The Compton Scattering Derivation

January 27, 2020

Consider a photon with energy $h\nu$ and momentum $h\nu/c$ striking an electron with energy m_0c^2 where m_0 is the rest mass of electron, and is at rest. Upon collision, you get a photon with a new energy $h\nu'$ and momentum $h\nu'/c$ at an angle θ and an electron with energy $\sqrt{p^2c^2 + m_0^2c^4}$ and momentum p at an angle ϕ. To find the difference in the wavelengths of the scattered and the incident photon we will use energy and momentum conservation laws.

1 Energy conservation

$$h\nu + m_0c^2 = h\nu' + \sqrt{p^2c^2 + m_0^2c^4} \tag{1}$$

2 Momentum conservation in the x direction

$$h\nu/c = h\nu' \cos(\theta)/c + p\cos(\phi) \tag{2}$$

3 Momentum conservation in the y direction

$$0 = h\nu' \sin(\theta)/c - p\sin(\phi) \implies p\sin(\phi) = h\nu' \sin(\theta)/c \tag{3}$$

$$\implies p = (h\nu' \sin(\theta)/\sin(\phi))/c \implies \csc(\theta) = \frac{pc}{h\nu' \sin(\theta)} \tag{4}$$

4 Deriving the equation for the wavelength

Substituting equation (4) in equation (2)

$$\implies h\nu/c = h\nu' \cos(\theta)/c + (h\nu' \sin(\theta)\cos(\phi))/c\sin(\phi) \tag{5}$$

$$\implies h\nu/c = h\nu' \cos(\theta)/c + (h\nu' \sin(\theta)\cot(\phi))/c \tag{6}$$

$$\implies \cot(\phi) = \frac{\nu - \nu' \cos(\theta)}{\nu' \sin(\theta} \tag{7}$$

Using the trigonometric equation:

$$\csc^2(\theta) = 1 + \cot^2(\theta) \implies \frac{p^2c^2}{(h\nu'\sin(\theta))^2} = 1 + \frac{(\nu - \nu'\cos(\theta))^2}{(\nu'\sin(\theta))^2} \tag{8}$$

$$p^2c^2 = h^2\nu'^2\sin^2(\theta) + h^2(\nu - \nu'\cos(\theta))^2 \tag{9}$$

Substituting eqn (9) in (1)

$$h\nu + m_0c^2 = h\nu' + \sqrt{h^2\nu'^2\sin^2(\theta) + h^2(\nu - \nu'\cos(\theta))^2 + m_0^2c^4} \tag{10}$$

$$\implies h^2(\nu-\nu')^2+m_0^2c^4+2(h(\nu-\nu')*m_0c^2) = h^2\nu'^2\sin^2(\theta)+h^2(\nu-\nu'\cos(\theta))^2+m_0^2c^4 \tag{11}$$

$$\implies h^2(\nu^2+\nu'^2-2\nu\nu')+2(h(\nu-\nu')*m_0c^2) = h^2\nu'^2\sin^2(\theta)+h^2(\nu^2+\nu'^2\cos^2(\theta)-2\nu\nu'\cos(\theta)) \tag{12}$$

$$\implies h^2\nu^2+h^2\nu'^2-2h^2\nu\nu'+2h\nu m_0c^2-2h\nu' m_0c^2 = h^2\nu'^2\sin^2(\theta)+h^2\nu^2+h^2\nu'^2\cos^2(\theta)-2h^2\nu\nu'\cos(\theta) \tag{13}$$

$$\implies 2h^2\nu\nu'(1 - \cos(\theta)) = 2hm_0c^2(\nu - \nu') \tag{14}$$

$$\nu = c/\lambda, \nu' = c/\lambda' \tag{15}$$

$$\implies \frac{4h^2c^2\sin^2(\theta/2)}{\lambda\lambda'} = 2h(c)(m_0c^2)(\frac{1}{\lambda} - \frac{1}{\lambda'}) \tag{16}$$

$$\implies \frac{2h\sin^2(\theta/2)}{\lambda\lambda'} = m_0c(\frac{\lambda' - \lambda}{\lambda'\lambda}) \tag{17}$$

$$\implies 2h\sin^2(\theta/2) = m_0c(\lambda' - \lambda) \tag{18}$$

$$\implies \lambda' - \lambda = \frac{2h\sin^2(\theta/2)}{m_0c} \tag{19}$$

We can also find the energy of the recoil electron

5 Energy of the recoil electron

$$E_e = m_0c^2 + h(\nu - \nu') \tag{20}$$

Gain in the energy of recoil electron

$$Gain = h(\nu - \nu') \tag{21}$$