Sheet#04 Proposed Solution MLPH_W24

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1 Sheet 5

1.1 1

(a)

$\frac{\partial\sigma(x))}{\partial x} = \frac{\partial}{\partial x}\frac{1}{1+\exp(-x)} = \frac{-(-\exp(-x))}{(1+\exp(-x))^2} = \exp(-x)\sigma(x)^2$

(b)

Consider $2\sigma(2x) - 1 = 2\frac{1}{1+\exp(-2x)} - 1 = 2\frac{\exp(x)}{\exp(x)+\exp(-x)} - 1 = \frac{2\exp(x)-\exp(x)-\exp(-x)}{\exp(x)+\exp(-x)} = \tanh(x)$.

(c)

$a = (1, 1)^{\mathrm{T}}$, $c = (2, 2)^{\mathrm{T}}$, $d = (1, 2)^{\mathrm{T}}$, $f = (2, 3)^{\mathrm{T}}$,

Consider $w = (4, -3)$, $b = 0$:

$$\sigma(a; w, b) = \sigma(1) > 0.5$$
$$\sigma(c; w, b) = \sigma(2) > 0.5$$
$$\sigma(d; w, b) = \sigma(-2) < 0.5$$
$$\sigma(f; w, b) = \sigma(-1) < 0.5$$

[1]:
```
import os
import pandas as pd
from sklearn.linear_model import LogisticRegression
import numpy as np
from matplotlib import pyplot as plt
```

1.2 2 Logistic regression: an LLM lie detector

Download the data from https://heibox.uni-heidelberg.de/f/38bd3f2a9b7944248cc2/
Unzip it and place the lie_detection folder in the folder named `data` to get the following structure: "data/lie_detection/datasets" and "data/lie_detection/acts".

This is how you can load a dataset of LLM activations. Use a new Datamanager if you want to have a new dataset. Use the same data manager if you want to combine datasets.

[2]:
```
from lie_detection_utils import DataManager

path_to_datasets = "data/datasets"
path_to_acts = "data/acts"
```

```
# check if the datasets and activations are available
assert os.path.exists(path_to_datasets), "The path to the datasets does not␣
↪exist."
assert os.path.exists(path_to_acts), "The path to the activations does not␣
↪exist."

# these are the different datasets containing true and false factual statements␣
↪about different topics
dataset_names = ["cities", "neg_cities", "sp_en_trans", "neg_sp_en_trans"]
dataset_name = dataset_names[0] # choose some dataset from the above datasets,␣
↪index "0" loads the "cities" dataset for example

# the dataloader automatically loads the training data for us
dm = DataManager()
dm.add_dataset(dataset_name, "Llama3", "8B", "chat", layer=12, split=0.8,␣
↪center=False,
                device='cpu', path_to_datasets=path_to_datasets,␣
↪path_to_acts=path_to_acts)
acts_train, labels_train = dm.get('train') # train set
acts_test, labels_test = dm.get('val')
print(acts_train.shape, labels_train.shape)
```

```
torch.Size([1196, 4096]) torch.Size([1196])
```

[3]:
```
# have a look at the statements that were fed to the LLM to produce the␣
↪activations:
df = pd.read_csv(f"{path_to_datasets}/{dataset_name}.csv")
print(df.head(10))
```

```
                                        statement  label       city  \
0              The city of Krasnodar is in Russia.      1  Krasnodar
1        The city of Krasnodar is in South Africa.      0  Krasnodar
2                   The city of Lodz is in Poland.      1       Lodz
3  The city of Lodz is in the Dominican Republic.      0       Lodz
4             The city of Maracay is in Venezuela.      1    Maracay
5                 The city of Maracay is in China.      0    Maracay
6               The city of Baku is in Azerbaijan.      1       Baku
7                  The city of Baku is in Ukraine.      0       Baku
8                   The city of Baoji is in China.      1      Baoji
9               The city of Baoji is in Guatemala.      0      Baoji

                  country correct_country
0                  Russia          Russia
1            South Africa          Russia
2                  Poland          Poland
3  the Dominican Republic          Poland
```

```
4              Venezuela       Venezuela
5                  China       Venezuela
6             Azerbaijan      Azerbaijan
7                Ukraine      Azerbaijan
8                  China           China
9              Guatemala           China
```

(a)

```
[4]: for dataset_name in dataset_names:
         dm = DataManager()
         dm.add_dataset(dataset_name, "Llama3", "8B", "chat", layer=12, split=0.8,␣
      ↪center=False,
                         device='cpu', path_to_datasets=path_to_datasets,␣
      ↪path_to_acts=path_to_acts)
         acts_train, labels_train = dm.get('train') # train set
         print(acts_train.mean(), acts_train.std())
         acts_test, labels_test = dm.get('val')
         print(acts_train.shape, labels_train.shape)

         logreg = LogisticRegression(n_jobs=1, C=1e6, random_state=0)

         logreg.fit(acts_train, labels_train)
         assert logreg.n_features_in_ == 4096

         print("score", dataset_name, logreg.score(acts_test, labels_test), np.
      ↪max(logreg.coef_))
```

```
tensor(0.0002) tensor(0.0985)
torch.Size([1196, 4096]) torch.Size([1196])
score cities 0.9966666666666667 0.9378154589707266
tensor(-0.0002) tensor(0.1108)
torch.Size([1196, 4096]) torch.Size([1196])
score neg_cities 1.0 0.48106144589465705
tensor(-0.0027) tensor(0.1095)
torch.Size([283, 4096]) torch.Size([283])
score sp_en_trans 1.0 0.6866930208929193
tensor(-0.0009) tensor(0.1103)
torch.Size([283, 4096]) torch.Size([283])
score neg_sp_en_trans 1.0 0.3806657749382361
```

If our inputs were linearly separable, we would expect at least one very large value in the weights vectors w, $y = \sigma(w \cdot x + b)$, because then the sigmoid turns into a step function along that dimension of data. Howver, the maximum is on the order of unity for all datasets, suggesting that in our chosen representation the classes are not linearly separable and have some overlap.

(b)

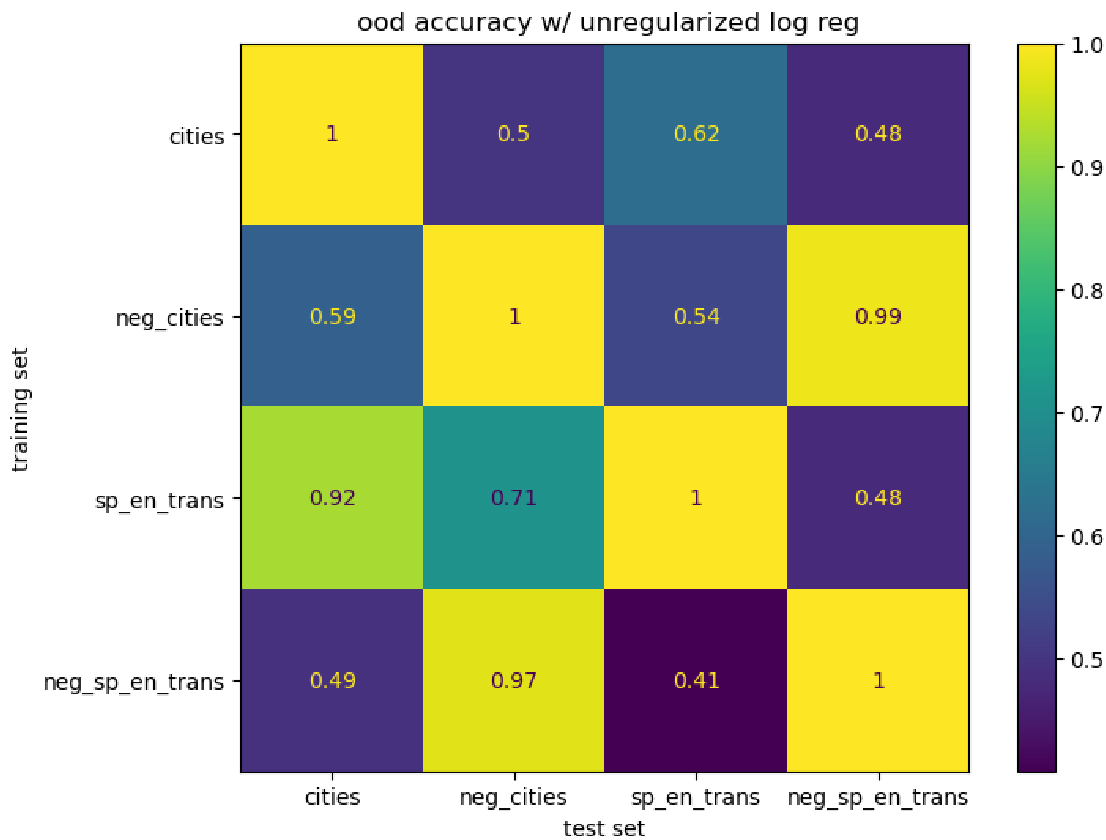
[5]:

```
from sklearn.metrics import ConfusionMatrixDisplay

# Plot of accuracy when train and test data come from different datasets,␣
↪without regularization

scores = []
for train_set_name in dataset_names:
    for test_set_name in dataset_names:

        dm1 = DataManager()
        dm1.add_dataset(train_set_name, "Llama3", "8B", "chat", layer=12,␣
↪split=0.8, center=False,
                        device='cpu', path_to_datasets=path_to_datasets,␣
↪path_to_acts=path_to_acts)
        acts_train, labels_train = dm1.get('train') # train set


        dm2 = DataManager()
        dm2.add_dataset(test_set_name, "Llama3", "8B", "chat", layer=12,␣
↪split=0.8, center=False,
                        device='cpu', path_to_datasets=path_to_datasets,␣
↪path_to_acts=path_to_acts)
        acts_test, labels_test = dm2.get('val')

        logreg = LogisticRegression(n_jobs=10, C=1e6, random_state=0)

        logreg.fit(acts_train, labels_train)
        score = logreg.score(acts_test, labels_test)
        print("score", train_set_name, test_set_name, score)
        scores.append(logreg.score(acts_test, labels_test))
```

```
score cities cities 1.0
score cities neg_cities 0.49666666666666665
score cities sp_en_trans 0.6197183098591549
score cities neg_sp_en_trans 0.4788732394366197
score neg_cities cities 0.59
score neg_cities neg_cities 1.0
score neg_cities sp_en_trans 0.5352112676056338
score neg_cities neg_sp_en_trans 0.9859154929577465
score sp_en_trans cities 0.9233333333333333
score sp_en_trans neg_cities 0.71
score sp_en_trans sp_en_trans 1.0
score sp_en_trans neg_sp_en_trans 0.4788732394366197
score neg_sp_en_trans cities 0.49333333333333335
score neg_sp_en_trans neg_cities 0.9733333333333334
score neg_sp_en_trans sp_en_trans 0.4084507042253521
score neg_sp_en_trans neg_sp_en_trans 1.0
```

```
[6]: fig, ax = plt.subplots(1,1,figsize=(8,6))
     cm = ConfusionMatrixDisplay(np.reshape(scores, (4,4)), display_labels =␣
      ↪dataset_names)

     cm.plot(ax=ax)
     ax.set_xlabel("test set")
     ax.set_ylabel("training set")
     ax.set_title(" ood accuracy w/ unregularized log reg")
```

```
[6]: Text(0.5, 1.0, ' ood accuracy w/ unregularized log reg')
```

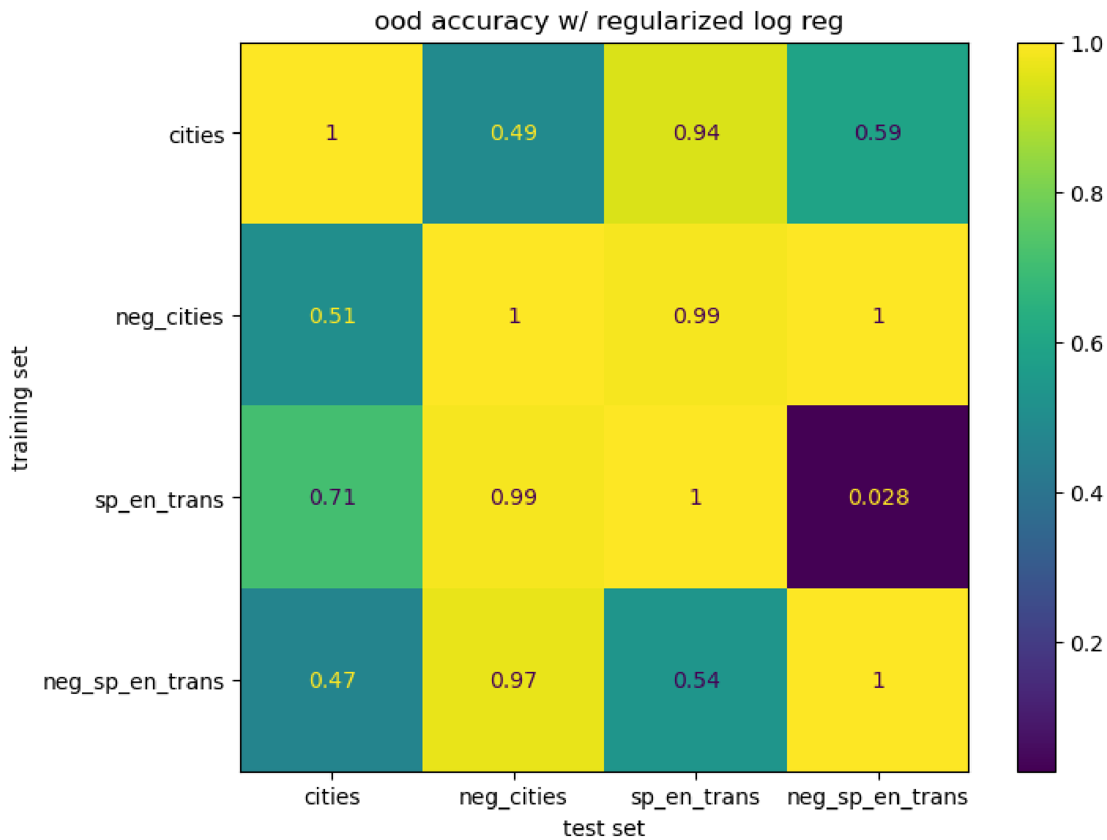



```
[7]: from sklearn.metrics import ConfusionMatrixDisplay

     # Plot of accuracy when train and test data come from different datasets,␣
      ↪without regularization

     scores = []
     for train_set_name in dataset_names:
         for test_set_name in dataset_names:
```

```
        dm1 = DataManager()
        dm1.add_dataset(train_set_name, "Llama3", "8B", "chat", layer=12,␣
↪split=0.8, center=False,
                                device='cpu', path_to_datasets=path_to_datasets,␣
↪path_to_acts=path_to_acts)
        acts_train, labels_train = dm1.get('train') # train set


        dm2 = DataManager()
        dm2.add_dataset(test_set_name, "Llama3", "8B", "chat", layer=12,␣
↪split=0.8, center=False,
                                device='cpu', path_to_datasets=path_to_datasets,␣
↪path_to_acts=path_to_acts)
        acts_test, labels_test = dm2.get('val')

        logreg = LogisticRegression(n_jobs=10, C=1, random_state=0)

        logreg.fit(acts_train, labels_train)
        score = logreg.score(acts_test, labels_test)
#         print("score", train_set_name, test_set_name, score)
        scores.append(logreg.score(acts_test, labels_test))
```

[8]:
```
fig, ax = plt.subplots(1,1,figsize=(8,6))
cm = ConfusionMatrixDisplay(np.reshape(scores, (4,4)), display_labels =␣
↪dataset_names)

cm.plot(ax=ax)
ax.set_xlabel("test set")
ax.set_ylabel("training set")
ax.set_title(" ood accuracy w/ regularized log reg")
```

[8]:
```
Text(0.5, 1.0, ' ood accuracy w/ regularized log reg')
```



We see that regularizing training on the `cities` dataset improves the out-of-distribution test accuracy, especially in the case of the `sp_en_trans` test set. Overall however the regularized model performs poorer than the unregularized model, for instance training on `sp_en_trans` and testing on `cities` or `neg_sp_en_trans` leads to lower accuracy (.91-> .77 and .52 -> 0.07) in the regularized case.

(c)

[9]:
```
training_sets = ["cities", "neg_cities"]
test_sets = ["sp_en_trans", "neg_sp_en_trans"]
# training on (neg-)cities data
dm = DataManager()
# adding both cities set to data
dm.add_dataset(training_sets[0], "Llama3", "8B", "chat", layer=12, split=0.8,␣
 ↪center=False,
                device='cpu', path_to_datasets=path_to_datasets,␣
 ↪path_to_acts=path_to_acts)
dm.add_dataset(training_sets[1], "Llama3", "8B", "chat", layer=12, split=0.8,␣
 ↪center=False,
                device='cpu', path_to_datasets=path_to_datasets,␣
 ↪path_to_acts=path_to_acts)
```

```
cities_train, labels_train = dm.get('train')
cities_test, labels_test = dm.get('val')


logreg = LogisticRegression(n_jobs=1, C=1e-1, random_state=0)

logreg.fit(cities_train, labels_train)
# verifying city training
print("city score", logreg.score(cities_test, labels_test))


dm = DataManager()
# adding both cities set to data
dm.add_dataset(test_sets[0], "Llama3", "8B", "chat", layer=12, split=0.8,␣
 ↪center=False,
                device='cpu', path_to_datasets=path_to_datasets,␣
 ↪path_to_acts=path_to_acts)
dm.add_dataset(test_sets[1], "Llama3", "8B", "chat", layer=12, split=0.8,␣
 ↪center=False,
                device='cpu', path_to_datasets=path_to_datasets,␣
 ↪path_to_acts=path_to_acts)

trans_test, trans_labels_test = dm.get('val')
print("trans score", logreg.score(trans_test, trans_labels_test))
```

```
city score 0.9983333333333333
trans score 0.9788732394366197
```

Training on both of the cities datasets shows a very high (<0.98) test score for both the cities and the trans test sets.

1.3 3 Log-sum-exp and soft(arg)max

1.3.1 (a)

(i) :

$$\text{softmax}(a\sigma, b\lambda)_k = \frac{\exp(ab\lambda\sigma_k)}{\sum_{i=1}^n \exp(ab\sigma_i\lambda)} \tag{1}$$

$$= \frac{(\exp(\lambda\sigma_k))^{ab}}{\sum_{i=1}^n (\exp(\sigma_i\lambda))^{ab}} \tag{2}$$

$$\neq \frac{\exp(\lambda\sigma_k)}{\sum_{i=1}^n \exp(\sigma_i\lambda)} \tag{3}$$

The function is not invariant under rescaling.

(ii) :

$$\begin{aligned}
\operatorname{softmax}(a\sigma, b\lambda)_k &= \frac{\exp(\lambda\sigma_k + c)}{\sum_{i=1}^{n}\exp(ab\sigma_i\lambda + c)} && (4)\\
&= \frac{\exp(c)\exp(\lambda\sigma_k)}{\sum_{i=1}^{n}\exp(c)\exp(\sigma_i\lambda)} && (5)\\
&= \frac{\exp(c)\exp(\lambda\sigma_k)}{\exp(c)\sum_{i=1}^{n}\exp(\sigma_i\lambda)} && (6)\\
&= \frac{\exp(\lambda\sigma_k)}{\sum_{i=1}^{n}\exp(\sigma_i\lambda)} && (7)
\end{aligned}$$

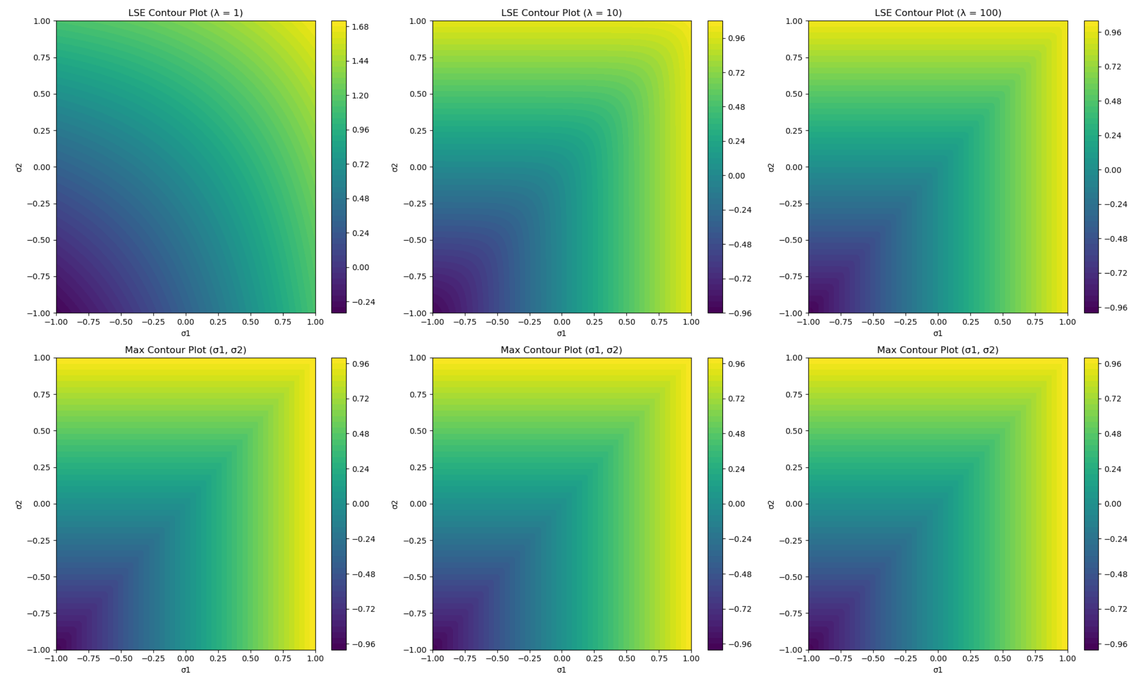
So the function is invariant under a constant offset. Therefore σ^1 and σ^2 produce the same result while σ^3 is diffrent.

1.3.2 (b)

[10]:

```python
def lse(sigma, lamda):
    return (1 / lamda) * np.log(np.sum(np.exp(np.array(sigma) * lamda)))


lambda_values = np.array([1, 10, 100])
sigma_values = np.linspace(-1, 1, 100)
sigma_1, sigma_2 = np.meshgrid(sigma_values, sigma_values)


def evaluate_lse(lamda):
    Z = np.zeros_like(sigma_1)
    for i in range(sigma_1.shape[0]):
        for j in range(sigma_1.shape[1]):
            Z[i, j] = lse([sigma_1[i, j], sigma_2[i, j]], lamda)
    return Z


def max_sigma(sigma_1, sigma_2):
    return np.maximum(sigma_1, sigma_2)


fig, axes = plt.subplots(2, len(lambda_values), figsize=(20, 12))


for idx, lam in enumerate(lambda_values):
    Z_lse = evaluate_lse(lam)
    ax = axes[0, idx]
    cp = ax.contourf(sigma_1, sigma_2, Z_lse, levels=50, cmap='viridis')
    fig.colorbar(cp, ax=ax)
    ax.set_title(f"LSE Contour Plot (  = {lam})")
    ax.set_xlabel(" 1")
```

```
    ax.set_ylabel(" 2")


Z_max = max_sigma(sigma_1, sigma_2)
for idx in range(len(lambda_values)):
    ax = axes[1, idx]
    cp = ax.contourf(sigma_1, sigma_2, Z_max, levels=50, cmap='viridis')
    fig.colorbar(cp, ax=ax)
    ax.set_title(f"Max Contour Plot ( 1,  2)")
    ax.set_xlabel(" 1")
    ax.set_ylabel(" 2")

plt.tight_layout()
plt.show()
```

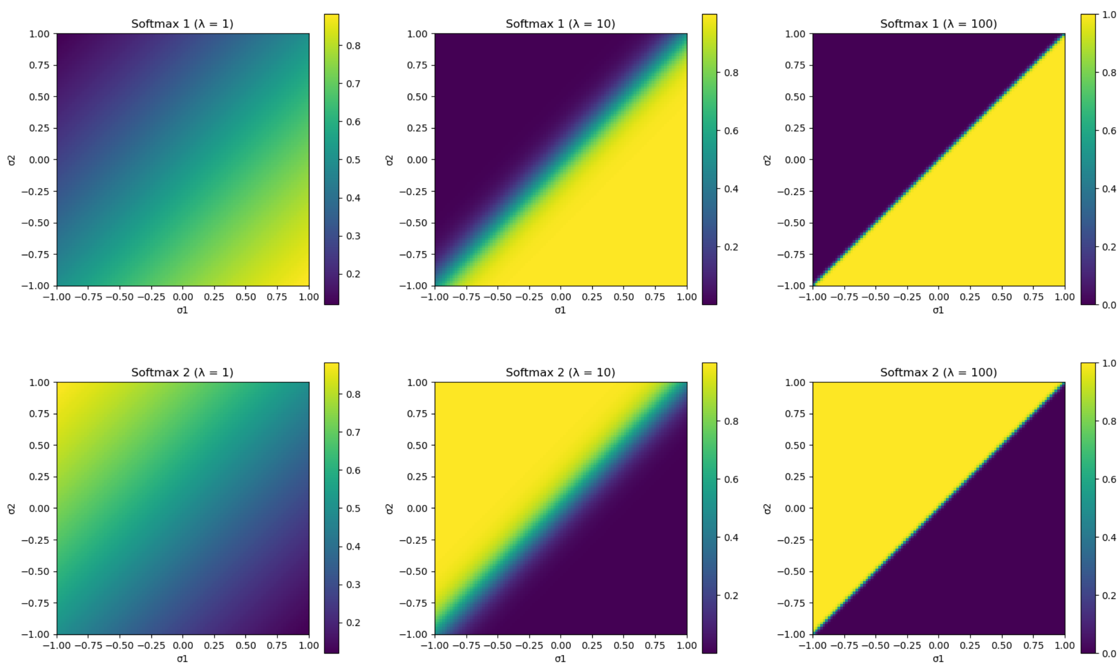


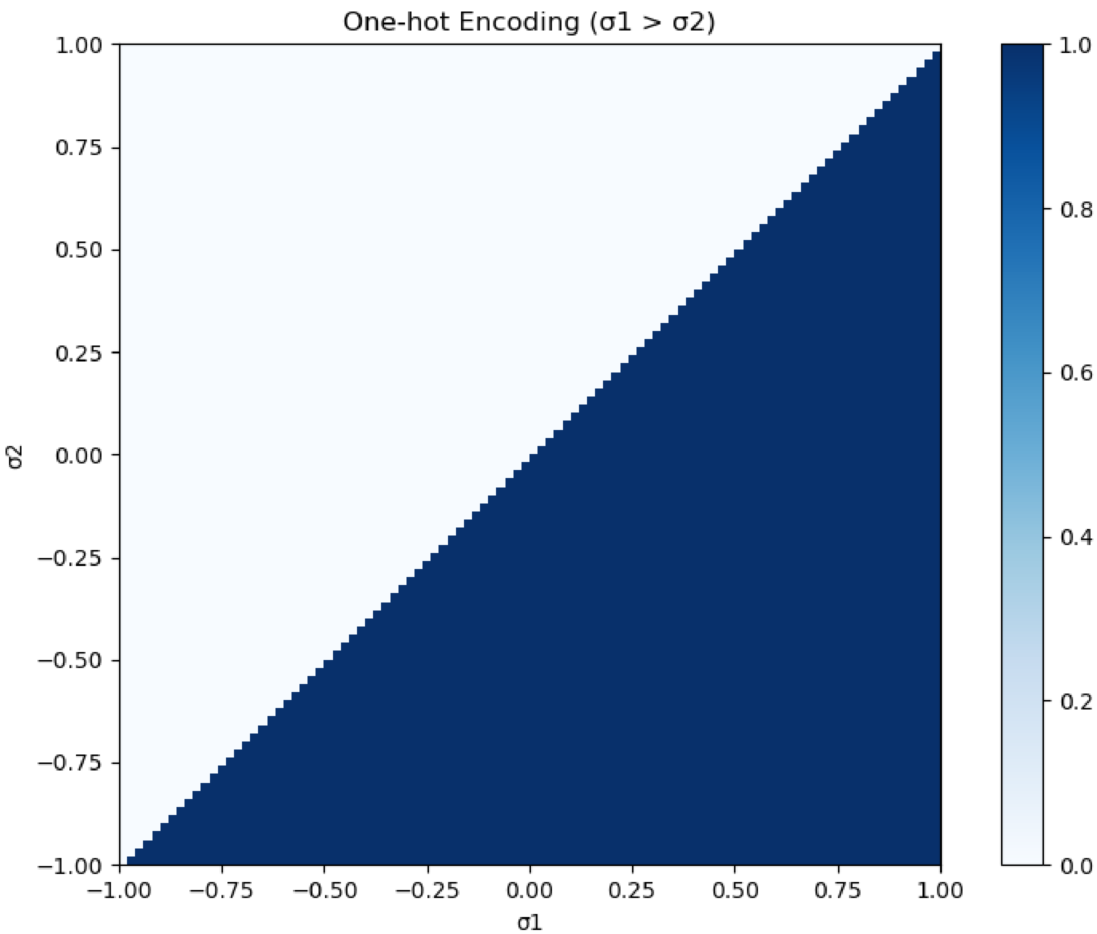
1.3.3 (c)

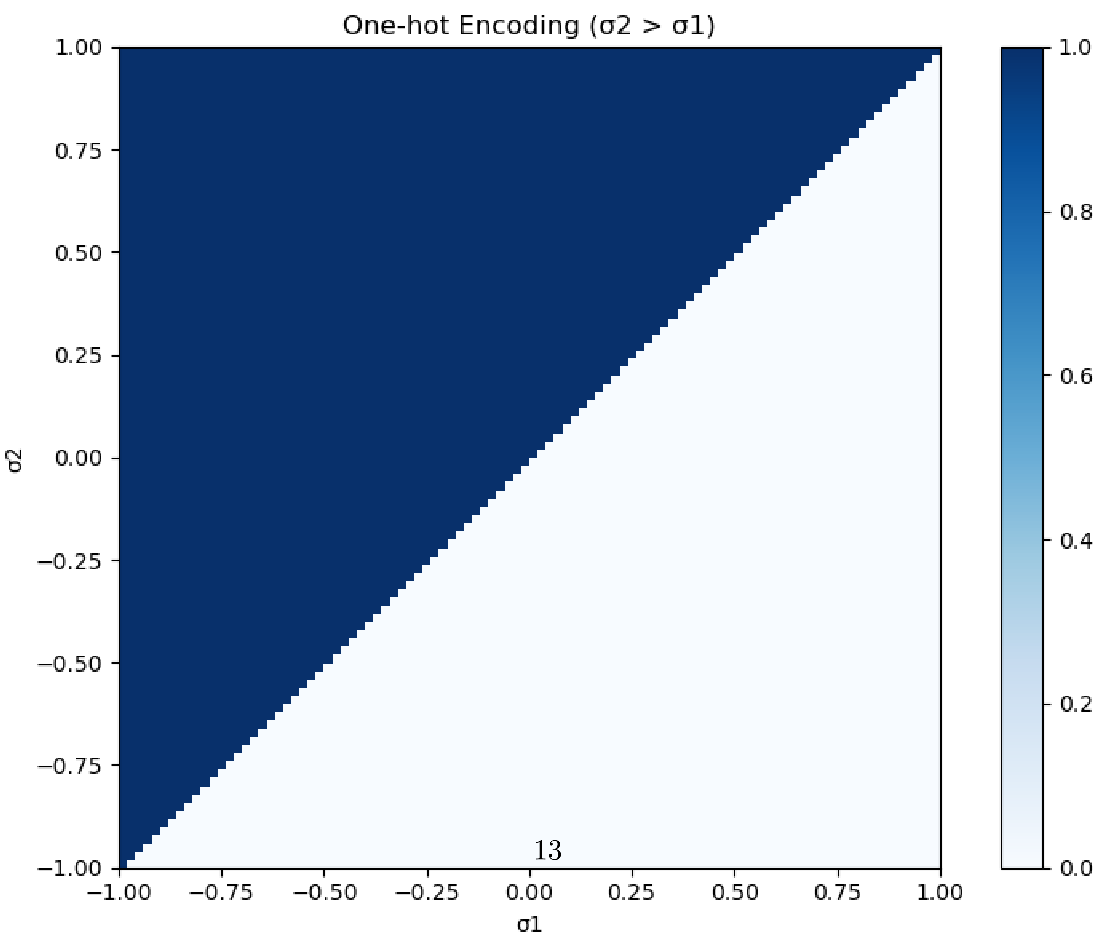
[11]:
```
def softmax_1(sigma, lamda):
    return np.exp(sigma[0] * lamda) / (np.exp(sigma[0] * lamda) + np.
 ↪exp(sigma[1] * lamda))

def softmax_2(sigma, lamda):
    return np.exp(sigma[1] * lamda) / (np.exp(sigma[0] * lamda) + np.
 ↪exp(sigma[1] * lamda))
```

```
def onehot_argmax(sigma):
    return np.array([1, 0] if sigma[0] > sigma[1] else [0, 1])

def evaluate_softmax_components(lamda):
    softmax_1_vals = np.zeros_like(sigma_1)
    softmax_2_vals = np.zeros_like(sigma_1)

    for i in range(sigma_1.shape[0]):
        for j in range(sigma_1.shape[1]):
            softmax_1_vals[i, j] = softmax_1([sigma_1[i, j], sigma_2[i, j]], lamda)
            softmax_2_vals[i, j] = softmax_2([sigma_1[i, j], sigma_2[i, j]], lamda)

    return softmax_1_vals, softmax_2_vals

def evaluate_onehot():
    onehot_vals_1 = np.zeros((sigma_1.shape[0], sigma_1.shape[1]))
    onehot_vals_2 = np.zeros_like(onehot_vals_1)

    for i in range(sigma_1.shape[0]):
        for j in range(sigma_1.shape[1]):
            onehot_vec = onehot_argmax([sigma_1[i, j], sigma_2[i, j]])
            onehot_vals_1[i, j] = onehot_vec[0]
            onehot_vals_2[i, j] = onehot_vec[1]

    return onehot_vals_1, onehot_vals_2

fig, axes = plt.subplots(2, len(lambda_values), figsize=(20, 12))

for idx, lam in enumerate(lambda_values):
    softmax_1_vals, softmax_2_vals = evaluate_softmax_components(lam)

    ax1 = axes[0, idx]
    im1 = ax1.imshow(softmax_1_vals, extent=[-1, 1, -1, 1], origin='lower', cmap='viridis')
    fig.colorbar(im1, ax=ax1)
    ax1.set_title(f"Softmax 1 (  = {lam})")
    ax1.set_xlabel(" 1")
    ax1.set_ylabel(" 2")

    ax2 = axes[1, idx]
    im2 = ax2.imshow(softmax_2_vals, extent=[-1, 1, -1, 1], origin='lower', cmap='viridis')
    fig.colorbar(im2, ax=ax2)
    ax2.set_title(f"Softmax 2 (  = {lam})")
```

```
    ax2.set_xlabel(" 1")
    ax2.set_ylabel(" 2")

onehot_vals_1, onehot_vals_2 = evaluate_onehot()

fig2, axes2 = plt.subplots(2, 1, figsize=(8, 12))

ax1 = axes2[0]
im1 = ax1.imshow(onehot_vals_1, extent=[-1, 1, -1, 1], origin='lower',␣
 ↪cmap='Blues', vmin=0, vmax=1)
fig2.colorbar(im1, ax=ax1)
ax1.set_title("One-hot Encoding ( 1 >  2)")
ax1.set_xlabel(" 1")
ax1.set_ylabel(" 2")

ax2 = axes2[1]
im2 = ax2.imshow(onehot_vals_2, extent=[-1, 1, -1, 1], origin='lower',␣
 ↪cmap='Blues', vmin=0, vmax=1)
fig2.colorbar(im2, ax=ax2)
ax2.set_title("One-hot Encoding ( 2 >  1)")
ax2.set_xlabel(" 1")
ax2.set_ylabel(" 2")

plt.tight_layout()
plt.show()
```




One-hot Encoding (σ1 > σ2)
σ2
σ1
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
1.0
0.8
0.6
0.4
0.2
0.0


One-hot Encoding (σ2 > σ1)
σ2
σ1
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
1.0
0.8
0.6
0.4
0.2
0.0

1.3.4 (d)

$$
\begin{aligned}
\frac{d\text{LSE}(\sigma_k,\lambda)}{d\sigma_k} &= \frac{d}{d\sigma_k}\left(\frac{1}{\lambda}\log\left(\sum_{i=1}^{n}\exp(\sigma_i\lambda)\right)\right) && (8)\\
&= \frac{1}{\lambda}\frac{1}{\sum_{i=1}^{n}\exp(\sigma_i\lambda)}\exp(\lambda\sigma_j\delta_{jk})\lambda && (9)\\
&= \frac{\exp(\lambda\sigma_k)}{\sum_{i=1}^{n}\exp(\sigma_i\lambda)} && (10)\\
&= \text{softmax}_2(\sigma,\lambda) && (11)
\end{aligned}
$$

1.4 4 Linear regions of MLPs

(a)

[12]:
```
import torch

from torch.nn import ReLU, Linear
```

[16]:
```
def shallow_model(x):
    x = x.astype("float32")
    x = x.T
    #assert np.shape(x)[0] == 2
    # initializing the input linear layer that broadcasts all the input to all␣
↪the hidden units
    torch.manual_seed(0)
    input_layer = Linear(in_features=2, out_features=20)

    # initialize activation function

    sigma = ReLU()

    # initializing the linear layer taht sums the activations into a scalar
    output_layer = Linear(in_features=20, out_features=1)

    # input vector
    #x = np.random.normal(size=(10,2)).astype("float32")

    # input vector as tensor
    input_as_tensor = torch.from_numpy(x)

    # apply the ReLU after applying linear transformation to the input
    activation = sigma(input_layer(input_as_tensor))

    # summing the activations to a scalar
```

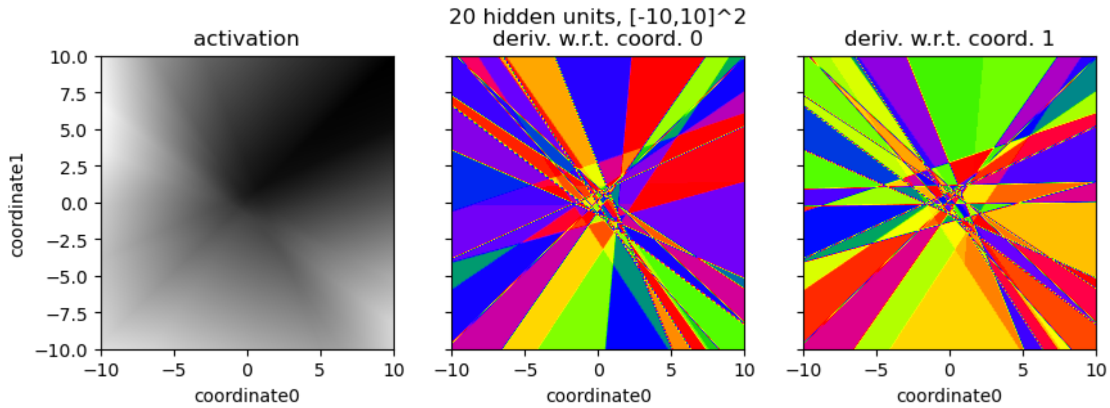
```
    y = output_layer(activation)

    return y
```

The model has 2 * 20 (w) + 20 (b) parameters on the input side and 1*20 + 1 parameters on the output. That makes a total of 81 parameters. It should be noted that only the linear layers have parameters and the ReLU has no parameter. Any scaling of the activation fct. would be absorbed into the data itself or explicit scaling layers.

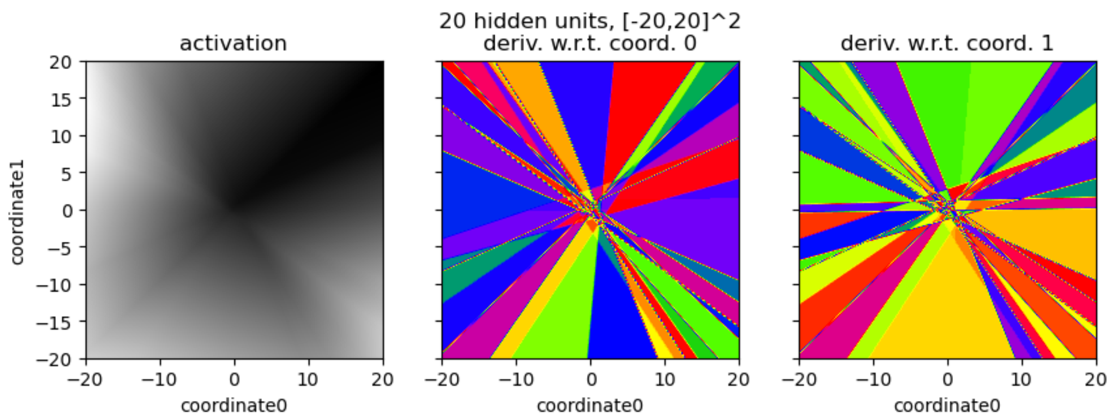
(b), (c)

[17]:
```
def plot_model_output(scale=10, model = shallow_model, title = ""):
    n_points = 2000
    scale = scale
    arr = np.mgrid[-n_points//2:n_points//2, -n_points//2:n_points//2] *␣
↪(scale*2/n_points)
    # print("arr.shape = ", arr.shape)
    y = model(arr).detach().numpy()
    # remove scalar dimension
    y = np.squeeze(y)
    # print("output shape= ", y.shape)
    extent = (-scale, scale, -scale, scale)
    fig, (ax0, ax1, ax2) = plt.subplots(1,3, figsize=(10,6), sharey = True)

    ax0.imshow(y, cmap="gray", extent = extent, origin="lower")
    ax1.imshow(np.gradient(y, axis=1), cmap="prism", extent = extent,␣
↪origin="lower")
    ax2.imshow(np.gradient(y, axis=0), cmap="prism", extent = extent,␣
↪origin="lower")

    ax0.set_ylabel("coordinate1")
    ax0.set_xlabel("coordinate0")
    ax1.set_xlabel("coordinate0")
    ax2.set_xlabel("coordinate0")
    ax0.set_title("activation")
    ax1.set_title("deriv. w.r.t. coord. 0")
    ax2.set_title("deriv. w.r.t. coord. 1")
    if title:
        ax1.set_title(title+"\n"+str(ax1.title.get_text()))
```

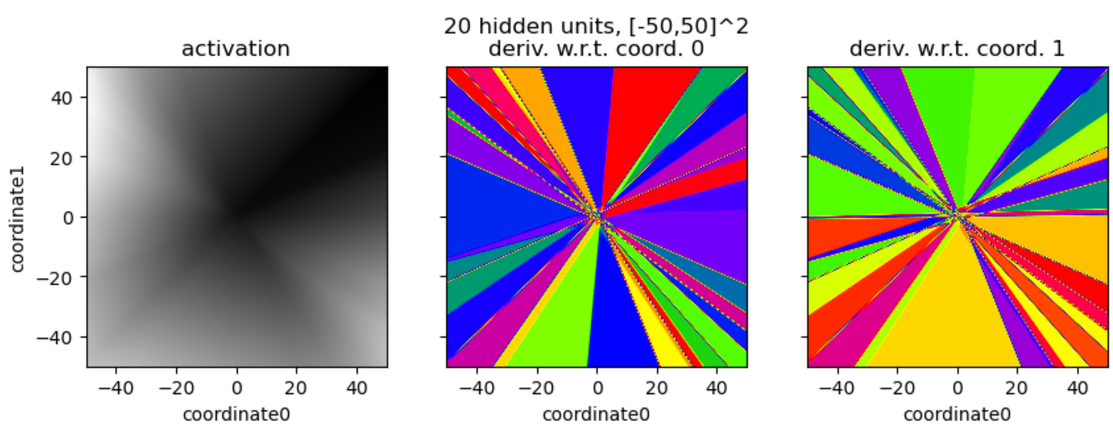
[18]:
```
plot_model_output(title="20 hidden units, [-10,10]^2")
```



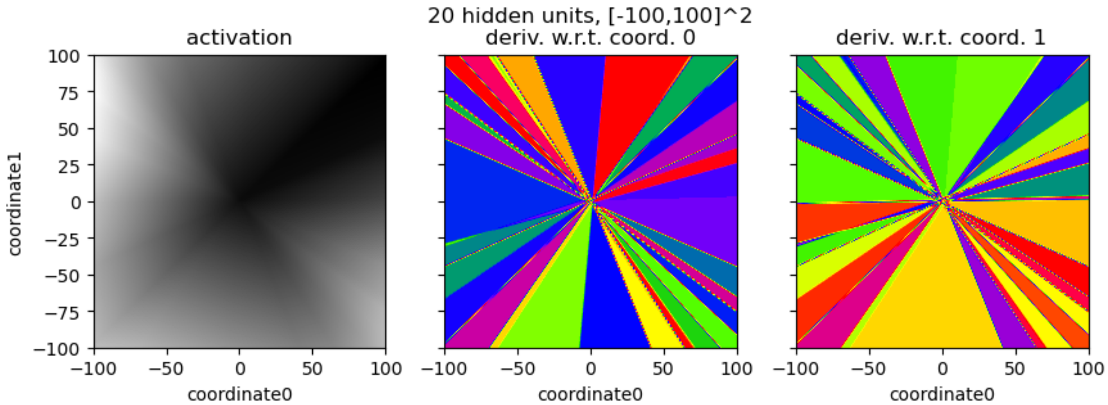
[19]:
```
plot_model_output(scale=20, title="20 hidden units, [-20,20]^2")
```



[20]:
```
plot_model_output(scale=50, title="20 hidden units, [-50,50]^2")
```



[21]:
```
plot_model_output(scale=100, title="20 hidden units, [-100,100]^2")
```



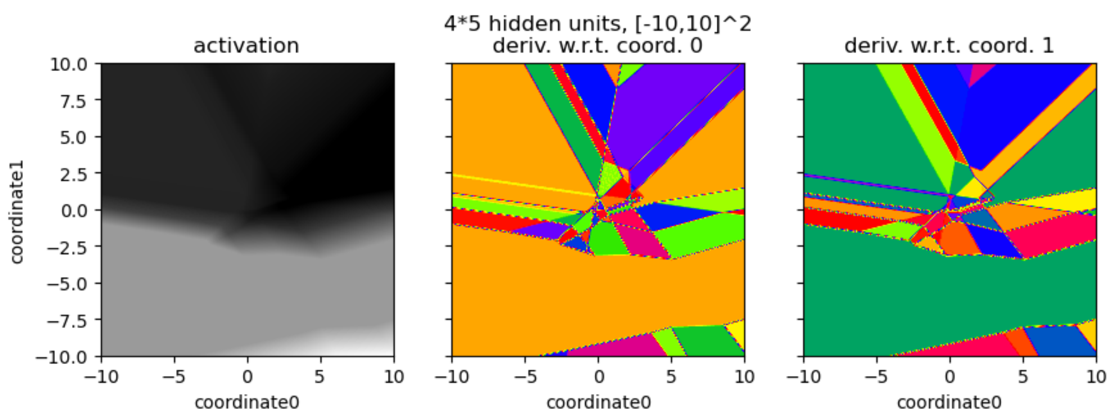
We conclude that the structure disappears at $[-100, 100]^2$ and is mostly centered around the origin within $[-5, 5]^2$, judging from the derivatives.

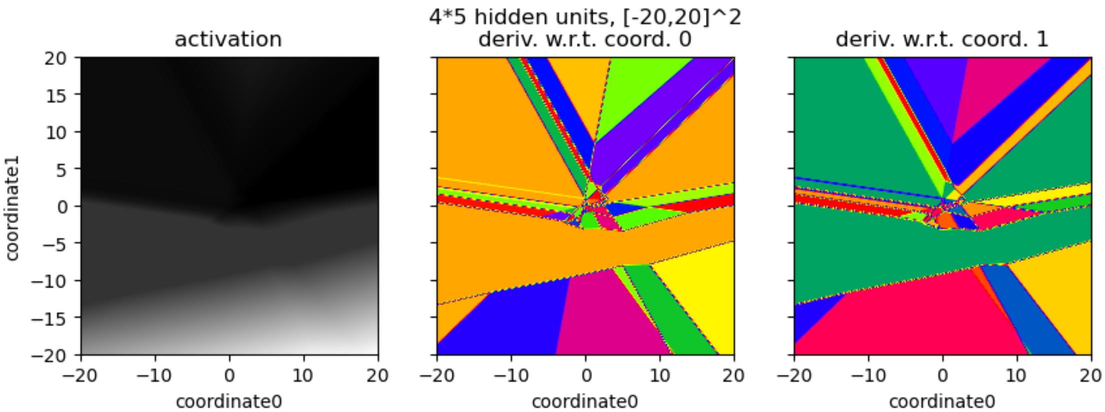
(d)

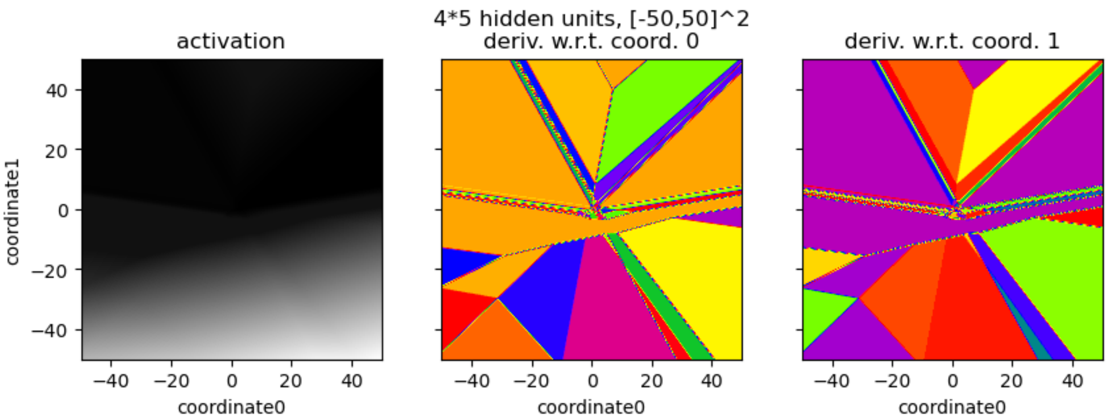
[22]:
```
def deep_model(x):

    x = x.astype("float32")
    x = x.T
    # input vector as tensor
    input_as_tensor = torch.from_numpy(x)

    # initialize activation function
    relu = ReLU()

    torch.manual_seed(2)
    # initializing the input linear layer that broadcasts all the input to all␣
↪the hidden units
    input_layer = Linear(in_features=2, out_features=5)

    # deep layers
    second_layer = Linear(in_features=5, out_features=5)
    third_layer = Linear(in_features=5, out_features=5)
    fourth_layer = Linear(in_features=5, out_features=5)

    # initializing the linear layer taht sums the activations into a scalar
    output_layer = Linear(in_features=5, out_features=1)

    # chaining
    y = output_layer(
```

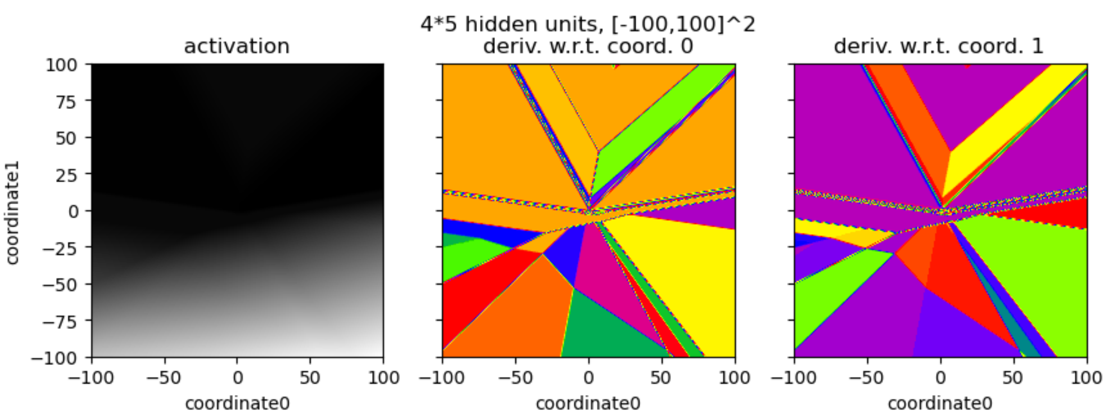
```
            relu(
                fourth_layer(
                    relu(
                        third_layer(
                            relu(
                                second_layer(
                                    relu(
                                        input_layer(input_as_tensor)
                                    )
                                )
                            )
                        )
                    )
                )
            )
        )

    return y
```

[23]:
```
plot_model_output(scale=10, model=deep_model, title="4*5 hidden units,␣
↪[-10,10]^2")
plot_model_output(scale=20, model=deep_model, title="4*5 hidden units,␣
↪[-20,20]^2")
plot_model_output(scale=50, model=deep_model, title="4*5 hidden units,␣
↪[-50,50]^2")
plot_model_output(scale=100, model=deep_model, title="4*5 hidden units,␣
↪[-100,100]^2")
```









For the deeper model with 4 hidden layers (the non-linear activation fct. is applied four times, hence four hidden layers, even though there is a fifth linear layer to the output), the shapes in the output appear more sophisticated judging from the derivatives. It seems like the model activation resembles curved shapes in some instances which is unseen in the linear model, though it is hard to tell with N =1 since the weights are initialized randomly. The structure also does not disappears at larger ranges and as opposed to the shallow model.

1.5 5 Number of linear regions

We consider the construction from the lecture where the non-linearity of the ReLU is represented by a line in two dimensional feature space. In that case, a MLP with no ReLUs has no lines dissecting it, thus it is characterized by a single partition, or linear region. By adding one line, two linear regions appear, such that the total number of regions has risen by one as compared to the case of none. If another line is added such that it intersects the first, the total number of regions grows by two. A third line can be added such that it intersects the first two, the number of intersections will grow by three since the two existing lines will intersect the new line in two points, creating three new regions. This rule generalizes: Any $n-1$ existing lines intersect the nth line $n-1$ times such that n new regions are created (think of the fencepost problem), assuming that none of the lines are parallel. Thus, the total number N of regions created by a single hidden layer with n units is given by

$$N = 1 + \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

where the 1 one the left is due to the fact that without lines there is exactly one partition.