Date: October 24th, 2025

Submission #: 7325

Article Title: Wasserstein distance between two life table age-at-death distributions equals the difference in life expectancy at birth whenever the survivorship functions do not crossover

This article discusses the application of a measure of distributional difference known as the *Wasserstein distance* to life table age-at-death distributions. By considering two death distributions $d_A(x)$ and $d_B(x)$, and specializing to the particular Wasserstein distance

$$W_1(d_A, d_B) = \int_0^\omega |F_A(x) - F_B(x)| \, dx$$

(where $F_i(x)$ is the cumulative distribution function of distribution $d_i(x)$), the authors show that

$$W_1(d_A, d_B) = e_{0,A} - e_{0,B},$$

provided the death distributions' corresponding survivorship functions, $l_A(x)$ and $l_B(x)$, do not cross over (i.e., $l_A(x) \geq l_B(x) \ \forall x \in [0, \omega]$). This is a nice result that connects a widely studied measure of distributional difference (the Wasserstein distance) to demographically-meaningful questions (e.g., differences in life expectancy between two deaths distributions satisfying the above hypotheses). The authors also show that under the same hypotheses,

$$W_1(d_A, d_B) = \overline{d_A} - \overline{d_B},$$

using the fact that life expectancy at birth is the mean of the deaths distribution. The proofs of these results are clear, logical, and reader-friendly. The authors then present an empirical analysis that illustrates their theoretical results using Human Mortality Database (HMD) data from the U.S.A. (Figure 1). They complement this with a case study of U.S.A. and Germany life table data showing how if the "no survival function crossover" condition is not met then the Wasserstein distance W_1 does not necessarily equal the difference of life expectancies. This is a useful addition to the manuscript.

In the remainder of the article, the authors conduct and discuss empirical analyses to larger HMD data sets. These show that even when the crossover condition is not necessarily met, the Wasserstein W_1 distances are close to the life expectancy differences (Figures 3 and 4). Finally, the authors conduct correlation studies showing that W_1 is strongly correlated with two other metrics of dissimilarity—the Kullback-Leibler divergence and the Jaccard Index (Figure 5).

The article is well-written, easy to follow, and presents a new formal relationship in demography connecting differences in life expectancies (associated with "no crossover" survival functions) to the Wasserstein W_1 distance. Furthermore, the relationship is complemented by one empirical analysis illustrating it (c.f., Figure 1). **I therefore recommend the article be accepted after the author(s) address the two comments below.** At the end of this document I also include an additional suggestion that would improve the manuscript. But I leave the decision of whether to accept that suggestion to the discretion of the author(s).

Comments to Address in a Revised Version.

1. In equation (19) the author(s) are careful to include the condition under which their $W_1(d_A, d_B) = \overline{d_A} - \overline{d_B}$ result holds. Please add that condition to equation (4) as well. This will help prevent readers from blindly using the result in equation (4) without first ensuring the relevant hypotheses used to derive it are met.

2. The captions in Figures 3–5 state that the author(s) performed the relevant calculations. However, the captions say that the graphs were taken from Sauerberg (2005). Please create all relevant graphs to accompany the empirical analyses in the article under review (as done in Figures 1-2), so that the article's figures are self-contained (at least those that are marketed as illustrating the article's empirical analyses).

Optional Additional Suggestion.

As a submission to the Formal Relationships special collection, the article's main contributions are the relationships in equations (4) and (19). Figure 1—and the accompanying text—illustrate equation (4) via a comparison of two U.S.A. life tables. Figure 2 illustrates the *non-applicability* of those equations (since the "no survival function crossover" condition is not met in the life table pair being used, as acknowledged). This is valuable, in part because it helps caution readers to not use equations (4) and (19) without first checking the "no survival function crossover" condition (otherwise, as shown by Figure 2, the two dissimilarity metrics W_1 and $e_{0,A} - e_{0,B}$ may be unequal). Up to this point the article is cohesive in its objectives—establish the formal relationship(s); illustrate them via empirical analyses; discuss the potential ramifications of their non-applicability.

The remaining empirical analyses, however, detract from the article's main findings. The discussion related to Figures 3–4 does not state whether the life table pairs used in the empirical analyses that produced them satisfy the "no survival function crossover" condition. (I am inferring that the answer is "no" most of the time, based on the exposition.) Figure 5 then presents comparisons between W_1 and two other distributional dissimilarity metrics, but once again, it is unclear how many life table pairs used in those analyses satisfy the "no survival function crossover" condition. This means that those three figures—and their accompanying discussions—are *potentially* illustrating equations (4) and (19), or *potentially* not; without a check of the "no survival function crossover" condition for each life table pair included in those analyses, one cannot be sure.

Now, I understand that in Figures 3-4 the author(s) want to argue that even when the "no survival function crossover" condition *potentially* fails, W_1 and the difference in life expectancies are close in value. Similarly, I understand that in Figure 5 the author(s) want to argue that W_1 is correlated with other metrics of distributional dissimilarity even when life table pairs *potentially* do not satisfy the crossover condition. But the constant "potentially" qualifier here detracts from the article's main findings—the formal relationships established in equations (4) and (19) via *insistence* on the "no survival function crossover" condition.

To keep the article focused on those formal relationships, I recommend keeping the first paragraph of Section 3 in the article along with the first five sentences of the second paragraph in that section. Then, I recommend either:

- Removing the rest of the exposition that starts with the sixth sentence ("Sauerberg (2025) has compared the W_1..."), and also removing Figures 3–5; or
- Keeping that exposition but still removing Figures 3–5.

(Of course, the Conclusion would need to be updated accordingly.) Those figures come from the Sauerberg (2025) article; they pertain to data sets which the reader is not told satisfy the "no survival function crossover" condition; and the Sauerberg (2025) article is not written as a Formal Relationships article and includes additional empirical analyses. That article therefore appears to be the more appropriate place

for Figures 3–5 and their related discussion. But again, I leave the decision of whether to incorporate these suggestions to the discretion of the author(s).