

Logic and Hybrid Systems

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Hybrid Systems

- ▶ Dynamical Systems exhibiting both discrete (jump) and continuous (flow) behaviors.
- ▶ Serve as models of physical systems, from thermostats to trains.
- ▶ Continuous dynamics specified using Differential Equations.

Differential Dynamic Logic (dL)

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- ▶ Practical deductive verification of hybrid systems.
- ▶ Introduces Hybrid Program - program notation for hybrid systems.
- ▶ Dynamic Logic for Hybrid Programs, a generalization of Dynamic Logic.
- ▶ Suited for automation.

Hybrid Automata

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- ▶ Nodes specify continuous dynamics. Edges describe discrete transitions.
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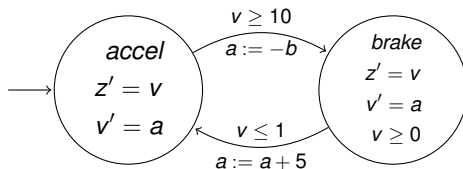


Figure: Hybrid Automata (simplified) of a Train Control System

Differential Dynamic Logic

Motivations

- ▶ **First Order Logic** - No builtin means for referring to state transitions.
- ▶ **Temporal Logics** - Modal operators allow referring to state transitions. But valid formulas only express generic facts.

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- ▶ **First Order Logic** - No builtin means for referring to state transitions.
- ▶ **Temporal Logics** - Modal operators allow referring to state transitions. But valid formulas only express generic facts.
- ▶ **Dynamic Logic (DL)** - Combines operational system models with operators for reasoning.
 - ▶ Provides parameterized modal operators, $[\alpha]$, $\langle\alpha\rangle$ that refer to states reachable by system α .
 - ▶ $[\alpha]\phi$ expresses all states reachable by α satisfy ϕ , allowing reasoning about discrete systems.
 - ▶ Say $(b > 0) \rightarrow [a := -b](a < 0)$ expresses a discrete transition. We can prove $(b > 0) \vdash (a < 0)[b/a]$ using DL's calculus.
 - ▶ No built in notion for describing or reasoning about continuous dynamics.

Differential Dynamic Logic

Motivations

- ▶ Generalize DL so operational models α can be used in modal formulas like $[\alpha]\phi$. dL refers to generalized models as “Hybrid Programs”.
- ▶ A compositional calculus for verification. Decompose $[\alpha]\phi$ into an equivalent formula $[\alpha_1]\phi_1 \wedge [\alpha_2]\phi_2$.
- ▶ Prove subsystems and subproperties $[\alpha_i]\phi_i$ independently and combine results conjunctively.
- ▶ Complete relative to handling of differential equations.

Differential Dynamic Logic

Syntax and Semantics

dL formulas built over

- ▶ V , set of real-valued logical variables and signature Σ containing functions, predicate symbols over reals, like $0, 1, +, \geq$.
- ▶ Signature Σ containing functions and predicates, like $0, 1, \geq$. Σ also contains *System State Variables*. Unlike rigid symbols, like $1, 2$, their interpretation can change from state to state.
- ▶ Set $\text{Trm}(\Sigma, V)$ of *terms* defined as classical FOL polynomial (or rational) expressions over V with additional skolem terms $s(X_1, \dots, X_n)$, where $X_1, \dots, X_n \in V$.

Differential Dynamic Logic

Syntax and Semantics

Hybrid Programs

Consider $x_i \in \Sigma$, $\theta_i, \vartheta_i \in \text{Trm}(\Sigma, V)$ for $1 \leq i \leq n$, χ a (Σ, V) FOL-formula, $\alpha, \beta \in \text{HP}(\Sigma, V)$ Set $\text{HP}(\Sigma, V)$, is defined inductively as -

- ▶ $(x_1 := \theta_1, \dots, x_n := \theta_n) \in \text{HP}(\Sigma, V)$
- ▶ $(x'_1 = \vartheta_1, \dots, x'_n = \vartheta_n) \& \chi \in \text{HP}(\Sigma, V)$. $x'_i = \vartheta_i$ is a differential equation where x'_i is the first order time derivative of x_i .
- ▶ $(?\chi) \in \text{HP}(\Sigma, V)$.
- ▶ $\alpha \cup \beta \in \text{HP}(\Sigma, V)$.
- ▶ $\alpha; \beta \in \text{HP}(\Sigma, V)$.
- ▶ $\alpha^* \in \text{HP}(\Sigma, V)$.

Differential Dynamic Logic

Hybrid Program Example

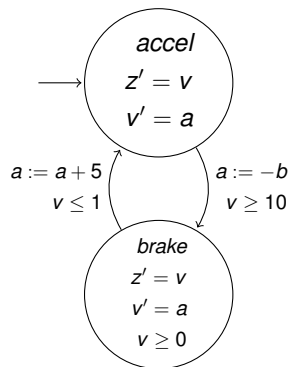
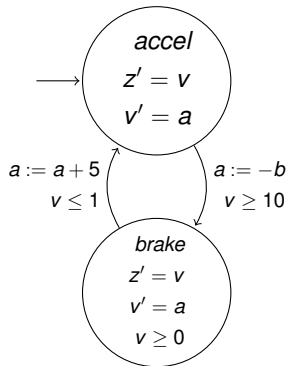


Figure: Hybrid Automata of Simple Train Control System

Differential Dynamic Logic

Hybrid Program Example



$q := accel;$
 $((?q = accel; z' = v, v' = a)$
 $\cup (?q = accel \wedge z \geq s; a := -b; q := brake; ?v \geq 0)$
 $\cup (?q = brake; z' = v, v' = a \& v \geq 0)$
 $\cup (?q = brake \wedge v \leq 1; a := a + 5; q := accel))^*$

Figure: Hybrid Automata of Simple Train Control System

Differential Dynamic Logic

Syntax and Semantics

dL Formulas

Differential Dynamic Logic

Syntax and Semantics

Some Notation

- ▶ Interpretation I assigns functions and relations over Reals to rigid symbols in Σ .
- ▶ A state is a map $\nu : \Sigma_{fl} \rightarrow \mathbb{R}$.
- ▶ Assignment of logical variables is a map $\eta : V \rightarrow \mathbb{R}$.
- ▶ Note the difference between *Logical* and *State Variables*. The evaluation of a state variable “evolves” across states. Logical Variables can be quantified over, but not state variables.
- ▶ The models of dL are Kripke Structures, where nodes are hybrid system states.

Differential Dynamic Logic

Syntax and Semantics

dL Valuation of dL Terms

Say $val_{I,\eta}(\nu, \cdot)$ is evaluation w.r.t. interpretation I , assignment η and state ν .

Differential Dynamic Logic

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- ▶ (Logical Variables) $val_{l,\eta}(\nu, x) = \eta(x), x \in V$
- ▶ (Rigid Symbols)
 $val_{l,\eta}(\nu, f(\theta_1, \dots, \theta_n)) = f_l(val_{l,\eta}(\nu, \theta_1), \dots, val_{l,\eta}(\nu, \theta_n))$
where f is n-ary rigid symbol in Σ

Differential Dynamic Logic

Syntax and Semantics

Valuation of dL Formulas

- ▶ $val_{l,\eta}(\nu, p(\theta_1, \dots, \theta_n)) = p_l(val_{l,\eta}(\nu, \theta_1), \dots, val_{l,\eta}(\nu, \theta_n))$
- ▶ $val_{l,\eta}(\nu, \varphi \wedge \psi) = \top$ iff $val_{l,\eta}(\nu, \varphi) = \top \wedge val_{l,\eta}(\nu, \psi) = \top$.
Similarly for \rightarrow, \neg, \vee
- ▶ $val_{l,\eta}(\nu, \exists x. \varphi) = \top$ iff $val_{l,\eta[x \mapsto d]}(\nu, \varphi) = \top$ for some $d \in \mathbb{R}$

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- ▶ $val_{l,\eta}(\nu, [\alpha]\varphi) = \top$ iff $val_{l,\eta}(\omega, \varphi) = \top$ for all states ω with $(\nu, \omega) \in \rho_{l,\eta}(\alpha)$.
- ▶ $val_{l,\eta}(\nu, \langle \alpha \rangle \varphi) = \top$ iff $val_{l,\eta}(\omega, \varphi) = \top$ for some state ω with $(\nu, \omega) \in \rho_{l,\eta}(\alpha)$.

Differential Dynamic Logic

Syntax and Semantics

Transition Semantics of Hybrid Programs

Evaluation $\rho_{l,\eta}(\alpha)$ of HP α . $(\nu, \omega) \in \rho_{l,\eta}(\alpha)$ means state ω is reachable from ν by operations of α .

- ▶ $(\nu, \omega) \in \rho_{l,\eta}(x_1 := \theta_1, \dots, x_n := \theta_n)$ iff
 $\nu[x_1 \mapsto \text{val}_{l,\eta}(\nu, \theta_1), \dots, x_n \mapsto \text{val}_{l,\eta}(\nu, \theta_n)] = \omega$ and
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- ▶ $\rho_{l,\eta}(\alpha; \beta) = \{(\nu, \omega) : (\nu, Z) \in \rho_{l,\eta}(\alpha) \wedge (Z, \omega) \in \rho_{l,\eta}(\beta) \text{ for some state } Z\}$

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- ▶ $(\nu, \omega) \in \rho_{l,\eta}(\alpha^*)$ iff for $n \in \mathbb{N}$, there are states $\nu = \nu_0, \dots, \nu_n = \omega$ s.t. $(\nu_i, \nu_{i+1}) \in \rho_{l,\eta}(\alpha)$ for $0 \leq i < n$.

Differential Dynamic Logic

Syntax and Semantics

Transition Semantics of Hybrid Programs

$(\nu, \omega) \in \rho_{l,\eta}((x'_1 = \vartheta_1 \dots x'_n = \vartheta_n) \& \chi)$ iff there is a flow of some duration $r \geq 0$, from ν to ω respecting the differential equations and evolution domain. Formally, there is a function

$f : [0, r] \rightarrow \text{Sta}(\Sigma)$ such that -

- ▶ $f(0) = \nu$ and $f(r) = \omega$.
- ▶ $\forall \delta : [0, r]. \text{val}_{l,\eta}(f(\delta), x_i)$ is continuous and $\text{val}_{l,\eta}(f(\delta), \chi) = \top$.
- ▶ $\forall \epsilon : (0, r). \text{val}_{l,\eta}(f(\epsilon), \dot{\vartheta}_i) = \dot{f}(\epsilon)$.
- ▶ For any $z \notin \{x_1, x_2, \dots, x_n\}$, $\text{val}_{l,\eta}(f(\zeta), z)$ remains constant for $\zeta \in [0, r]$. In other words, all other state variables remain unchanged.

Differential Dynamic Logic

Syntax and Semantics

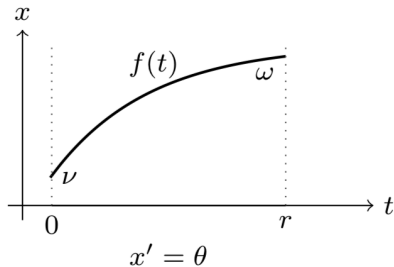


Figure: Unbounded Evolution

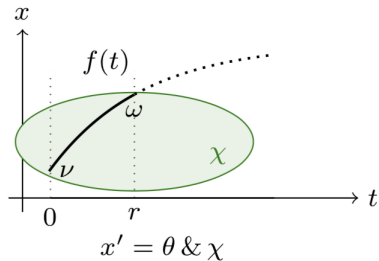


Figure: Evolution bound by χ

Differential Dynamic Logic

Examples

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Differential Dynamic Logic

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- ▶ Loosely speaking, an ES or CPS is built upon the integration of Computation (Controller) and Physical Systems (Plant). Examples include autopilot, ABS (Anti Lock Braking Systems), fly-by-wire (human interaction).
- ▶ Expressed in dL as an HP $\alpha \equiv (ctrl; \{plant\})^*$.

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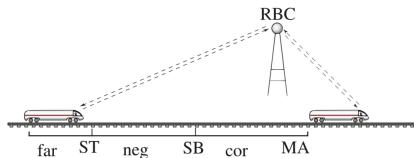
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- ▶ Expressed in dL as an HP $\alpha \equiv (ctrl; \{plant\})^*$.
- ▶ Formulas of interest are of the form $\varphi \rightarrow [\alpha]\psi$.
- ▶ Formula above says that all states satisfying φ will always transition to states satisfying ψ , where α dictates the transition relation.

Differential Dynamic Logic

Motivating Example

European Train Control System (ETCS)



- ▶ Goal of ECTS is to avoid collision and maximize throughput.
- ▶ Discards usage of traditional fixed segments of track with mutual exclusion. Instead uses decentralized Radio Block Controllers (RBCs) for access.