CSC418 F2013 A1 part A Solutions.

1) a) Let $p_i = (p_{ix}, p_{iy})$ and $p_{i+1} = (p_{(i+1)x}, p_{(i+1)y})$.

[3 marks]
① for forming vector
② for rotating properly

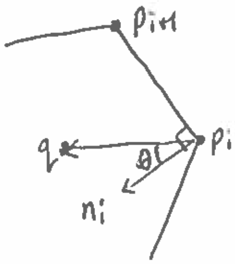
Form the vector $p_{i+1} - p_i$ and rotate it counterclockwise to get the inward-facing normal.

$$p_{i+1} - p_i = (p_{(i+1)x} - p_{ix},\ p_{(i+1)y} - p_{iy})$$

$$\vec{n}_i = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Counter-clockwise rotation by } 90^\circ} (p_{i+1} - p_i) = (p_{iy} - p_{(i+1)y},\ p_{(i+1)x} - p_{ix})$$

b)

[3 marks]
① for setting up vectors
② for dot product inequality



q is on the inward side if $\theta < 90$.

Since $(q - p_i) \cdot \vec{n}_i = \cos\theta \, \|q - p_i\| \cdot \|\vec{n}_i\|$

then q is on the inward side if

$$(q - p_i) \cdot \vec{n}_i > 0$$

c) Assume that if the point is on a boundary, then it is considered inside.

[4 marks]
② for testing against each edge
② for proper logic for the inner portion

```
function is_inside(q):
    for each i in 1..n:
        s = p((i%n)+1)
        n = (p_iy - s_y, s_x - p_ix)
        if (q - p_i) · n < 0
            return false

    inside_inner = true
    for each i in 1..m:
        s = r((i%m)+1)
        n = (r_iy - s_y, s_x - r_ix)
        if (q - r_i) · n < 0
            inside_inner = false
    return not inside_inner
```

[8 marks]

2) a) They do not commute. Counterexample:

~~[6 marks]~~
~~(3) for choosing proper rotations~~
~~(3) for showing that it's correct for arbitrary shear values~~
(2) for specific or general counterexample

$$S = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{uniform scale by } 2} \qquad T = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{translation by } (1,0)}$$

$$ST\begin{bmatrix}1\\0\\1\end{bmatrix} = S\begin{bmatrix}2\\0\\1\end{bmatrix} = \begin{bmatrix}4\\0\\1\end{bmatrix} \neq \begin{bmatrix}3\\0\\1\end{bmatrix} = T\begin{bmatrix}2\\0\\1\end{bmatrix} = TS\begin{bmatrix}1\\0\\1\end{bmatrix}$$

b) They commute: (Consider the affine portions)

(2) for proving general case

Let $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $S = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

$$RS = \begin{bmatrix} s\cos\theta & -s\sin\theta \\ s\sin\theta & s\cos\theta \end{bmatrix} = SR$$

c) They do not commute:

(2) for specific or general counterexample

Let $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$

$$RS = \begin{bmatrix} s_x\cos\theta & -s_y\sin\theta \\ s_x\sin\theta & s_y\cos\theta \end{bmatrix} \neq \begin{bmatrix} s_x\cos\theta & -s_x\sin\theta \\ s_y\sin\theta & s_y\cos\theta \end{bmatrix} = SR$$

d) They do commute:

(2) for proving general case

Let $S_1 = \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix}$, $S_2 = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$

$$S_1 S_2 = \begin{bmatrix} 1 & h+g \\ 0 & 1 \end{bmatrix} = S_2 S_1$$

3) b) Choose a shear in x by s_1, followed by a shear in y by s_2, followed by a shear in x by s_3.

[6 marks]

① for composing a matrix product

① for proper matrix multiplication

① for equating with the form of a rotation matrix

① for solving

② for verifying the top-right value

Let $T = \begin{bmatrix} 1 & s_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & s_3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+s_1s_2 & s_1 \\ s_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & s_3 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+s_1s_2 & s_3+s_1s_2s_3+s_1 \\ s_2 & s_2s_3+1 \end{bmatrix}$

We want to show this matrix is of the form

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for any θ

Then $1+s_1s_2 = 1+s_2s_3$ so $s_1 = s_3$.

and $T = \begin{bmatrix} 1+s_1s_2 & 2s_1+s_1^2s_2 \\ s_2 & 1+s_1s_2 \end{bmatrix}$

From the bottom-left corner, $s_2 = \sin\theta$.

From the top-left and bottom-right,

$$1+s_1\sin\theta = \cos\theta$$

$$s_1 = \frac{\cos\theta - 1}{\sin\theta}$$

We need to check that the upper-right corner equals $-\sin\theta$

$$2s_1+s_1^2s_2 = \frac{2\cos\theta-2}{\sin\theta} + \left(\frac{\cos^2\theta-2\cos\theta+1}{\sin^2\theta}\right)\sin\theta$$

$$= \frac{2\cos\theta-2+\cos^2\theta-2\cos\theta+1}{\sin\theta} = \frac{\cos^2\theta-1}{\sin\theta} = \frac{-\sin^2\theta}{\sin\theta}$$ since $\sin^2\theta+\cos^2\theta=1$

$= \sin\theta$

The transforms are $\begin{bmatrix} 1 & \frac{\cos\theta-1}{\sin\theta} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \sin\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{\cos\theta-1}{\sin\theta} \\ 0 & 1 \end{bmatrix}$

Note that this is not possible when $\sin\theta = 0$.

3) a)

[6 marks]

③ for choosing proper rotations

③ for showing that it's correct for arbitrary shear values

$$\underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{specific rotation}}\underbrace{\begin{bmatrix} 1 & h & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{arbitrary shear in } x \text{ by } h}\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{specific rotation}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & h & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -h & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{shear in } y \text{ by } -h}$$

4) Note that

[10 marks]

(2) for computing piecewise derivatives or expressing as function of $\operatorname{sgn}'(t)$

(1) for noting tangent is undefined at 4 points

(3) for computing correct directions of all 4 tangents and noting that the vector scales as you change t

(1) for computing the normal vectors

(1) for specifying area and perimeter

(2) for general numeric idea for approximate area and perimeter

$$\operatorname{sgn}(\cos 2\pi t) = \begin{cases} -1 & \text{if } \tfrac{1}{4}<t<\tfrac{3}{4} \\ 1 & \text{if } t<\tfrac{1}{4} \text{ or } t>\tfrac{3}{4} \\ 0 & \text{if } t=\tfrac{1}{4} \text{ or } t=\tfrac{3}{4} \end{cases}$$

$$\operatorname{sgn}(\sin 2\pi t) = \begin{cases} -1 & \text{if } \tfrac{1}{2}<t<1 \\ 1 & \text{if } 0<t<\tfrac{1}{2} \\ 0 & \text{if } t=0 \text{ or } t=\tfrac{1}{2} \text{ or } t=1 \end{cases}$$

Let $f(t) = (x(t), y(t)) = (\operatorname{sgn}(\cos 2\pi t)\, a \cos^2 2\pi t,\ \operatorname{sgn}(\sin 2\pi t)\, b \sin^2 2\pi t)$

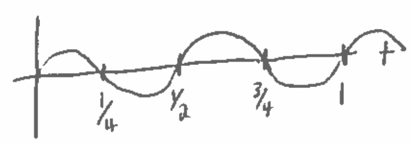
then
$$f(t) = \begin{cases} (a,0) & \text{if } t=0 \\ (a\cos^2 2\pi t,\ b\sin^2 2\pi t) & \text{if } 0<t<\tfrac{1}{4} \\ (0,b) & \text{if } t=\tfrac{1}{4} \\ (-a\cos^2 2\pi t,\ b\sin^2 2\pi t) & \text{if } \tfrac{1}{4}<t<\tfrac{1}{2} \\ (-a,0) & \text{if } t=\tfrac{1}{2} \\ (-a\cos^2 2\pi t,\ -b\sin^2 2\pi t) & \text{if } \tfrac{1}{2}<t<\tfrac{3}{4} \\ (0,-b) & \text{if } t=\tfrac{3}{4} \\ (a\cos^2 2\pi t,\ -b\sin^2 2\pi t) & \text{if } \tfrac{3}{4}<t<1 \\ (a,0) & \text{if } t=1 \end{cases}$$

Note that $\operatorname{sgn}'(t)$ is undefined when $t=0$,

so
$$f'(t) = \begin{cases} g(t)\,(-a,b) & \text{if } 0<t<\tfrac{1}{4} \\ g(t)\,(a,b) & \text{if } \tfrac{1}{4}<t<\tfrac{1}{2} \\ g(t)\,(a,-b) & \text{if } \tfrac{1}{2}<t<\tfrac{3}{4} \\ g(t)\,(-a,-b) & \text{if } \tfrac{3}{4}<t<1 \end{cases}$$

where $g(t) = 4\pi \cos 2\pi t \sin 2\pi t$

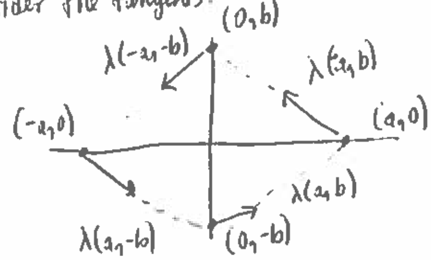
Note the shape of $g(t)$:



when $0<t<\tfrac{1}{4}$ or $\tfrac{1}{2}<t<\tfrac{3}{4}$, $g(t)>0$

and when $\tfrac{1}{4}<t<\tfrac{1}{2}$ or $\tfrac{3}{4}<t<1$, $g(t)<0$

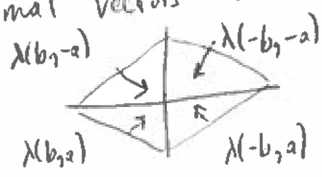
Consider the tangents:



for some scalar $\lambda(t)>0$

The area is $2ab$. The perimeter is $4\sqrt{a^2+b^2}$.

The normal vectors are:



(continued)

4) (continued)

Consider computing the area of a closed parametric curve $(x(t), y(t))$, $0 \le t \le 1$

Consider first the area under $y(t)$:

$\cancel{\int_0^1 y(x)\,dx}$

$A = \int_0^1 y(x)\,dx$

Now think of $x(t)$ as a substitution in this integral.

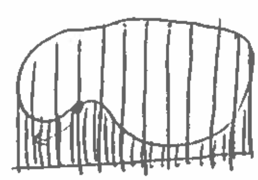
Assume $x(\alpha)=0$ an $x(\beta)=0$ for some α and β.

We need $dx = x'(t)\,dt$

So $A = \int_\alpha^\beta y(t)\,x'(t)\,dt$

This can be computed approximately be estimating the integral numerically.

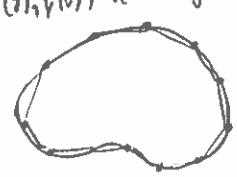
Geometric intuition



Compute area under y curve of bars of width x', subtracting when $x'<0$.

Another way to compute it approximately is to rasterize the curve's polygonal approximation (say, using OpenGL) and count the pixels.

A way to estimate the perimeter is to compute the perimeter of the polygonal approximation. This polygon can be computed by sampling $(x(t), y(t))$ at regularly spaced intervals.



5) a) The tangent is given by $(x'(t), y'(t))$

$$x'(t) = a$$
$$y'(t) = -gt + b$$

So a possible normal is $(b - gt, -a)$

[4 marks]

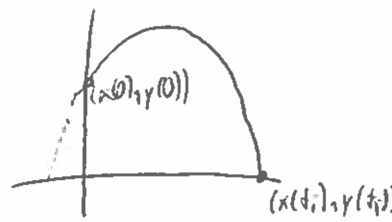
(2) for computing derivatives properly

(2) for calculating the normal

b) The impact occurs when $y(t_i) = 0$

$$-\frac{1}{2}g t_i^2 + b t_i + h = 0$$

$$t_i = \frac{-b \pm \sqrt{b^2 + 2hg}}{-g} = \frac{b \pm \sqrt{b^2+2hg}}{g}$$



we want $t_i > 0$

so $t_i = \frac{b + \sqrt{b^2+2hg}}{g}$

At the time of impact, the location is

$$\left(\frac{a\left(b + \sqrt{b^2+2hg}\right)}{g},\ -\frac{1}{2}g\left(\frac{b+\sqrt{b^2+2hg}}{g}\right)^2 + \frac{b\left(b+\sqrt{b^2+2hg}\right)}{g} + h \right)$$

and the velocity is

$$\left(a, -\left(b + \sqrt{b^2+2hg}\right) + b\right)$$

[6 marks]

(1) for setting up the quadratic equation

(1) for using the quadratic formula

(1) for choosing the positive root

(2) for computing the location at impact

(1) for computing the velocity at impact