Quantum Erasure in a Michelson's Interferometer: Theory

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1 Theory

1.1 Operators for Quarter Wave Plate and Half Wave Plate

A wave plate is a birefringent element that can add a phase between two orthogonal polarization components. If a wave plate's birefringent optic axis is along the horizontal or the vertical direction, the operator describing its action can be written as in the H,V basis as a Jones matrix $\hat{W}(\phi)$

$$\hat{W}(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

where ϕ is the phase added to the vertical component relative to the horizontal component. For a Quarter Wave Plate (QWP), $\phi = \pi/2$. To calculate the QWP operator, we will rotate the state of light by θ, act with the Jones matrix and rotate it back. The rotation operator $\hat{R}(\theta)$ is:

$$\hat{R}(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$Q\hat{W}P(\theta) = \hat{R}(\theta)^T \hat{W}(\phi) \hat{R}(\theta)$$

where $\hat{R}(\theta)^T$ is the transpose of the rotation matrix

$$Q\hat{W}P(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$Q\hat{W}P(\theta) = \begin{bmatrix} \cos^2(\theta) + i\sin^2(\theta) & (1-i)\cos(\theta)\sin(\theta) \\ (1-i)\cos(\theta)\sin(\theta) & \sin^2(\theta) + i\cos^2(\theta) \end{bmatrix} \tag{1}$$

For a Half Wave Plate (HWP), $\phi = \pi$.

$$H\hat{W}P(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$H\hat{W}P(\theta) = \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) & \sin^2(\theta) - \cos^2(\theta) \end{bmatrix}$$

We can simplify the HWP operator further using trigonometric identities

$$H\hat{W}P(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \tag{2}$$

1.2 Phase difference between the two modes of the incoming state

A 50/50 beam splitter divides an incoming state $|\psi\rangle$ into two modes (reflected R and transmitted T) with equal amplitude. The reflected mode gets a $\pi/2$ relative phase shift, and the reflection also induces a polarization-dependent shift. In this section, we will find the phase difference between the two modes once they returned to the beam splitter as a function of wavelength and path length imbalance ΔL.

The Jones matrix associated with the mirror $\hat{r_m}$ and with the beam splitter $\hat{r_{beam}}$

$$\hat{r_m} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{r_{beam}} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$d = e^{i4\pi\Delta L/\lambda} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To get the transmitted state, the incoming state is directly multiplied with the $\hat{r_m}$ as the beam splitter does not act on it. For the transmitted state:

$$|\psi_{transmitted}\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To get the reflected state, the incoming state is multiplied with the the path length d and then with $\hat{beam}$ and then with $\hat{r_m}$. For the reflected state:

$$|\psi_{reflected}\rangle = e^{i4\pi\Delta L/\lambda} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\psi_{reflected}\rangle = e^{i(4\pi\Delta L/\lambda + \pi/2)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Phase $= \frac{4\pi\Delta L}{\lambda} + \frac{\pi}{2}$

1.3 Calculating the state at Output 1 of the Michelson Interferometer

As shown in figure 1, at Output 1, there is a second reflection from $|\psi_{transmitted}\rangle$ where it goes through the beam splitter again. $|\psi_{reflected}\rangle$ in this case is the same as we computed in section 1.2.

$$|\psi_{transmitted}\rangle = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 0 \end{bmatrix}$$

$$|\psi_{reflected}\rangle = e^{i(4\pi\Delta L)/\lambda} \begin{bmatrix} i \\ 0 \end{bmatrix}$$

$$E_{out} = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} e^{i(4\pi\Delta L)/\lambda} \begin{bmatrix} i \\ 0 \end{bmatrix}$$

$$E_{out} = \frac{1}{\sqrt{2}} (1 + e^{i(4\pi\Delta L)/\lambda}) \begin{bmatrix} i \\ 0 \end{bmatrix}$$

$$E^*_{out} = \frac{1}{\sqrt{2}} (1 + e^{-i(4\pi\Delta L)/\lambda}) \begin{bmatrix} i & 0 \end{bmatrix}$$

$$I_{out} = E^*_{out} E_{out}$$

$$I_{out} = \frac{1}{\sqrt{2}} (1 + e^{i(4\pi\Delta L)/\lambda}) \begin{bmatrix} i \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} (1 + e^{-i(4\pi\Delta L)/\lambda}) \begin{bmatrix} i & 0 \end{bmatrix}$$

$$I_{out} = \frac{1}{2} (1 + e^{i(4\pi\Delta L)/\lambda} + 1 + e^{-i(4\pi\Delta L)/\lambda})]$$

$$e^{i(4\pi\Delta L)/\lambda} + e^{-i(4\pi\Delta L)/\lambda} = 2\cos(4\pi\Delta L)/\lambda)$$

$$I_{out} = \frac{1}{2} (2 + 2\cos(4\pi\Delta L)/\lambda))$$

Taking the 2 common, we can apply the following trig identity

$$2\cos^2(2\pi\Delta L)/\lambda = 1 + \cos(2(2\pi\Delta L)/\lambda))$$

$$I_{out} = 2\cos^2(2\pi\Delta L)/\lambda$$

1.4 Determining the optic axis angles

If horizontally polarized light is incident on a quarter-wave plate, the optic axis should be set at 0 degrees to keep the light horizontally polarized.

1.5 Proving equivalence of two setups

Two quarter-wave plates with the same optic axis setting, θ_1 in a row behaves as a half-wave plate also set at θ_1. Provided, that one starts with a horizontal or vertical polarization state, this set up is equivalent to pasing through a quarter-wave plate, bouncing off a mirror and passing through the same QWP again.

Proof:

Setup 1: Two QWPs acting as a HWP:

$$\hat{R_\theta}^T W(\pi/2)\hat{R_\theta}\hat{R_\theta}^T W(\pi/2)\hat{R_\theta}$$

From (1), for a QWP operator:

$$\begin{bmatrix} \cos^2(\theta)+i\sin^2(\theta) & (1-i)\cos(\theta)\sin(\theta) \\ (1-i)\cos(\theta)\sin(\theta) & \sin^2(\theta)+i\cos^2(\theta) \end{bmatrix} \begin{bmatrix} \cos^2(\theta)+i\sin^2(\theta) & (1-i)\cos(\theta)\sin(\theta) \\ (1-i)\cos(\theta)\sin(\theta) & \sin^2(\theta)+i\cos^2(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$$

This matches the HWP operator we derived in eqn (2)

Setup 2: QWP $->$ Mirror $->$ -QWP

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} \cos^2(\theta)+i\sin^2(\theta) & (1-i)\cos(\theta)\sin(\theta) \\ (1-i)\cos(\theta)\sin(\theta) & \sin^2(\theta)+i\cos^2(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (\cos^2(\theta)+i\sin^2(\theta))^2 - 2i\sin^2(\theta)\cos^2(\theta) \\ (1-i)\cos(\theta)\sin(\theta)(-i\cos^2(\theta)-\sin^2(\theta)) - (1-i)\cos(\theta)\sin(\theta)(\cos^2(\theta)+i\sin^2(\theta)) \end{bmatrix}$$

This can be simplified using trig identity

$$= \begin{bmatrix} \cos(2\theta) \\ -\sin(2\theta) \end{bmatrix}$$

To change a horizontally polarized light into vertically polarized light, we need to set the QWP angle to 45 degrees.

1.6 Calculating visibility of the Interferometer

Let us consider the case in which the two arms of the Michelson interferometer are orthogonally polarized by inserting a QWP in one of the arms with their angles set to 45 degrees.

Reflected path with delay:

$$\begin{bmatrix} ie^{\frac{4pi\pi\Delta L}{\lambda}} \\ 0 \end{bmatrix}$$

Transmitted path with QWP:

$$\begin{bmatrix} 0 \\ i \end{bmatrix}$$

$\Delta L = 0$: Sum of reflected and transmitted paths:

$$\begin{bmatrix} i \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} i \\ i \end{bmatrix}$$

$\Delta L = \lambda/4$: Sum of reflected and transmitted paths:

$$\begin{bmatrix} -i \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ i \end{bmatrix}$$

$$Visibility = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

In this case the visibility =0. Now, we calculated the visibility after adding the polarizer in ouput 1. The state after the polarizer is added:

$$\begin{bmatrix} i\cos(\theta)(e^{\frac{4pi\pi\Delta L}{\lambda}}\cos(\theta) + \sin(\theta)) \\ i\sin(\theta)(e^{\frac{4pi\pi\Delta L}{\lambda}}\cos(\theta) + \sin(\theta)) \end{bmatrix}$$

Intensity of the sum of two paths with $\Delta L = 0$:

H-Component:

$$\begin{bmatrix} \cos^2(\theta)(\cos(\theta) + \sin(\theta))^2 \end{bmatrix}$$

V-Component:

$$\begin{bmatrix} \sin^2(\theta)(\cos(\theta) + \sin(\theta))^2 \end{bmatrix}$$

Intensity of the sum of two paths with $\Delta L = 0$:

H-Component:

$$\left[\cos^2(\theta)(\cos(\theta)-\sin(\theta))^2\right]$$

V-Component:

$$\left[\sin^2(\theta)(\cos(\theta)-\sin(\theta))^2\right]$$

$$Visibility = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = 2\cos\theta\sin\theta = |\sin 2\theta|$$