane's method, the acceleration expressions need to be formulated in order to use the Kane's equation.

Another plus of the Euler-Lagrange equation is that one has to find the expressions for the kinetic and potential energy, which can later be used to make sure that the energy conservation law is being obeyed, like it was done in Section IV-A here.

C. Based on your simulation, are there any scenarios that you anticipate causing issues for the actual haptic trainer?

One of the possible problems that I can think of is numerical inaccuracies in the simulation which can make it a bit different from the actual polisher. Also, we are using more an ideal model of the polisher, ignoring friction, air drag, etc., but the actual panel polisher would have all these and its dynamics would be very different from the dynamics the trainees would be used to through the simulator.

D. Is the resultant force perpendicular to the planar surface minimized or close to zero?

The resultant force perpendicular to the surface will be close to zero, because the motor torques are acting so as to cancel the constraint force λ_1.