1 Distribution of the posterior of a finite basis expansion with Gaussian coefficients

Lemma 1. *Let $X^T = \{X_t : t \in [0, T]\}$ be an observation of*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

where b is equipped with the prior distribution defined by

$$b = \sum_{j=1}^{k} \theta_j \phi_j,$$

where $\{\phi_1, \dots, \phi_k\}$ is a linearly independent basis, and $\theta = (\theta_1, \dots, \theta_k)^t$ has multivariate normal distribution $N(\mu, \Sigma)$ and σ is a positive measurable function. Then the posterior distribution of θ is $N((S + \Sigma^{-1})^{-1}(m + \mu), (S + \Sigma^{-1})^{-1})$, where the vector $m = (m_1, \dots, m_k)^t$ is defined by

$$m_l = \int_0^T \frac{\phi_l(X_t)}{\sigma(X_t)^2} dX_t, \quad l = 1, \dots, k,$$

and the symmetric $k \times k$-matrix S is given by

$$S_{l,l'} = \int_0^T \frac{\phi_l(X_t)\phi_{l'}(X_t)}{\sigma^2(X_t)} dt, \quad l, l' = 1, \dots, k,$$

provided $S + \Sigma^{-1}$ is invertible.

Proof. Almost surely we have by Girsanov's theorem

$$p(X^T \mid \theta) = \exp\left(\int_0^T \frac{b(X_t)}{\sigma(X_t)^2} dX_t - \frac{1}{2} \int_0^T \left(\frac{b(X_t)}{\sigma(X_t)} \right)^2 dt \right), \tag{1}$$

with respect to the Wiener measure. So $\log p(X^T \mid b) = \theta^t m - \frac{1}{2}\theta^t S \theta$. And the log of the distribution of θ with respect to the Lebesgue measure on $\mathbb{R}^k$ is given by

$$\begin{aligned}\log p(\theta) =& C_1 - \frac{1}{2}(\theta - \mu)^t \Sigma^{-1} (\theta - \mu) \\ =& C_2 - \frac{1}{2}\theta \Sigma^{-1} \theta + \theta^t \Sigma^{-1} \mu,\end{aligned}$$

for some constants C_1 and C_2.

So, by the Bayes formula, for some constant C_3, the posterior density of θ is given by

$$\begin{aligned}\log p(\theta \mid X^T) =& C_3 + \theta^t m - \frac{1}{2}\theta^t S \theta - \frac{1}{2}\theta \Sigma^{-1} \theta + \theta^t \Sigma^{-1} \mu \\ =& C_3 + \theta^t (m + \Sigma^{-1}\mu) - \frac{1}{2}\theta^t (S + \Sigma^{-1})\theta \\ =& C_3 + \theta^t (S + \Sigma^{-1}) \Big((S + \Sigma^{-1})^{-1} (m + \Sigma^{-1}\mu) \Big) - \frac{1}{2}\theta^t (S + \Sigma^{-1})\theta.\end{aligned}$$

It follows that $p(\theta \mid X^T)$ is normally distributed with mean

$$(S + \Sigma^{-1})^{-1}(m + \mu).$$

and covariance matrix

$$(S + \Sigma^{-1})^{-1}.$$

□

2 Distribution of the posterior of a finite basis expansion with Gaussian coefficients

Lemma 2. *Let $X^T = \{X_t : t \in [0, T]\}$ be an observation of*

$$dX_t = b(X_t)dt + dW_t,$$

where b is equipped with the prior distribution defined by

$$b = \sum_{j=1}^{k} \theta_j \phi_j,$$

where $\{\phi_1, \dots, \phi_k\}$ is a linearly independent basis, and $\theta = (\theta_1, \dots, \theta_k)^t$ has multivariate normal distribution $N(\mu, \Sigma)$. Then the posterior distribution of θ is $N((S+\Sigma^{-1})^{-1}(m+\mu), (S+\Sigma^{-1})^{-1})$, where the vector $m = (m_1, \dots, m_k)^t$ is defined by

$$m_l = \int_0^T \phi_l(X_t)dX_t, \quad l = 1, \dots, k,$$

and the symmetric $k \times k$-matrix S is given by

$$S_{l,l'} = \int_0^T \phi_l(X_t)\phi_{l'}(X_t)dt, \quad l, l' = 1, \dots, k,$$

provided $S + \Sigma^{-1}$ is invertible.

Proof. Almost surely we have by Girsanov's theorem

$$p(X^T \mid \theta) = \exp\left(\int_0^T b(X_t)dX_t - \frac{1}{2}\int_0^T b(X_t)^2 dt\right), \tag{2}$$

with respect to the Wiener measure. So $\log p(X^T \mid b) = \theta^t m - \frac{1}{2}\theta^t S\theta$. And the log of the distribution of θ with respect to the Lebesgue measure on $\mathbb{R}^k$ is given by

$$\begin{aligned}\log p(\theta) =& C_1 - \frac{1}{2}(\theta - \mu)^t\Sigma^{-1}(\theta - \mu)\\ =& C_2 - \frac{1}{2}\theta\Sigma^{-1}\theta + \theta^t\Sigma^{-1}\mu,\end{aligned}$$

for some constants C_1 and C_2.

So, by the Bayes formula, for some constant C_3, the posterior density of θ is given by

$$\begin{aligned}\log p(\theta \mid X^T) =& C_3 + \theta^t m - \frac{1}{2}\theta^t S\theta - \frac{1}{2}\theta\Sigma^{-1}\theta + \theta^t\Sigma^{-1}\mu\\ =& C_3 + \theta^t(m + \Sigma^{-1}\mu) - \frac{1}{2}\theta^t(S + \Sigma^{-1})\theta\\ =& C_3 + \theta^t(S + \Sigma^{-1})\Big((S + \Sigma^{-1})^{-1}(m + \Sigma^{-1}\mu)\Big) - \frac{1}{2}\theta^t(S + \Sigma^{-1})\theta.\end{aligned}$$

It follows that $p(\theta \mid X^T)$ is normally distributed with mean

$$(S + \Sigma^{-1})^{-1}(m + \mu).$$

and covariance matrix

$$(S + \Sigma^{-1})^{-1}.$$

$\square$