Proof of the arithmetic mean and geometric mean (AM-GM) inequality

MSciMath

The Problem

We need to prove that for any positive numbers a, b we have:

$$\frac{a+b}{2} \geq \sqrt{a \cdot b},$$

and so, if $a \geq 0$ and $b \geq 0$, then

$$\frac{1}{2}(a+b) \geq \sqrt{a \cdot b}$$

The Proof

Case 1: $a = b$

If $a = b$, then

$$\frac{1}{2}(a+a) \geq \sqrt{a * a}$$

Left:	Right:
$\frac{1}{2}(a+a) = \frac{1}{2} \cdot 2a = a$	$\sqrt{a \cdot a} = \sqrt{a^2} = \lvert a \rvert = a$

Case 2: $a = 0$ or $b = 0$

if $a = 0$

$$\frac{1}{2}(0+b) \geq \sqrt{0 \cdot b}$$

Left:	Right:	
$\frac{1}{2}(0+b) = \frac{1}{2}b = \frac{b}{2}$	$\sqrt{0 \cdot b} = \sqrt{0} = 0$	, so:

$$\frac{b}{2} \geq 0$$

if $b = 0$

$$\frac{1}{2}(a+0) \geq \sqrt{a \cdot 0}$$

Left: $\frac{1}{2}(a+0) = \frac{1}{2}a = \frac{a}{2}$ Right: $\sqrt{a \cdot 0} = \sqrt{0} = 0$, so:

$$\frac{a}{2} \geq 0$$

Case 3: $a \neq b$

If $a \neq b$, then:

$$\frac{1}{2}(a+b) \geq \sqrt{a \cdot b}$$

$$a + b \geq 2\sqrt{a \cdot b}$$

$$(a+b)^2 \geq 4\sqrt{a \cdot b}^2$$

$$a^2 + 2ab + b^2 \geq 4 \cdot a \cdot b$$

$$a^2 - 2ab + b^2 \geq 0$$

$$(a-b)^2 \geq 0$$

Conclusion

Since the square of any real number is always non-negative, this means that the original inequality is also always true.