

Lazy Bayesian Approach to Normal Model Using Gibbs Sampling

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Question

Let $X_1, \dots, X_n \stackrel{iid}{\sim} \mathcal{N}(\mu, \tau^2)$, τ being the precision of the normal distribution. Suppose we wish to derive the Bayes estimators of μ and τ w.r.t. independent $\mathcal{N}(\mu_0, t^2)$ (t is the precision) and $\text{Gamma}(\alpha, \beta)$ priors for (μ, τ) . Identify the conditional posterior distributions $p(\mu|\tau, X)$ and $p(\tau|\mu, X)$. Consider the data on systolic BP levels from 20 individuals: 98, 160, 138, 128, 130, 114, 123, 134, 128, 107, 123, 125, 129, 132, 154, 115, 126, 132, 136, 130. Implement a Gibbs sampling procedure for the data set. You may not use WINBUGS. Specify (and justify) your selections of prior distributions. Plot the marginal posterior distributions of μ and τ , and provide the summary statistics in each case.

Derivation of marginal posteriors

Note that the joint distribution of the data is given by

$$f(x_1, \dots, x_n | \mu, \tau) \propto \tau^{\frac{n}{2}} \exp\left(-\frac{t}{2} \sum_{i=1}^n (x_i - \mu)^2\right)$$

and the joint prior is

$$\pi(\mu, \tau) = \pi(\mu)\pi(\tau) \propto \tau^{\alpha-1} \exp\left(-\frac{t}{2}(\mu - \mu_0)^2 - \frac{\tau}{\beta}\right)$$

Thus the posterior is given by

$$g(\mu, \tau | x_1, \dots, x_n) \propto \tau^{\alpha+n/2-1} \exp\left(\frac{t}{2} \sum_{i=1}^n (x_i - \mu)^2 - \frac{t}{2}(\mu - \mu_0)^2 - \frac{\tau}{\beta}\right)$$

Note that the full conditionals are given by

$$\begin{aligned} g(\mu | \tau, x_1, \dots, x_n) &\propto \exp\left(\frac{t}{2} \sum_{i=1}^n (x_i - \mu)^2 - \frac{t}{2}(\mu - \mu_0)^2\right) \\ &\propto \exp\left[\frac{1}{2}(n\tau(\mu - \bar{x})^2 + \frac{t}{2}(\mu - \mu_0)^2)\right] \\ &\propto \exp\left(-\frac{n\tau + t}{2}\left(\mu - \frac{n\tau\bar{x} + t\mu_0}{n\tau + t}\right)^2\right) \\ &\rightarrow \mu | \tau, x_1, \dots, x_n \sim \mathcal{N}\left(\frac{n\tau\bar{x} + t\mu_0}{n\tau + t}, \frac{1}{n\tau + t}\right) \end{aligned}$$

Similarly,

$$\begin{aligned}
g(\tau|\mu, x_1, \dots, x_n) &\propto \tau^{\alpha+n/2-1} \exp\left[-\frac{\tau}{2} \sum_{i=1}^n (x_i - \mu)^2 - \frac{\tau}{\beta}\right] \\
&\propto \tau^{\alpha+n/2-1} \exp\left(-\tau\left(\frac{1}{2} \sum_{i=1}^n (x_i - \mu)^2 + \frac{1}{\beta}\right)\right) \\
&\rightarrow \tau|\mu, x_1, \dots, x_n \sim \text{Gamma}\left(\alpha + \frac{n}{2}, \left(\frac{1}{2} \sum_{i=1}^n (x_i - \mu)^2 + \frac{1}{\beta}\right)^{-1}\right)
\end{aligned}$$

Choice of priors

Density plots for the marginal posterior distributions of μ and τ as well as summary statistics are presented below. We chose a gamma prior for τ with parameters $\alpha=15$ and $\beta=1$. We chose this as we wanted a relatively diffuse prior. Moreover, having α greater than β allowed the distribution to appear like a skewed bell curve with mean=15. I believe the variance most likely falls on the lower tail as high blood pressure may be considered an abnormal event or one that can be treated with medication. This prior allows us to capture that as more density is present on the lower tail whereas less density is present on the upper tail. For choice of μ_0 and τ_0 , I researched recent papers reporting SBP population estimates. In a 2020 study with 1457 participants¹, parameters were estimated as $\mu_{SBP}=111.3$ and $\sigma_{SBP}^2=10$. Therefore, I used 111.3 as my prior for μ and 1/100 as my prior precision τ^2

Gibbs sampler

```

#load data
X <- c(98, 160, 138, 128, 130, 114, 123, 134, 128, 107,
      123, 125, 129, 132, 154, 115, 126, 132, 136, 130)
n <- length(X)
nxbar <- sum(X)

#define algorithm parameters
burn_in <- 1000
loops <- 10000
intervals <- 15

#define prior parameters
beta <- 1
alpha <- 20

#initialize algorithm
mu <- rep(NA, loops)
tau <- rep(NA, loops)
tau[1] <- 1
mu0 <- 111.3
tau0 <- 1/100

#run Gibbs sampler
for (i in 2:loops){
  mu[i] <- rnorm(1, mean = (nxbar*tau[i-1] + mu0*tau0) / ((n*tau[i-1]) + (1*tau0)),
                sd = 1/sqrt(n*tau[i-1] + 1*tau0))

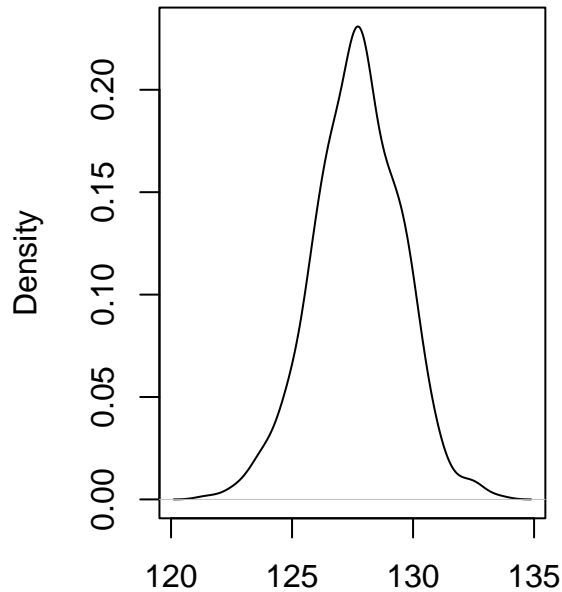
  tau[i] <- rgamma(1, n/2 + alpha, scale=1/(sum((X - mu[i])^2)/2 + 1/beta))
}

```

¹<https://jamanetwork.com/journals/jamacardiology/fullarticle/2766469>

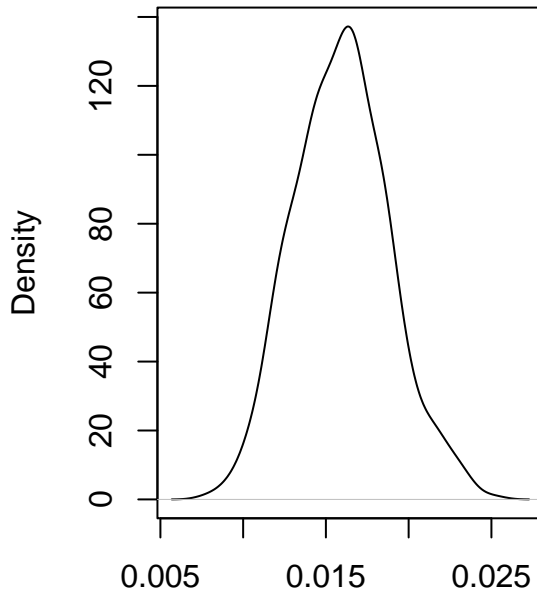
```
#remove burn-in period and record every 15th point
mu <- mu[-(1:burn_in)]
tau <- tau[-(1:burn_in)]
musample <- mu[seq(1,length(mu),intervals)]
tausample <- tau[seq(1,length(tau),intervals)]
```

Marginal posterior of mu



N = 600 Bandwidth = 0.453

Marginal posterior of tau



N = 600 Bandwidth = 0.0007168

Summary statistics for mu

```
t1 <- rbind(round(mean(musample),3), round(sd(musample),3),
            round(median(musample),3), round(IQR(musample),3))
d1 <- data.frame(t(t1))
colnames(d1) <- c("Mean", "SD", "Median", "IQR")
d1
```

```
##      Mean    SD  Median  IQR
## 1 127.694 1.809 127.676 2.45
```

Summary statistics for tau

```
t2 <- rbind(round(mean(tausample),3), round(sd(tausample),3),
            round(median(tausample),3), round(IQR(tausample),3))
d2 <- data.frame(t(t2))
colnames(d2) <- c("Mean", "SD", "Median", "IQR")
d2
```

##		Mean	SD	Median	IQR
##	1	0.016	0.003	0.016	0.004