

**Problem 1**

Let $X_C(n), X_L(n)$ be the events representing the # of heads Consuela or Larry have by trial n , respectively, so that the event where Consuela flips more heads than Larry on her $n+1$ toss is

$E = \{X_C(n+1) > X_L(n)\}$. There are three possible events that could exist after n coin flips:

$$A = \{X_C(n) = X_L(n)\}, B = \{X_C(n) < X_L(n)\}, C = \{X_C(n) > X_L(n)\}$$

Let $P(A) = p, P(B) = q, P(C) = q$. Since these events comprise the entire event space,

$$P(A) + P(B) + P(C) = 1, \text{ and } 2p + q = 1.$$

We can condition on the past n trials and use the law of total probability to compute $P(E)$:

$$P(E) = P(E|A)P(A) + P(E|B)P(B) + P(E|C)P(C)$$

If Consuela is behind Larry (event B), she can't make it up in one coin flip so $P(E|B) = 0$. If

Consuela is already ahead (event C), so that $P(E|C) = 1$. If the # of heads is equal on toss n , (event

A), then $P(E|A) = P(\text{Consuela flips heads on } n+1) = \frac{1}{2}$. Combining these equations gives:

$$P(E) = \left(\frac{1}{2}\right)p + q = \left(\frac{1}{2}\right)(2p + q) = \left(\frac{1}{2}\right).$$

Reference: answer adapted from "The Cuny Math Contest" (<http://math.cisdd.org/samples.php>)

Problem 2

(a) Let $X_t \in \{1, 2, 3, 4, 5\}$ be the stochastic process that represents the door John takes at time t . The joint probability of the history of doors taken at time t is given as:

$$P(X_1, \dots, X_t) = \prod_{k=1}^t P(X_k | X_{k-1}, \dots, X_1) = \prod_{k=1}^t P(X_k | X_{k-1})$$

Because the joint probability of the process can be factored like this, X_t is a Markov chain.

(b) Assume the doors are arranged in a circle, so each one has a left and right neighbor; door 5 is the left neighbor of 1. The transition probability matrix, where $T_{ij} = P(X_{k+1} = j | X_k = i)$ is given as:

$$T = \begin{bmatrix} 0 & p & 0 & 0 & (1-p) \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ p & 0 & 0 & 1-p & 0 \end{bmatrix}$$



(c) The steady state probability is computed as $T^\infty = \lim_{n \rightarrow \infty} \prod_1^n T$. Maybe there's a smart way of analytically computing the matrix power, like diagonalizing it first by finding the eigenvectors... but 5x5 matrices are kind of frustrating to deal with, so I'll provide a numerical result. A small R program was written to approximate the steady state transition matrix:

```
steady_state = function(n, p)
{
  q = 1 - p
  T = array(0, c(5, 5))
  T[1, 2] = p; T[2, 3] = p; T[3, 4] = p; T[4, 5] = p; T[5, 1] = p;
  T[1, 5] = q; T[2, 1] = q; T[3, 2] = q; T[4, 3] = q; T[5, 4] = q ;
  Tn = T
  for (k in 1:n) { Tn = Tn%%T }
  print(T)
  print(Tn)
}
```

This program gives the following matrix for $n=100, p \in (0,1)$:

$$T = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

Problem 3

(a) The statement to prove/disprove is:

$$X_1 \perp X_4 | \{X_2, X_3\} \text{ implies } X_2 \perp X_3 | \{X_1, X_4\}$$

This statement is false, a counterexample is given by the following graphical model:

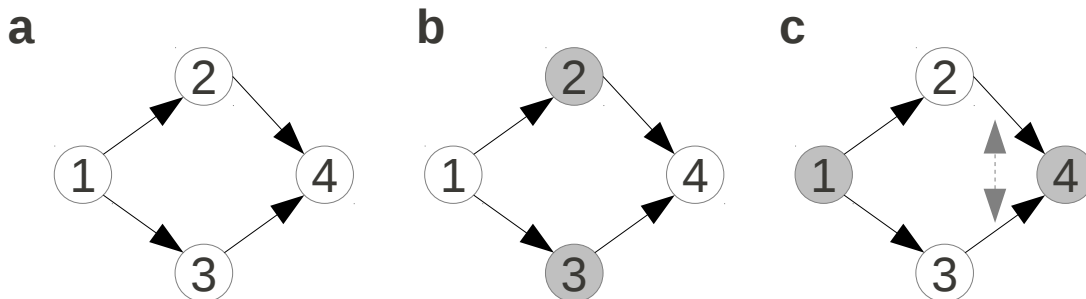


Figure a shows the model without conditioning. b shows $X_1 \perp X_4 | \{X_2, X_3\}$, and no Baye's balls can pass from 1 to 4. However, c shows conditioning on $\{X_1, X_4\}$ and allows a Baye's ball to pass through the “explaining away” subgraph on $\{X_2, X_3, X_4\}$, illustrated by the gray bidirectional arrow.

(b) The statement to prove/disprove is:

$$X_1 \perp X_2 | X_4 \text{ and } X_1 \perp X_3 | X_4 \text{ implies } X_1 \perp \{X_2, X_3\} | X_4$$

This statement can be shown as false, the intuition being that pairwise conditional independence does not imply joint conditional independence. A counterexample can be provided, by letting X_i be a binary random variable, and define each $i=1,2,3,4$ as:

$$X_1 = 1$$

$$X_2 = X_3 = \text{Binary Coin Toss with } p = \frac{1}{2}$$

$$X_4 = X_1(X_2 + X_3) \text{ with modulo 2 addition}$$

To prove that the joint conditional independence implication is violated, we need to show that:

$$P(X_1, X_2, X_3 | X_4) \neq P(X_1 | X_4) P(X_2, X_3 | X_4)$$

The joint conditional probabilities are given as follows:

(X_1, X_2, X_3)	$P(X_1, X_2, X_3 X_4 = 0)$
0, 0, 0	1/6
0, 0, 1	1/6
0, 1, 0	1/6
0, 1, 1	1/6
1, 0, 0	1/6
1, 0, 1	0
1, 1, 0	0
1, 1, 1	1/6

(X_1, X_2, X_3)	$P(X_1, X_2, X_3 X_4 = 1)$
0, 0, 0	0
0, 0, 1	0
0, 1, 0	0
0, 1, 1	0
1, 0, 0	0
1, 0, 1	1/2
1, 1, 0	1/2
1, 1, 1	0

(X_2, X_3)	$P(X_2, X_3 X_4 = 0)$
0, 0	1/3
0, 1	1/6
1, 0	1/6
1, 1	1/3

(X_2, X_3)	$P(X_2, X_3 X_4 = 1)$
0, 0	0
0, 1	1/2
1, 0	1/2
1, 1	0

Evaluating at $(X_1, X_2, X_3) = (0, 0, 0)$ gives:

$$P(X_1, X_2, X_3 = (0, 0, 0) | X_4 = 0) = 1/6$$

$$P(X_1 = 0 | X_4 = 0) = 2/3, \quad P(X_1, X_2 = (0, 0) | X_4 = 0) = 1/3$$

$$P(X_1 = 0 | X_4 = 0) P(X_1, X_2 = (0, 0) | X_4 = 0) = 2/9$$

so that $P(X_1, X_2, X_3 = (0, 0, 0) | X_4 = 0) \neq P(X_1 = 0 | X_4 = 0) P(X_1, X_2 = (0, 0) | X_4 = 0)$, the joint conditional independence assumption is violated, and the original implication is false.

(c) The statement to prove/disprove is:

$$X_1 \perp \{X_2, X_3\} | X_4 \text{ implies } X_1 \perp X_2 | X_4$$

Ran out of time!

(d) Show these statements are equivalent: (i) $X \perp Y | Z$, (ii) $p(x, y, z) = h(x, z)k(y, z)$,
(iii) $p(x, y, z) = p(x, z)p(y, z)/p(z)$.

(i) \Rightarrow (iii) :

$$p(x, y, z) = p(x, y | z) p(z)$$

$$p(x, y, z) = p(x | z) p(y | z) p(z) \text{ by (i)}$$

$$p(x, y, z) = p(x, z) p(y, z) / p(z)$$

(i) \Rightarrow (ii) :

$$p(x, y | z) = p(x | z) p(y | z)$$

$$p(x, y, z) = p(x | z) p(z) p(y | z)$$

$$\text{Let } h(x, z) = p(x | z) p(z), k(y, z) = p(y | z)$$

(iii) \Rightarrow (i) :

$$p(x, z) p(y, z) / p(z) = p(x | z) p(z) p(y | z) p(z) / p(z) \text{ Let } h(x, z) = p(x, z) / p(z), k(y, z) = p(y, z)$$

$$p(x, z) p(y, z) / p(z) = p(x | z) p(y | z) p(z)$$

(ii) \Rightarrow (i) :

$$p(x, y, z) = p(x | y, z) p(y | z) p(z)$$

If $p(x, y, z) = h(x, z)k(y, z)$ then

$$p(x | y, z) = p(x | z)$$

$$p(x, y, z) / p(z) = p(x | z) p(y | z)$$

$$p(x, y | z) = p(x | z) p(y | z)$$

(ii) \Rightarrow (iii) :

$$p(x, y, z) = p(x | y, z) p(y | z) p(z)$$

$$p(x, y, z) = p(x | z) p(y | z) p(z)$$

Problem 4

(a) No, the ordering $\{1, \dots, 10\}$ is not topological, because edge $(10, 3)$ exists but 3 comes before 10 in the sort. One topological ordering for this graph is $\{1, 2, 6, 10, 3, 4, 7, 8, 9, 5\}$.

(b) Standard factorization (let $i = X_i$):

$$p(1 \dots 10) = p(5|9) p(9|3, 7, 8) p(8) p(7) p(4|2, 7) p(3|10) p(10|2) p(6|1, 2) p(2) p(1)$$

(c) For which (i, j) does $X_i \perp X_j$ hold?

$(1, 2), (1, 7), (1, 8), (1, 10), (1, 9), (1, 3), (1, 5)$

$(2, 1), (2, 7), (2, 8)$

$(3, 1), (3, 7), (3, 8), (3, 5)$

$(4, 8)$

$(5, 1)$

$(6, 7), (6, 8)$

$(7, 1), (7, 2), (7, 3), (7, 6), (7, 10), (7, 8)$

$(8, 1), (8, 2), (8, 3), (8, 4), (8, 6), (8, 7), (8, 10)$

$(9, 1)$

$(10, 1), (10, 7), (10, 8)$

(d) What is the largest set A such that $X_1 \perp X_A | \{X_2, X_9\}$ holds?

$A = \{10, 3, 7, 8, 5\}$

(e) What is the largest set B where $X_8 \perp X_B | \{X_2, X_9\}$?

$B = \{1, 6, 5\}$

(f) How to draw from $p(X_5)$ without computing marginal?



I'm not sure?!? Maybe it involves using the Elimination algorithm.

Problem 5

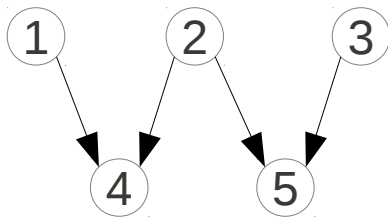
Given $X_i = \text{Coin Toss Lands on Heads}$, $p(X_i) = q$, $i \in \{1, 2, 3\}$, and $Z_4 = X_1 + X_2$, $Z_5 = X_2 + X_3$ with modulo 2 addition.

(a)

(X_2, X_3)	$p(X_2, X_3 Z_5 = 0)$
0, 0	$q - q^2$
0, 1	0
1, 0	0
1, 1	$q - q^2$

(X_2, X_3)	$p(X_2, X_3 Z_5 = 1)$
0, 0	0
0, 1	$q - q^2$
1, 0	$q - q^2$
1, 1	0

(b)



$$X_1 \perp X_2, X_1 \perp X_3, X_1 \perp Z_4, X_1 \perp Z_5$$

$$X_2 \perp X_3, X_2 \perp Z_4, X_2 \perp Z_5$$

$$X_3 \perp Z_4, X_3 \perp Z_5$$

$$X_1 \perp X_2 \perp X_3$$

(c)

Not sure how to draw an undirected graphical model for these relationships. I don't see how conditional dependence relationships for $X_1, X_2 | Z_4$ and $X_2, X_3 | Z_5$ can be represented in an undirectional way.

(d)

First we'll compute the marginal distribution for Z_4 and conditional distribution for $(X_1, Z_4) | X_2$:

$$p(Z_4 = 0) = p(X_1 = 1, X_2 = 1) + p(X_1 = 0, X_2 = 0) = 2q^2 - 2q + 1$$

$$p(Z_4 = 1) = p(X_1 = 1, X_2 = 0) + p(X_1 = 0, X_2 = 1) = 2q - 2q^2$$

(X_1, Z_4)	$p(X_1, Z_4 X_2 = 0)$
(0, 0)	$(1 - q)$
(0, 1)	0
(1, 0)	0
(1, 1)	q

(X_1, Z_4)	$p(X_1, Z_4 X_2 = 1)$
(0, 0)	0
(0, 1)	$(1 - q)$
(1, 0)	q
(1, 1)	0

The joint distribution $p(X_1, Z_4) = p(X_1, Z_4 | X_2 = 0) p(X_2 = 0) + p(X_1, Z_4 | X_2 = 1) p(X_2 = 1)$.

Independence implies $p(X_1, Z_4) = p(X_1)p(Z_4)$. Given an example of (X_1, Z_4) , such as

$(X_1, Z_4) = (0, 0)$, the computation of the joint distribution implies:

$$p(X_1=0, Z_4=0) = p(X_1=0, Z_4=0|X_2=0)p(X_2=0) + p(X_1=0, Z_4=0|X_2=1)p(X_2=1)$$

$$p(X_1=0, Z_4=0) = (1-q)(1-q)$$

Setting this equal to $p(X_1=0)p(Z_4=0) = (1-q)(2q^2-2q+1)$ gives:

$$(1-q)(1-q) = (1-q)(2q^2-2q+1)$$

$$1-q = 2q^2-2q+1$$

$$2q^2-q=0$$

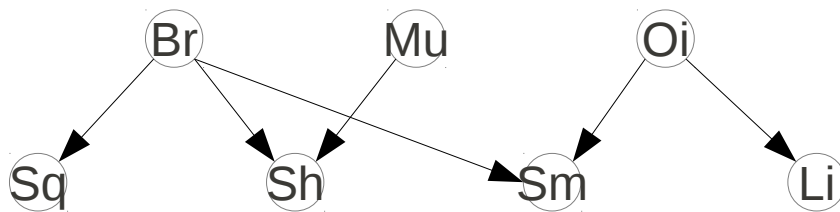
$$2q-1=0$$

$$q=1/2$$

This shows a value of $q=1/2$ or $q=0$ implies $X_1 \perp Z_4$.

Problem 6

(a)



(b)



Given Li, $P(Br|Li) = P(Li|Br)P(Br)/P(Li)$, since $P(Li|Br)=0$, $P(Br|Li)=0$. By a similar argument, $P(Mu|Li)=0$. However, $P(Oi|Li) = P(Li|Oi)P(Oi)/P(Li) \neq 0$. An engine light gives information only about an oil problem.

(c) Given Sm, Sh, what does Li tell us about Br or Mu?

We have $P(Br|Sm, Sh) = P(Sm, Sh|Br)P(Br)/P(Sm, Sh)$ and

$$P(Br|Sm, Sh, Li) = P(Sm, Sh, Li|Br)P(Br)/P(Sm, Sh, Li)$$



It is not necessarily the case that $P(Sm, Sh, Li|Br)/P(Sm, Sh, Li) = P(Sm, Sh|Br)/P(Sm, Sh)$, so the engine light could reduce the probability of a brake problem or increase it. The same argument applies to a muffler problem.

(d) Given Br, a Bayes ball cannot pass between Sm and Mu, so $Sm \perp Mu|Br$, measuring Sm is irrelevant for determining a muffler problem.