UC Berkeley
Department of Electrical Engineering and Computer Science
Department of Statistics

EECS 281A / STAT 241A Statistical Learning Theory

Problem Set 1

Fall 2011

Issued: Thurs, September 8, 2011 **Due:** Monday, September 19, 2011

Reading: For this problem set: Chapters 2–4.

Problem 1.1

Larry and Consuela love to challenge each other to coin flipping contests. Suppose that Consuela flips $n+1$ coins, and Larry flips with n coins, and that all coins are fair. Letting E be the event that Consuela flips more heads than Larry, show that $\mathbb{P}[E] = \frac{1}{2}$.

Problem 1.2

John's office is in a building that has 5 doors. Due to personal peculiarity, John refuses to use the same door twice in a row. In fact, he will choose the door to the left or the right of the last door he used, with probability p and $1-p$ respectively, and independently of what he has done in the past. For example, if he just chose door 5, there is a probability p that he will choose door 4, and a probability $1-p$ that he will choose door 1 next.

(a) Explain why the above process is a Markov chain.

(b) Find the transition probability matrix.

(c) Find the steady state probabilities.

Problem 1.3

Let X_1, X_2, X_3, X_4 be random variables. For each of the following statements, either give a proof of its correctness, or a counterexample to show incorrectness.

(a) If $X_1 \perp X_4 \mid \{X_2, X_3\}$, then $X_2 \perp X_3 \mid \{X_1, X_4\}$.

(b) If $X_1 \perp X_2 \mid X_4$ and $X_1 \perp X_3 \mid X_4$, then $X_1 \perp (X_2, X_3) \mid X_4$.

(c) If $X_1 \perp (X_2, X_3) \mid X_4$, then $X_1 \perp X_2 \mid X_4$.

(d) Given three discrete random variables (X, Y, Z) and their PMF p, the following three statements are all equivalent (that is, $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$)

(i) $X \perp Y \mid Z$

(ii) $p(x, y, z) = h(x, z)k(y, z)$ for some functions h and k.

(iii) $p(x, y, z) = p(x, z)p(y, z)/p(z)$.

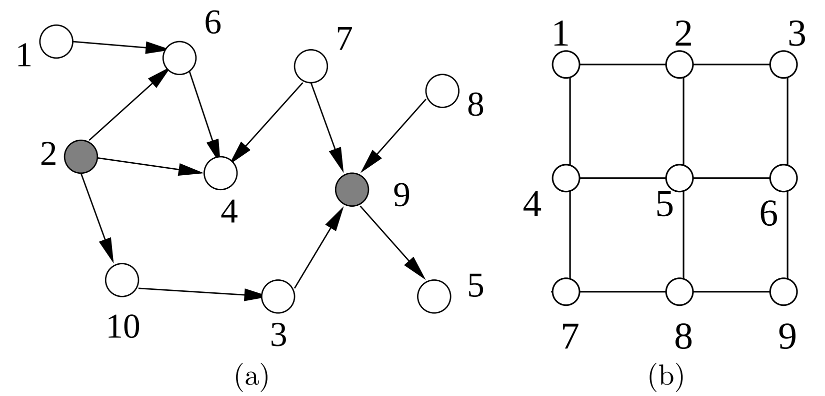

Figure 1: (a) A directed graph. (b) An undirected graphical model: a 3×3 grid or lattice graph. (Two-dimensional lattices frequently arise in spatial statistics.)

Problem 1.4

Directed graphical models: Consider the graph in Figure 1(a).

(a) Is the ordering $\{1, 2, \ldots, 10\}$ topological? If yes, justify your answer; if not, find a topological ordering.

(b) Write down the standard factorization for the given graph.

(c) For what pairs (i, j) does the statement $X_i \perp X_j$ hold? (Don't assume any conditioning in this part.)

(d) Suppose that we condition on $\{X_2, X_9\}$, shown shaded in the graph. What is the largest set A for which the statement $X_1 \perp X_A \mid \{X_2, X_9\}$ holds. The Bayes ball algorithm could be helpful.

(e) What is the largest set B for which $X_8 \perp X_B \mid \{X_2, X_9\}$ holds?

(f) Suppose that I wanted to draw a sample from the marginal distribution $p(x_5) = \mathbb{P}[X_5 = x_5]$. (Don't assume that X_2 and X_9 are observed.) Describe an efficient algorithm to do so without actually computing the marginal. (Hint: Use the factorization from (b).)

Problem 1.5

Graphs and independence relations: For $i = 1, 2, 3$, let X_i be an indicator variable for the event that a toin coss comes up heads (which occurs with probability q). Supposing that that the X_i are independent, define $Z_4 = X_1 \oplus X_2$ and $Z_5 = X_2 \oplus X_3$ where $\oplus$ denotes addition in modulo two arithmetic.

(a) Compute the conditional distribution of (X_2, X_3) given $Z_5 = 0$; then, compute the conditional distribution of (X_2, X_3) given $Z_5 = 1$.

(b) Draw a directed graphical model (the graph and conditional probability tables) for these five random variables. What independence relations does the graph imply?

(c) Draw an undirected graphical model (the graph and compatibility functions) for these five variables. What independence relations does it imply?

(d) Under what conditions on q do we have $Z_5 \perp X_3$ and $Z_4 \perp X_1$? Are either of these marginal independence assertions implied by the graphs in (b) or (c)?

Problem 1.6

Car problems: Alice takes her her car to a dishonest mechanic, who claims that it requires \$2500 dollars in repairs. Doubting this diagnosis, Alice decides to verify it with a graphical model. Suppose that the car can have three possible problems: brake trouble (Br), muffler trouble (Mu), and low oil (Oi), all which are a priori marginally independent. The diagnosis is performed by testing for four possible symptoms: squealing noises (Sq), smoke (Sm), shaking (Sh), and engine light (Li). The conditional probabilities of these symptoms are related to the underlying causes as follows. Squealing depends only on brake problems, whereas smoke depends on brake problems and low oil. Shaking is related to brake problems and muffler problems, whereas the engine light depends only on the oil level.

(a) Draw a directed graphical model (graph only) for this problem.

(b) For which car problems do we gain information by observing the engine light? Suppose that we see the car belching oily clouds of smoke. What does seeing the engine light tell us now?

(c) Suppose that, in addition to smoking, the car also shakes violently. What does the flickering engine light tell us about brake problems or muffler problems?

(d) Now suppose that Alice knows that the car has a brake problem. How useful is it to measure the smoke level in assessing whether it also has a muffler problem?