Back to Basix

Construction of arbitrary order elements on polygons and polyhedrons in FEniCSx

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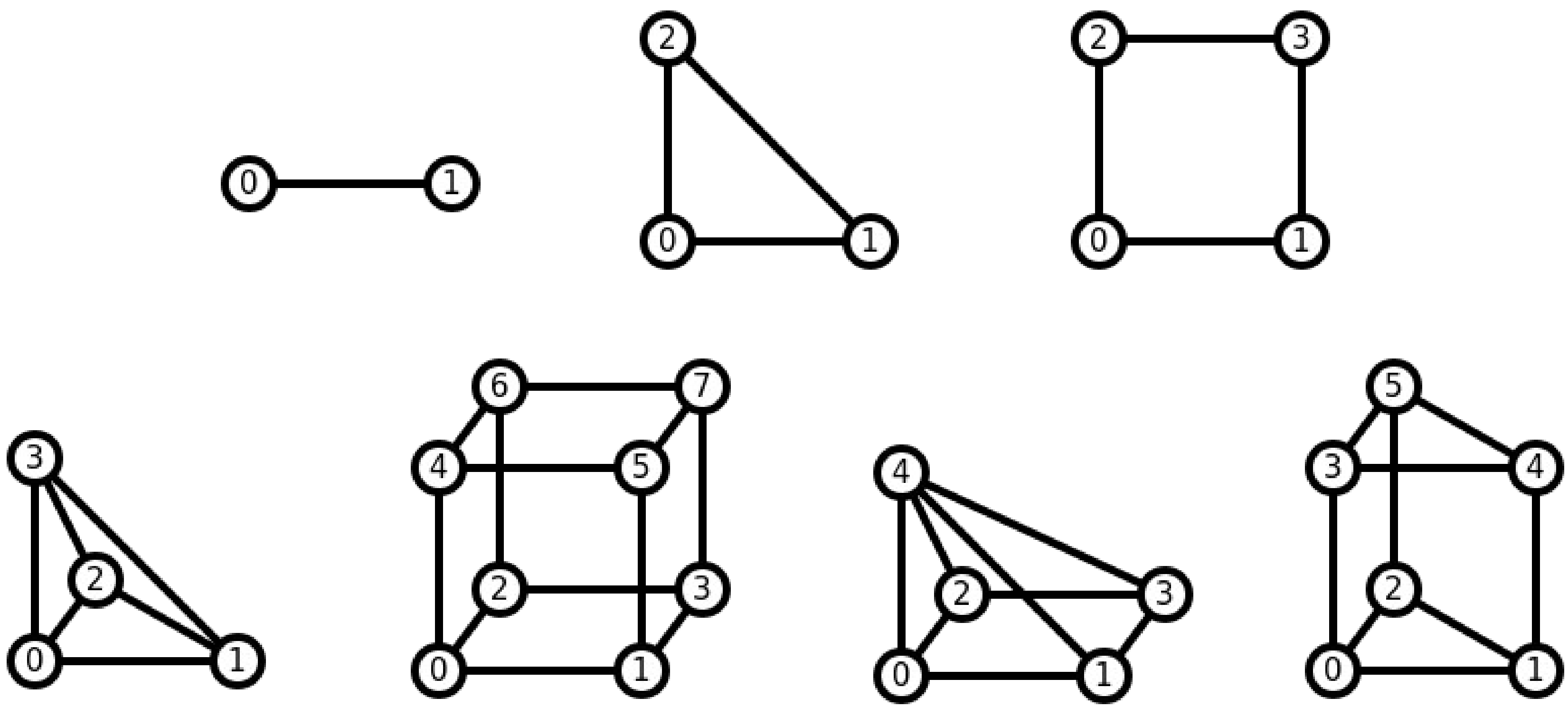
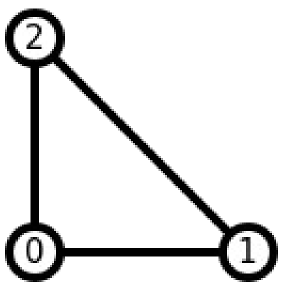
Tabulation in FEniCSx

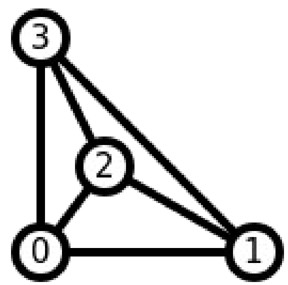
- FEniCS uses FIAT.
- We want:
  - Tabulation at runtime.
  - C++.

Basix
github.com/FEniCS/basix

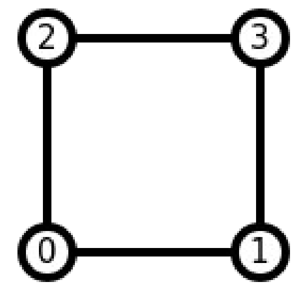
- C++ with Python interface

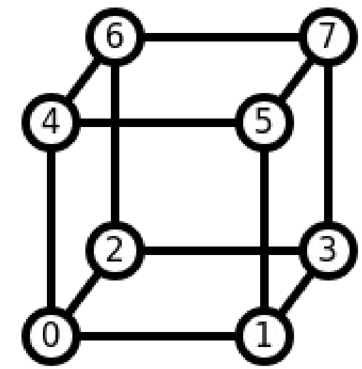






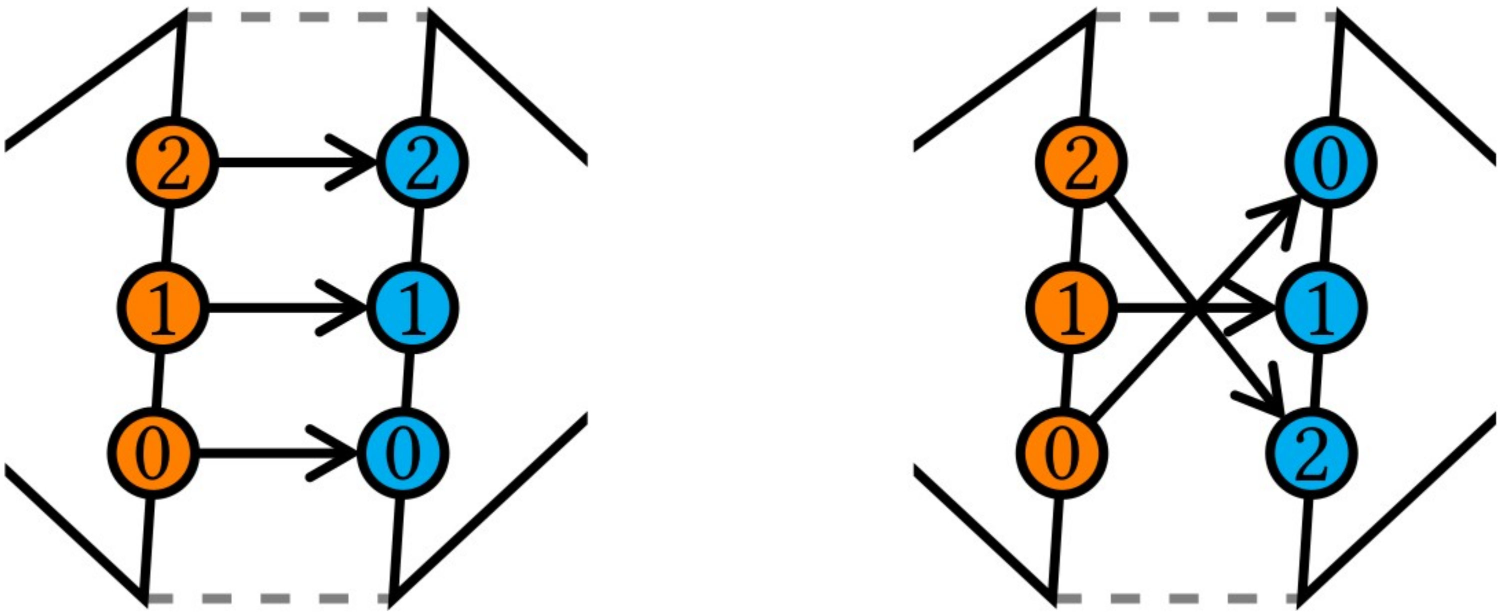
Lagrange
Nédélec (first kind)
Nédélec (second kind)
Raviart–Thomas
Brezzi–Douglas–Marini
Bubble
Crouzeix–Raviart
Regge





Lagrange (Q)
Nédélec
Raviart–Thomas
Bubble
DPC
Serendipity

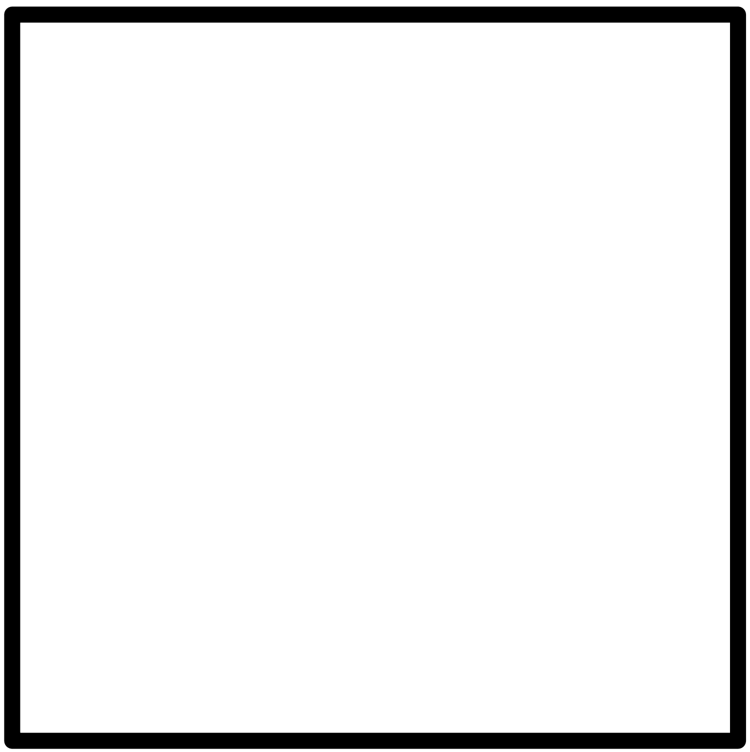
Higher order spaces

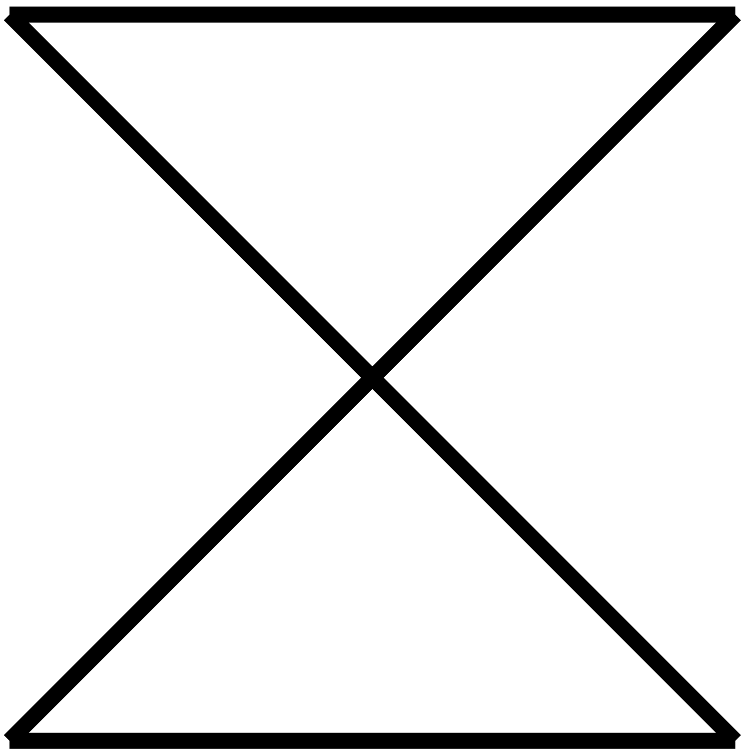


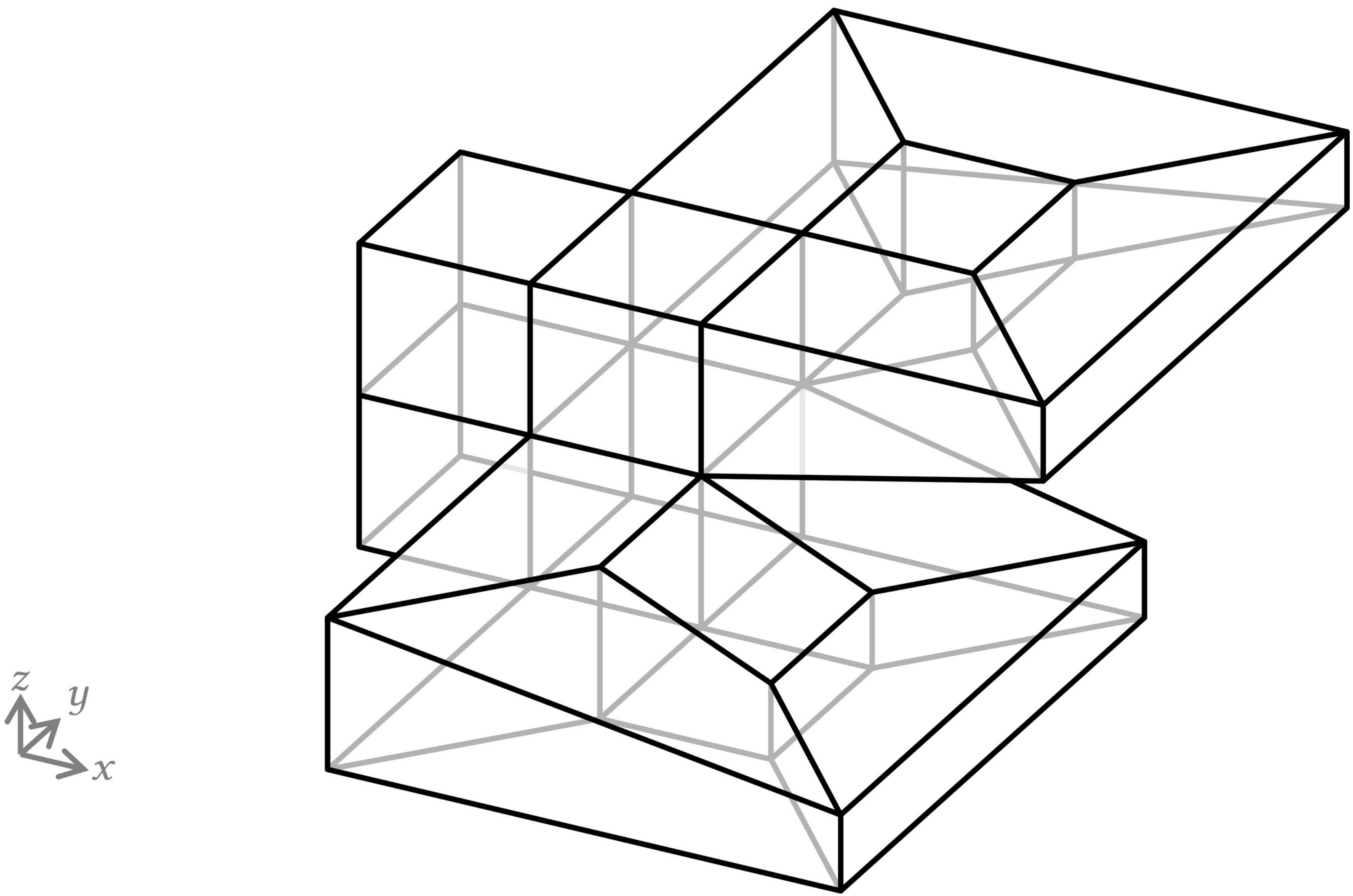
DOF transformations

- Solution?: order cells in the mesh

Input cells		Cells
`[0, 5, 1]`		`[0, 1, 5]`
`[1, 2, 3]`	→	`[1, 2, 3]`
`[5, 3, 0]`		`[0, 3, 5]`
`[5, 1, 2]`		`[1, 2, 5]`

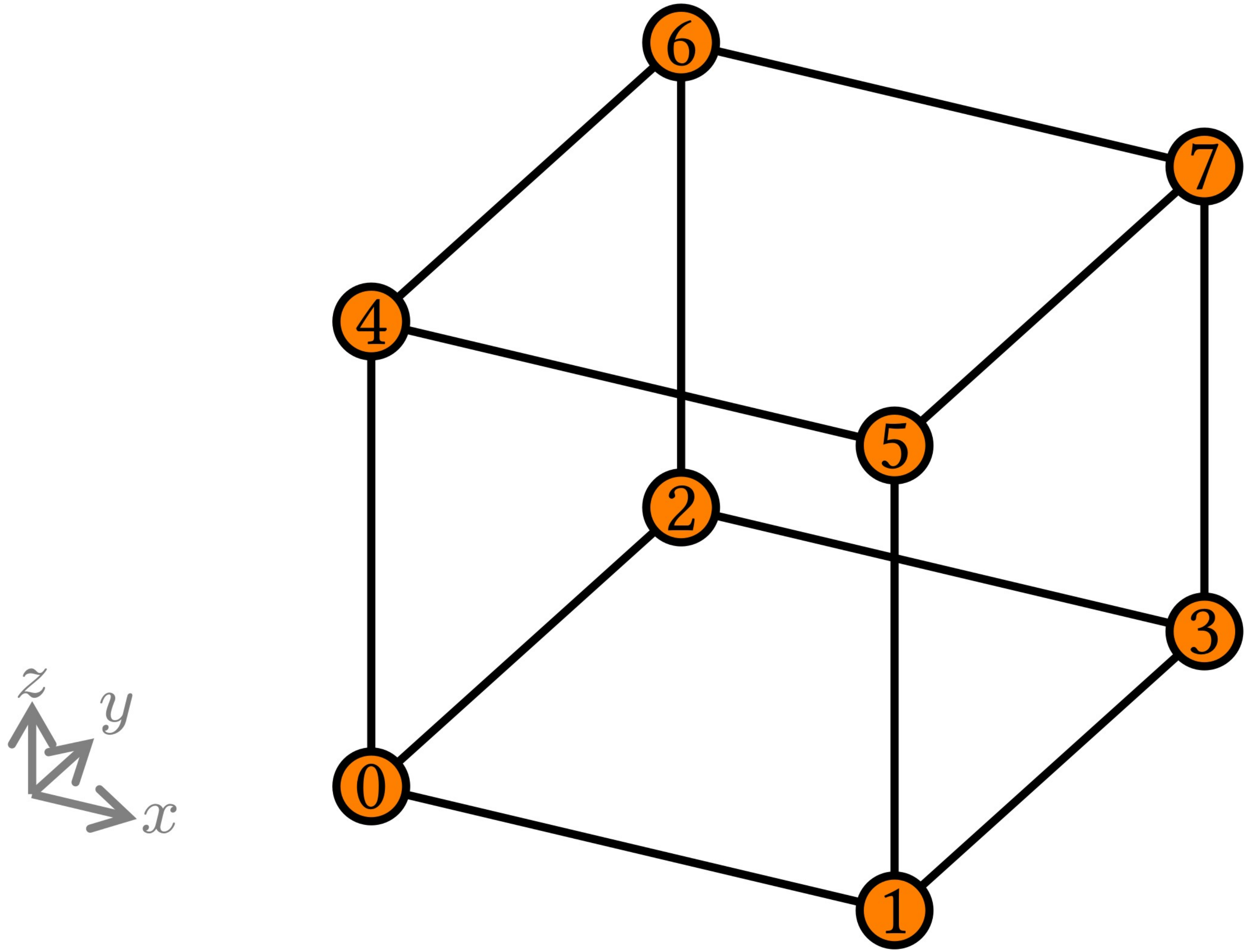




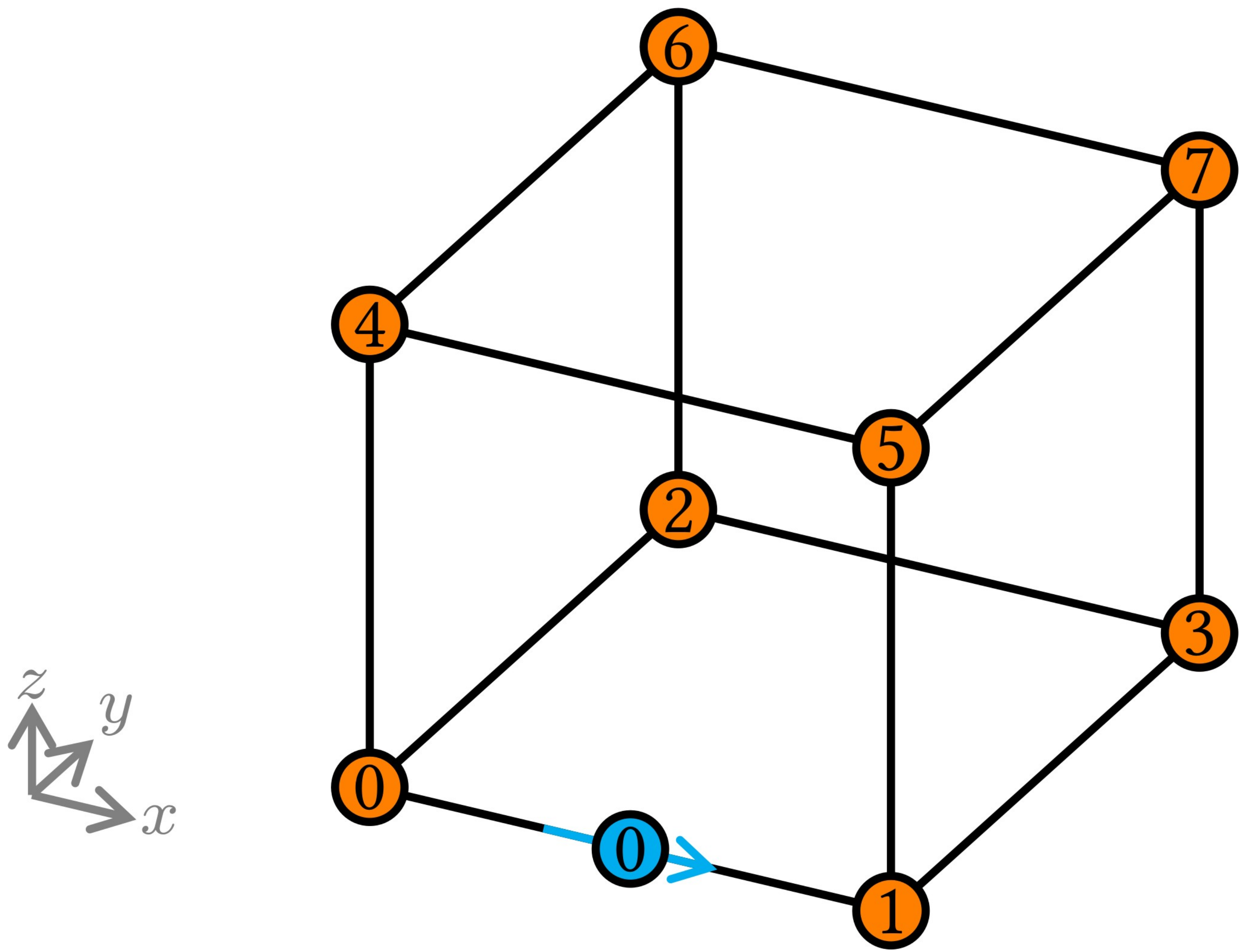


Rainer Agelek, Michael Anderson, Wolfgang Bangerth, and William L. Barth.
On orienting edges of unstructured two- and three-dimensional meshes. (2017)
ACM Transactions on Mathematical Software 44, 1, 5:1–5:22. DOI: **10.1145/3061708**

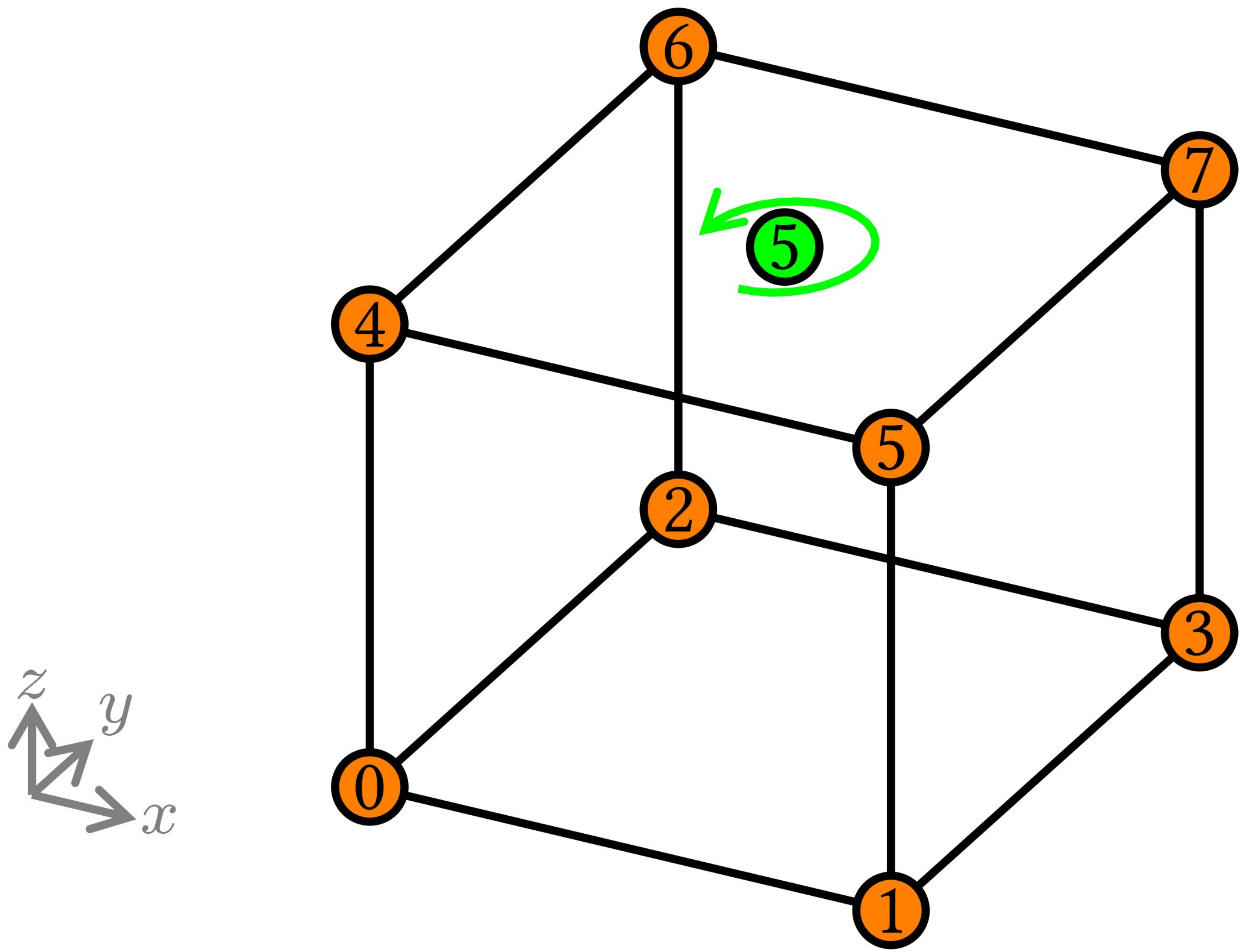
Example: Hexahedron



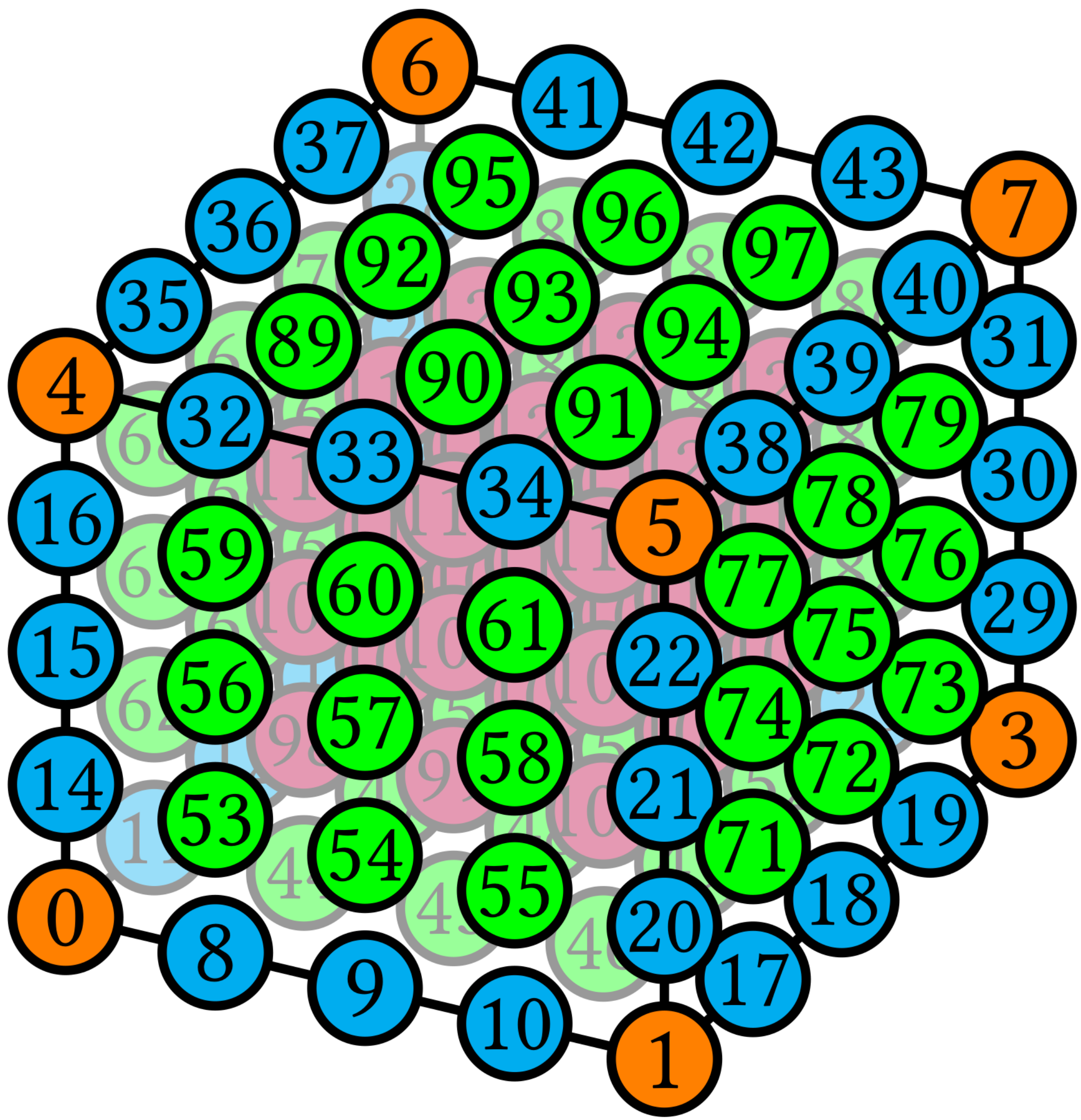
Example: Hexahedron



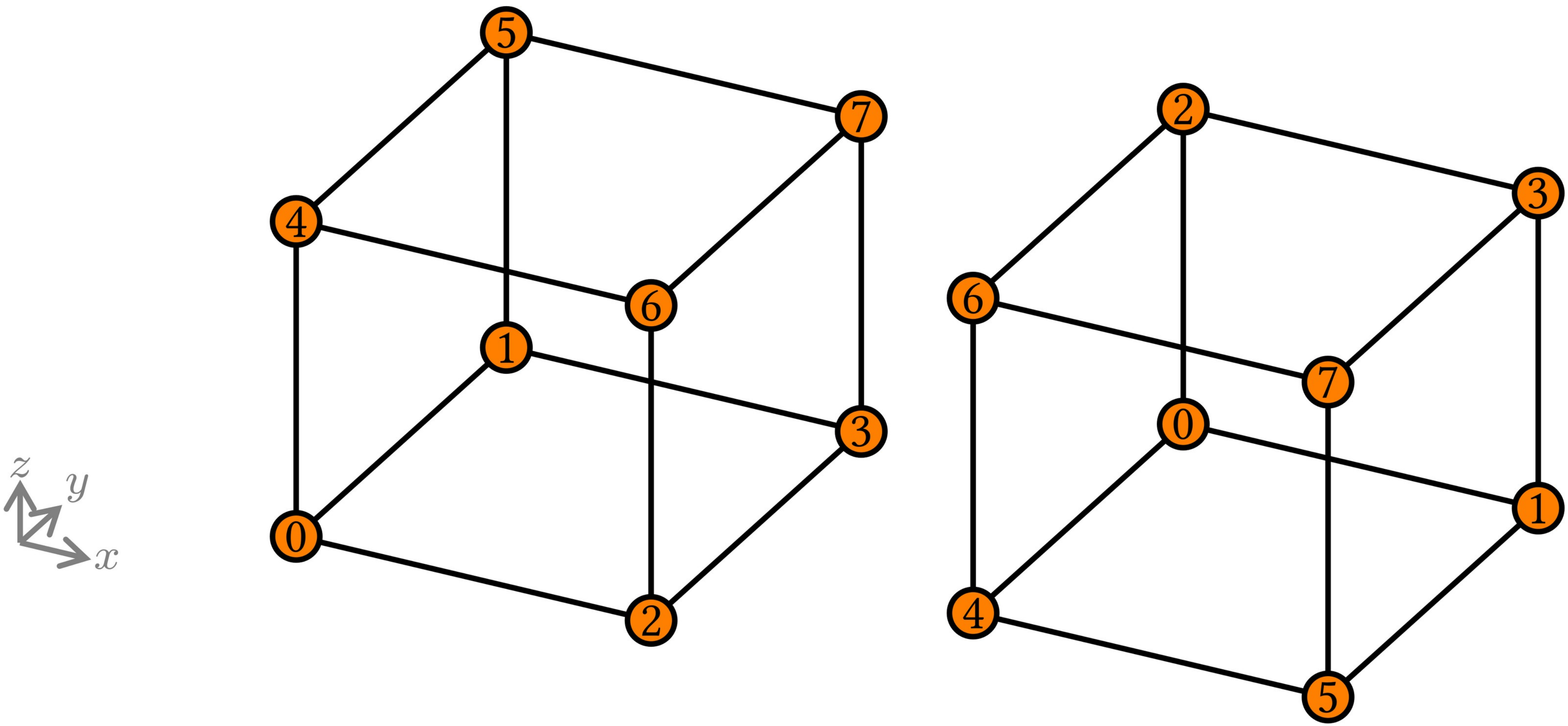
Example: Hexahedron



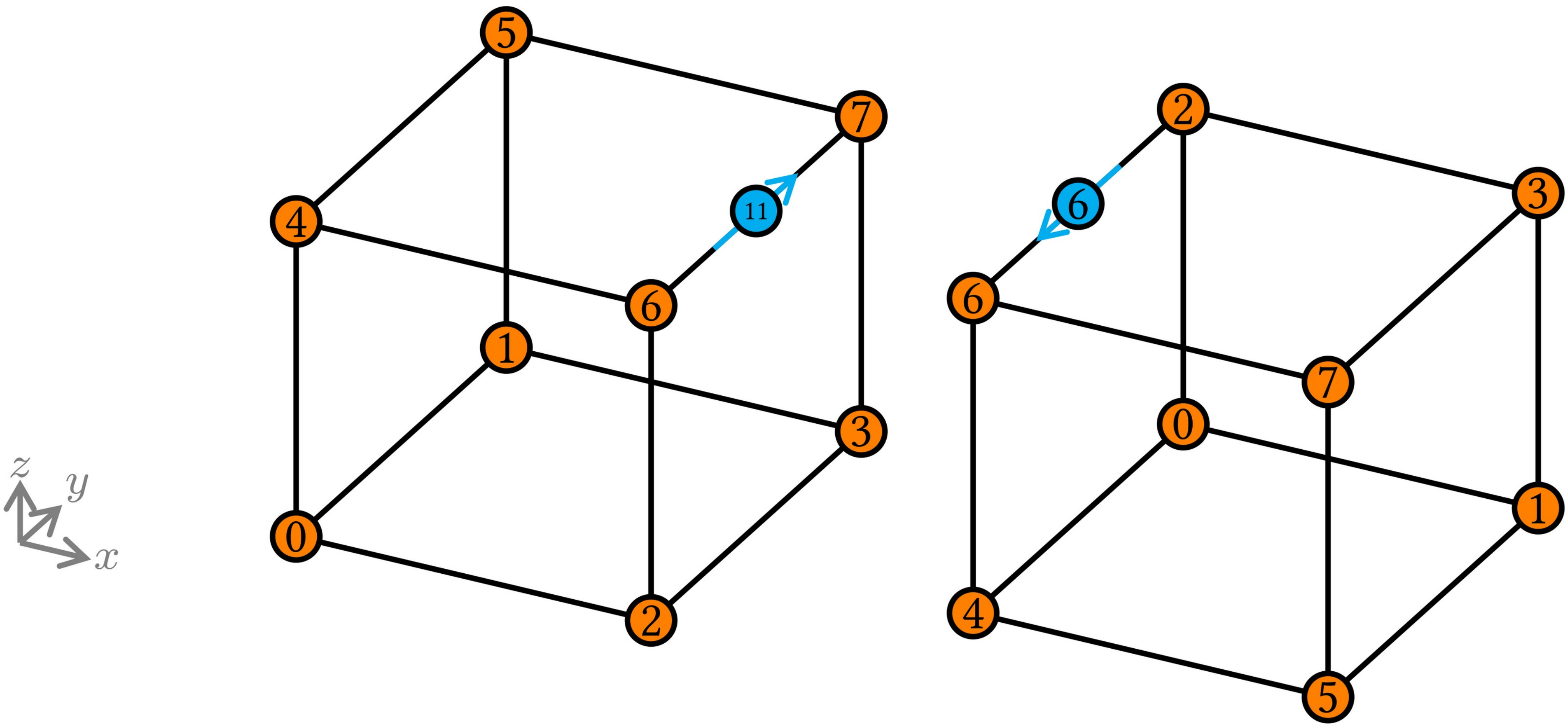
Example: Order 4 Lagrange



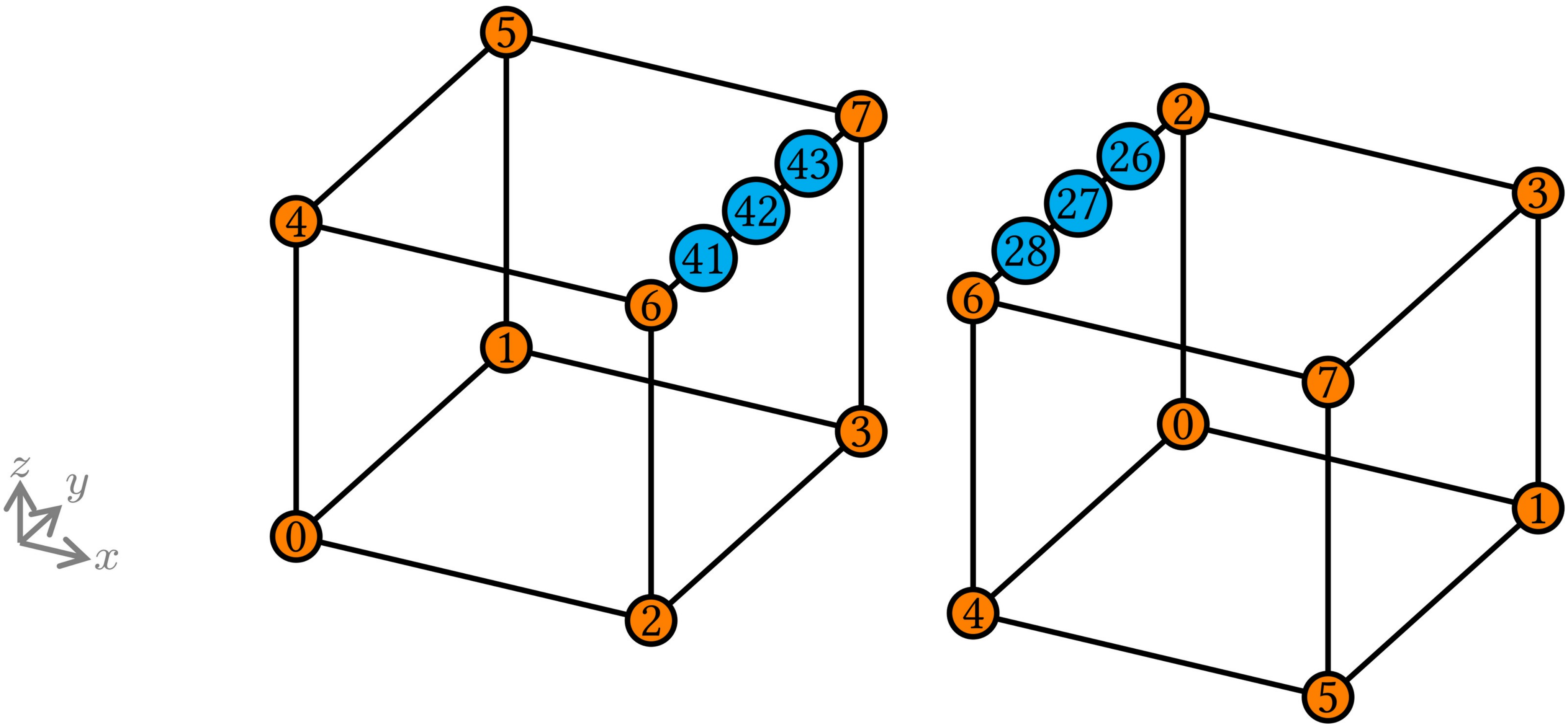
Example: Order 4 Lagrange



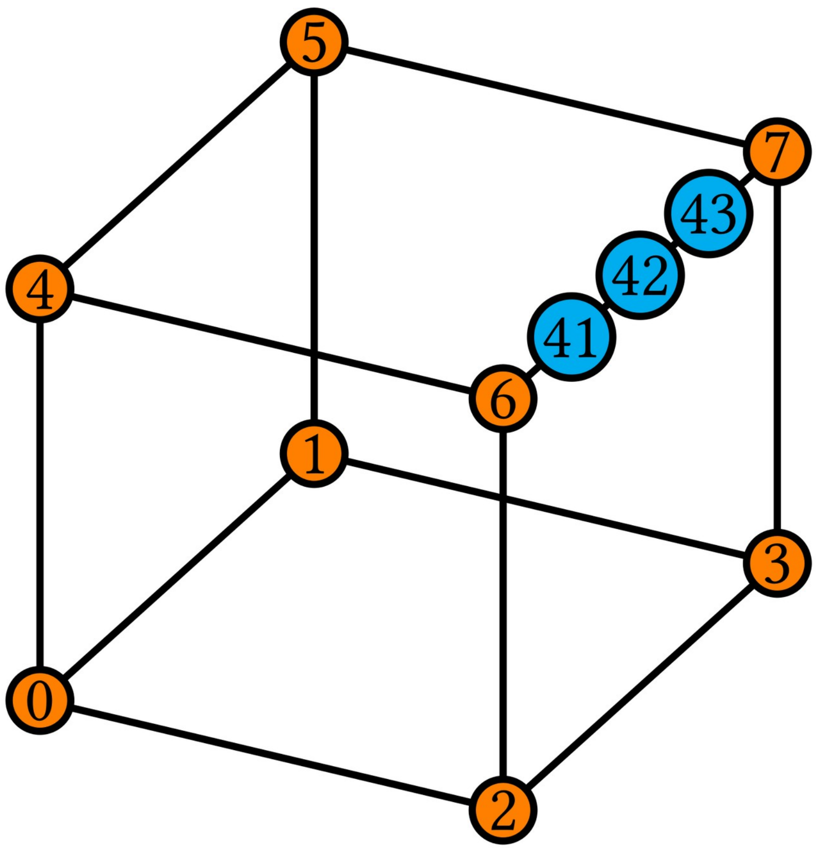
Example: Order 4 Lagrange



Example: Order 4 Lagrange

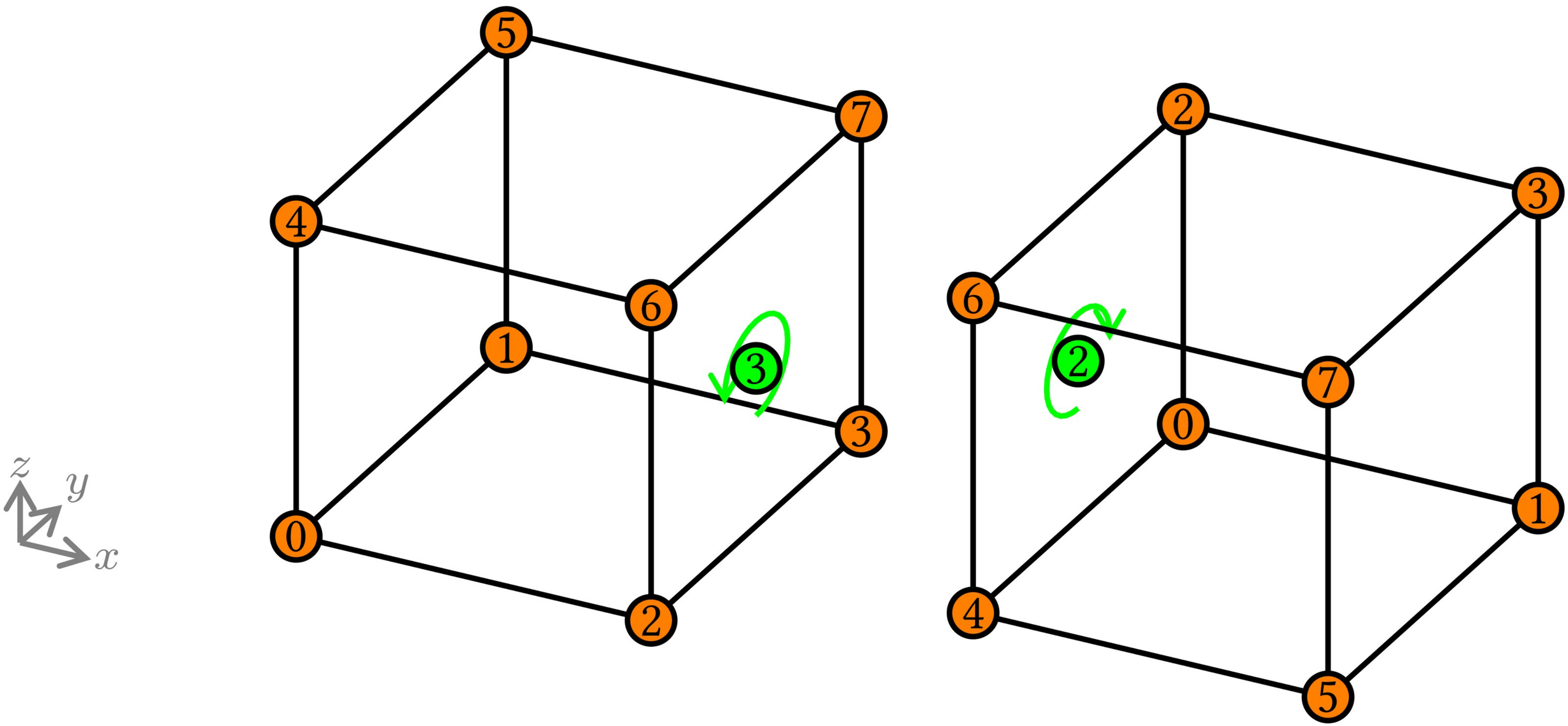


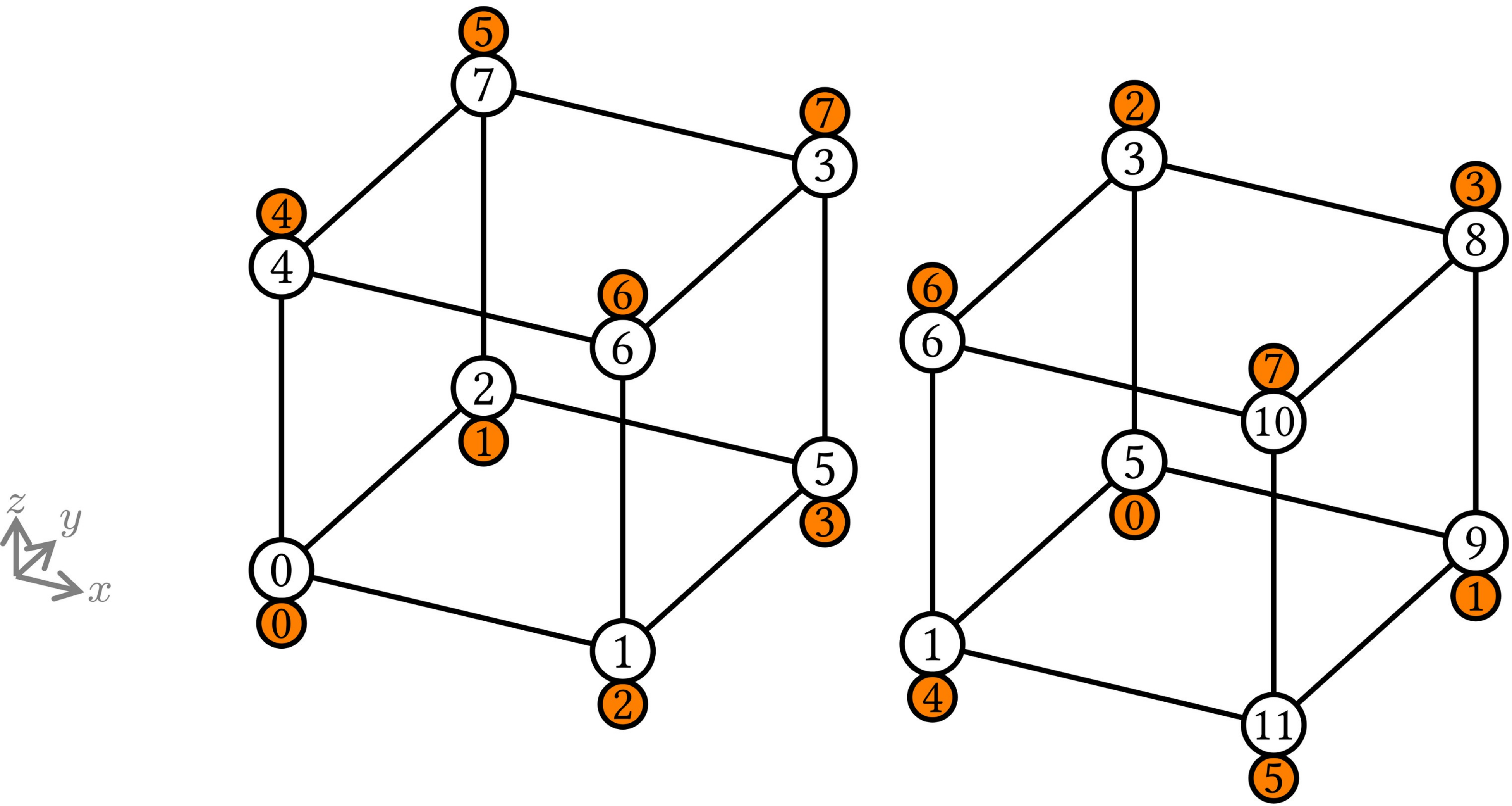
Example: Order 4 Lagrange



$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & 0 & 1 & & & \\ & & & 0 & 1 & 0 & & & \\ & & & 1 & 0 & 0 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix}$$

Example: Order 4 Lagrange





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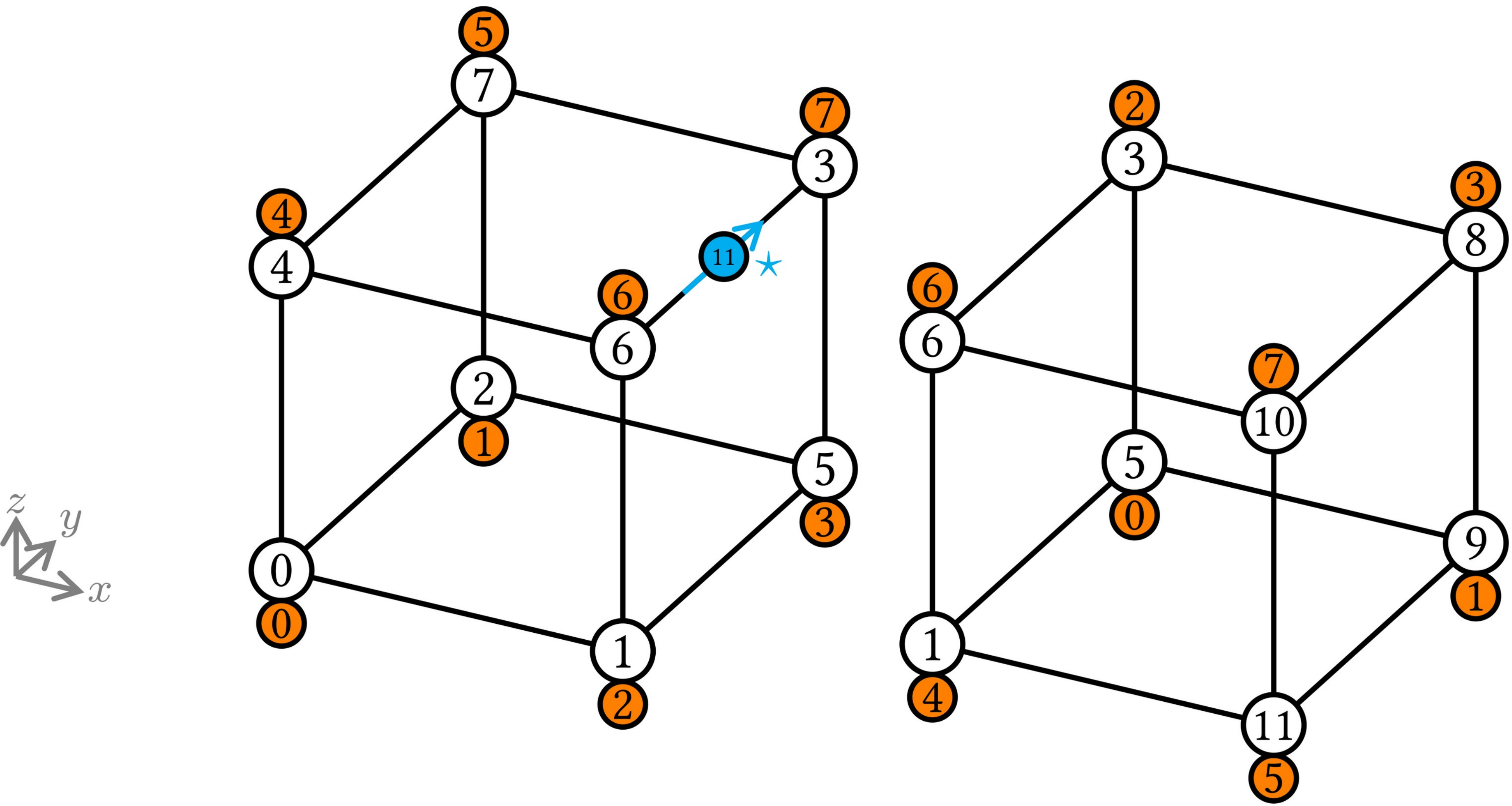

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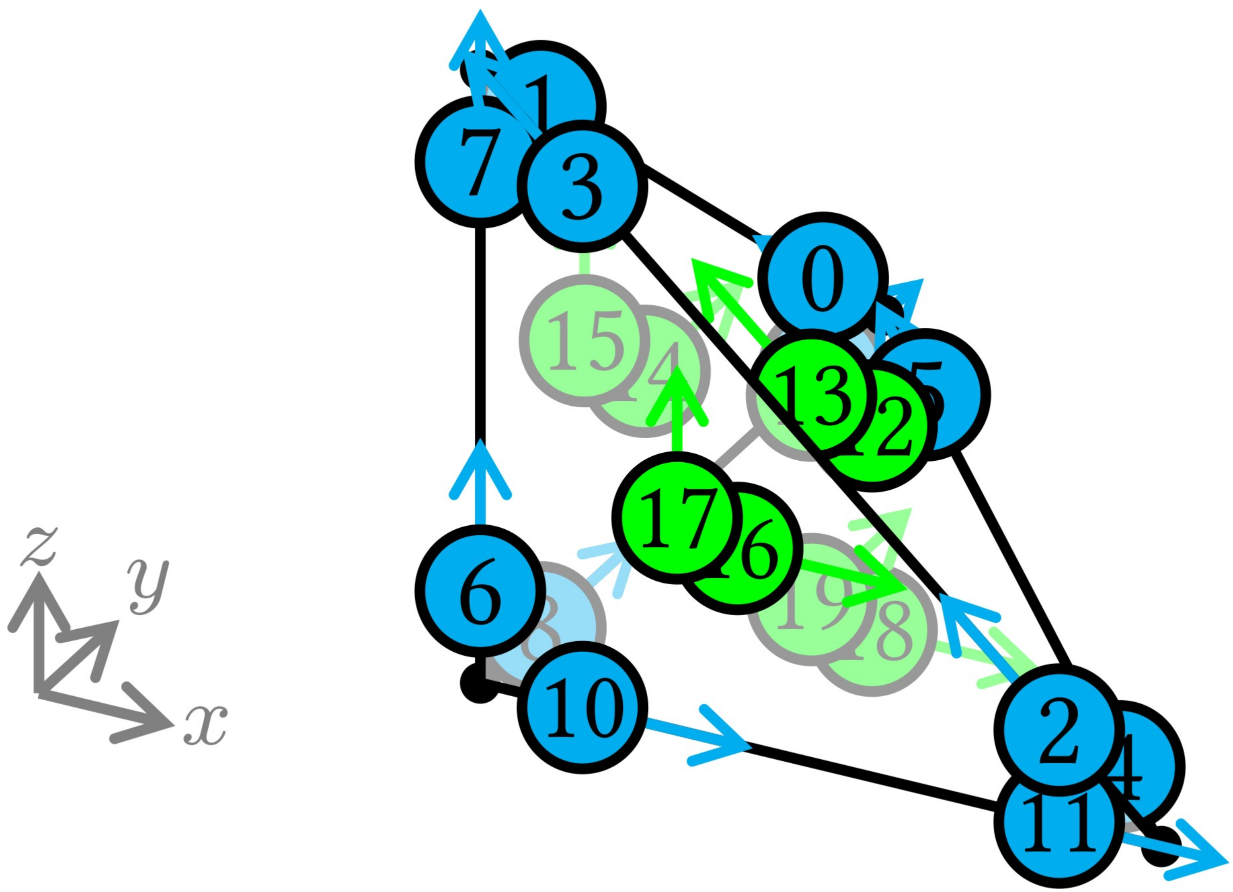
In general

- Precompute:
  - 1 matrix for each edge
  - 2 matrices for each face
- Compare local and global numbers to decide which matrices to apply

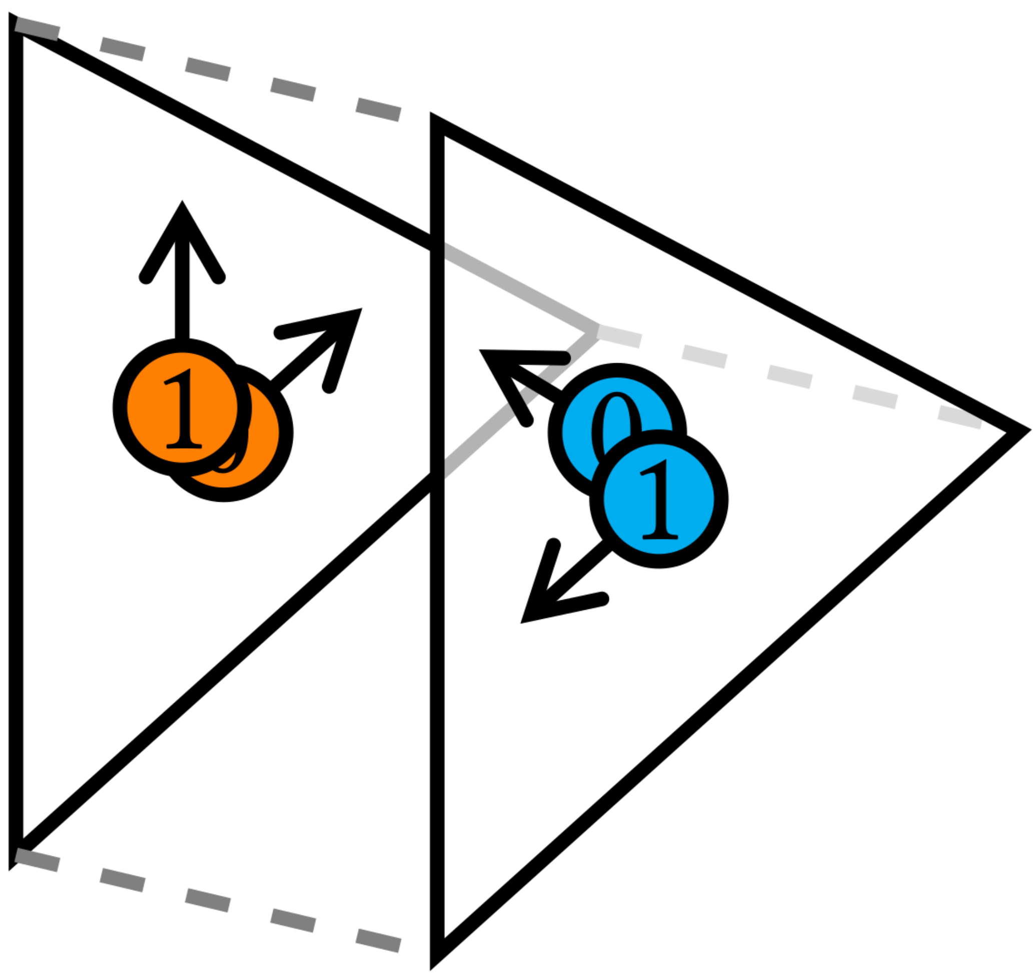
Example: Order 2 Nédélec

(first kind)



Example: Order 2 Nédélec

(first kind)

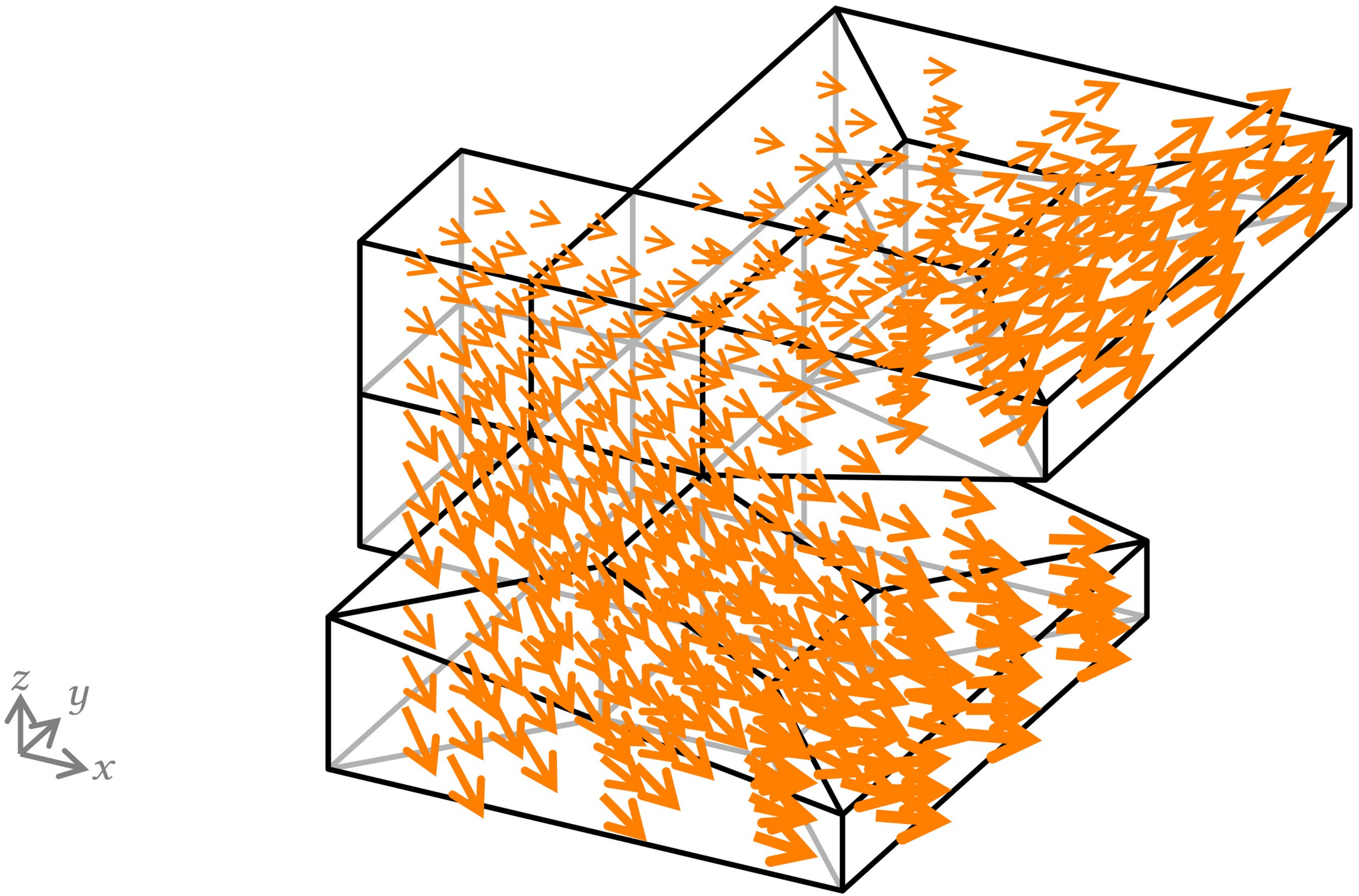


$$
\begin{pmatrix}
\ddots & & & \\
& -1 & -1 & \\
& 1 & 0 & \\
& & & \ddots
\end{pmatrix}
$$

Basix

```
import basix
element = basix.create_element(
    "Nedelec 1st kind H(curl)", "tetrahedron", 2)
print(element.base_transformations)
```

Example: Order 3 Nédélec



arxiv.org/abs/2102.11901

Construction of arbitrary order finite element degree-of-freedom maps on polygonal and polyhedral cell meshes

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CHRIS N. RICHARDSON, BP Institute, University of Cambridge, United Kingdom
GARTH N. WELLS, Department of Engineering, University of Cambridge, United Kingdom

We develop an approach to generating degree-of-freedom maps for arbitrary order Ciarlet-type finite element spaces for any cell shape. The approach is based on the composition of permutations and transformations by cell sub-entity. Current approaches to generating degree-of-freedom maps for arbitrary order problems typically rely on a consistent orientation of cell entities that permits the definition of a common local coordinate system on shared edges and faces. However, while orientation of a mesh is straightforward for simplex cells and is a local operation, it is not a strictly local operation for quadrilateral cells and in the case of hexahedral cells not all meshes are orientable. The permutation and transformation approach is developed for a range of element types, including arbitrary degree Lagrange, serendipity, and divergence- and curl-conforming elements, and for a range of cell shapes. The approach is local and can be applied to cells of any shape, including general polytopes and meshes with mixed cell types. A number of examples are presented and the developed approach has been implemented in open-source libraries.

CCS Concepts: • **Mathematics of computing** → **Discretization**; *Partial differential equations.*

Additional Key Words and Phrases: finite element methods, degrees-of-freedom, polyhedral cells

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DefElement

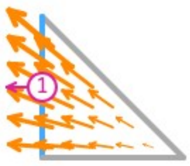
an encyclopedia of finite element definitions

Welcome to DefElement: an encyclopedia of finite element definitions.

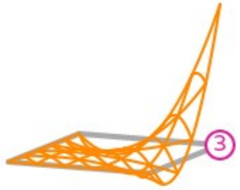
This website contains a collection of definitions of finite elements, including commonly used elements such as Lagrange, Raviart–Thomas, Nédélec (first kind) and Nédélec (second kind) elements, and more exotic elements such as serendipity H(div), serendipity H(curl) and Regge elements.

You can:

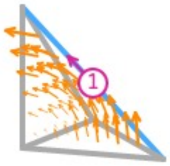
- view the full alphabetical list of elements
- view the elements by category
- view the elements by reference element
- view the elements by family



A basis function of an order 1 Raviart–Thomas space on a triangle



A basis function of an order 2 Q space on a quadrilateral



A basis function of an order 1 Nédélec (first kind) space on a tetrahedron

The finite element method

The finite element method is a numerical method that involves discretising a problem using a finite dimensional function space. These function spaces are commonly defined using a finite element on a reference element to derive basis functions for the space. This website contains a collection of finite elements, and examples of the basis functions they define.

Following the Ciarlet definition of a finite element, the elements on this website are defined using a reference element, a polynomial space, and a set of functionals. Each element's page describes how these are defined for that element, and gives examples of these and the basis functions they lead to for a selection of low-order spaces.

You can read a detailed description of how the finite element definitions can be understood here.

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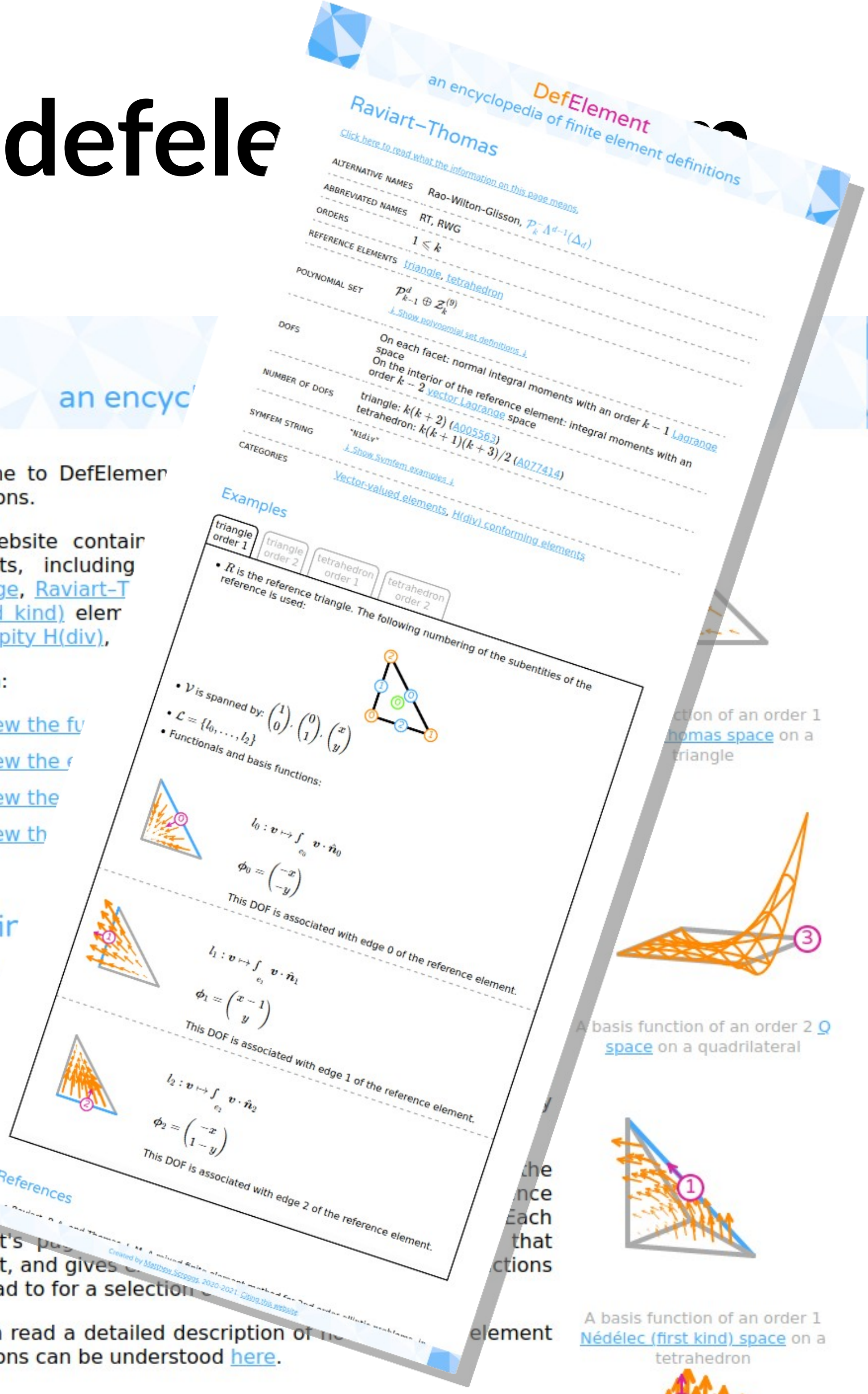

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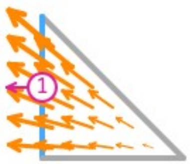
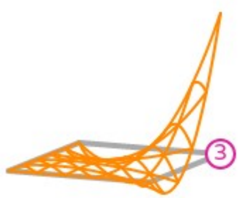
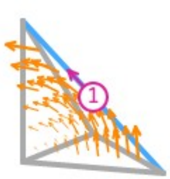


A basis function of an order 1 Raviart–Thomas space on a triangle



A basis function of an order 2 Q space on a quadrilateral



A basis function of an order 1 Nédélec (first kind) space on a tetrahedron

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Nédélec (first kind) space on a
tetrahedron
DefElement
an encyclopedia of finite element definitions
Raviart–Thomas
Click here to read what the information on this page means.
ALTERNATIVE NAMES
Rao-Wilton-Glisson, $\mathcal{P}_k^-\Lambda^{d-1}(\Delta_d)$
ABBREVIATED NAMES
RT, RWG
ORDERS
$1\leqslant k$
REFERENCE ELEMENTS
triangle, tetrahedron
POLYNOMIAL SET
$\mathcal{P}_{k-1}^d \oplus \mathcal{Z}_k^{(9)}$
↓ Show polynomial set definitions ↓
DOFS
On each facet: normal integral moments with an order $k-1$ Lagrange space
On the interior of the reference element: integral moments with an order $k-2$ vector Lagrange space
NUMBER OF DOFS
triangle: $k(k+2)$ (A005563)
tetrahedron: $k(k+1)(k+3)/2$ (A077414)
SYMFEM STRING
"N1div"
↓ Show Symfem examples ↓
CATEGORIES
Vector-valued elements, H(div) conforming elements
Examples
triangle order 1
triangle order 2
tetrahedron order 1
tetrahedron order 2
• R is the reference triangle. The following numbering of the subentities of the reference is used:
• $\mathcal{V}$ is spanned by: $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}$
• $\mathcal{L}=\{l_0,\dots,l_2\}$
• Functionals and basis functions:
$l_0:\boldsymbol{v}\mapsto\int_{e_0}\boldsymbol{v}\cdot\hat{\boldsymbol{n}}_0$
$\phi_0=\begin{pmatrix}-x\\-y\end{pmatrix}$
This DOF is associated with edge 0 of the reference element.
$l_1:\boldsymbol{v}\mapsto\int_{e_1}\boldsymbol{v}\cdot\hat{\boldsymbol{n}}_1$
$\phi_1=\begin{pmatrix}x-1\\y\end{pmatrix}$
This DOF is associated with edge 1 of the reference element.
$l_2:\boldsymbol{v}\mapsto\int_{e_2}\boldsymbol{v}\cdot\hat{\boldsymbol{n}}_2$
$\phi_2=\begin{pmatrix}-x\\1-y\end{pmatrix}$
This DOF is associated with edge 2 of the reference element.
References

Thanks for listening!

arxiv.org/abs/2102.11901

defelement.com

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