# Consensus ADMM for Inverse Problems Governed by Multiple PDE Models

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# Motivation

- PDEs are often dependent on unknown (or difficult to measure) parameters associated with physical systems and can be estimated via an inverse problem.
- Inverse problems are often-illposed: there's not enough data to recover the parameter.
- Regularization selects one solution among many possible solutions
- Non-smooth regularization reinforces certain "nice" properties in solutions: TV enforces sharp edges

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- Inverse problems are often-illposed: there's not enough data to recover the parameter.
- Regularization selects one solution among many possible solutions
- Non-smooth regularization reinforces certain "nice" properties in solutions: TV enforces sharp edges
- ADMM provides a natural way of splitting these inverse problems into smaller problems.
  - The subproblems related to the PDEs can be solved efficiently using INCG, which requires a smooth objective term.
  - The term related to the regularization can be solved for separately using other proximal methods.
- FEniCS is used for efficient discretization of these variational problems.

# ADMM Description

- Equality between solutions of subproblems is reinforced with a consensus term.

[1] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein,Distributed Optimization andStatistical Learning via the Alternating Direction Method of Multipliers, Foundation andTrends in Machine Learning, Vol. 3, No. 1 (2010).

[2] D. Gabay and B. Mercier,A dual algorithm for the solution of nonlinear variational problemsvia finite element approximations, Computers and Mathematics with Applications, Vol. 2,No. 1 (1976)

[3] Y. Wang, J. Yang, W. Yin, and Y. Zhang,A New Alternating Minimization Algorithm forTotal Variation Image Reconstruction, SIAM Journal on Imaging Sciences, (2007)

# ADMM Description

- Equality between solutions of subproblems is reinforced with a consensus term.
- ADMM will only reach moderate accuracy in a few iterations and requires many following iterations for high-precision convergence[1].
- This is sufficient for most large-scale applications including
  - Machine learning
  - Continuum mechanics[2]
  - Imaging[3]

[1] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein,Distributed Optimization andStatistical Learning via the Alternating Direction Method of Multipliers, Foundation andTrends in Machine Learning, Vol. 3, No. 1 (2010).

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# Setting

Consider the minimization problem

$$\min_{m \in \mathcal{M},} \quad \mathcal{L}(m) + \mathcal{R}(m)$$

- $\mathcal{M}$ is a possibly infinite-dimensional Hilbert space.
- $\mathcal{L} : \mathcal{M} \mapsto \mathbb{R}$ is twice differentiable, may be expensive to evaluate
- $\mathcal{R} : \mathcal{M} \mapsto \mathbb{R}$ is assumed convex and non-smooth

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Introduce a consensus variable $z \in \mathcal{M}$.

$$\begin{aligned} \min_{m,z \in \mathcal{M},} \quad & \mathcal{L}(m) + \mathcal{R}(z), \\ s.t. \quad & m - z = 0 \end{aligned}$$

# Consensus ADMM

We introduce the *augmented Lagrangian* for some $\rho > 0$

$$L_\rho(m, z, y) = \mathcal{L}(m) + \mathcal{R}(z) + \langle y, m - z \rangle + \frac{\rho}{2}||m - z||^2$$

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**Algorithm 2:** Consensus ADMM

Begin with starting points $(m^0, z^0, y^0)$

**while** *While convergence criterion is not met* **do**

$m^{k+1} = \text{argmin}_m \, L_\rho(m, z^k, y^k)$

$z^{k+1} = \text{argmin}_z \, L_\rho(m^{k+1}, z, y^k)$

$y^{k+1} = y^k + \rho(m^{k+1} - z^{k+1})$

**end**

# Inverse problems governed by PDEs

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- Goal of estimating a parameter $m$ given a measurement $\boldsymbol{d} \in \mathcal{D}$ where

$$\boldsymbol{d} = \mathcal{F}(m) + \boldsymbol{e},$$

- $\mathcal{F}$ is the composition of a PDE solver and observation operator $\mathcal{B} : \mathcal{U} \to \mathcal{D}$.

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- $\mathcal{F}$ is the composition of a PDE solver and observation operator $\mathcal{B} : \mathcal{U} \to \mathcal{D}$.
- Introduce the *state variable* $u \in \mathcal{U}$ s.t.

$$\mathcal{F}(m) = B(u(m)), \; r(m, u) = 0$$

$$\min_{m \in \mathcal{M}, u \in \mathcal{U}} \quad \mathcal{J}(m) = \frac{1}{2}\|\mathcal{B}(u) - \boldsymbol{d}\|^2 + \mathcal{R}(m)$$

For a Newton type solution method

- Using the Lagrangian formalism, gradient computation requires solving two PDEs: the forward & adjoint problems
- Each Hessian action requires solving two linearized PDEs: the incremental forward & incremental adjoint problems.

# The proposed consensus ADMM

**Algorithm 3:** The mean based scaled ADMM for parameter inversion with multiple PDEs

Let $q$ be the number of PDEs

Begin with starting points $(\{m_i^0\}_{i=1}^q, z^0, \{y_i^0\}_{i=1}^q)$

**while** *While convergence criterion is not met,* $k = 1, \ldots$ **do**

  **for** $i = 1, \ldots, q$ **do**

    $m_i^{k+1} = \text{argmin}_{m_i} \frac{1}{2q}||\mathcal{F}_i(m_i) - \boldsymbol{d}_i||^2 + \frac{\rho^k}{2q}||m_i - z^k + y_i^k||^2$

  **end**

  Set $\bar{m} = \frac{1}{q}\sum_{i=1}^q m_i^{k+1}$ and, $\bar{y} = \frac{1}{q}\sum_{i=1}^q y_i^{k+1}$

  $z^{k+1} = \text{argmin}_z \mathcal{R}(z) + \frac{\rho}{2}||\bar{m} - z + \bar{y}||^2$

  **for** $i = 1, \ldots, q$ **do**

    $y_i^{k+1} = y_i^k + (m_i^{k+1} - z^{k+1})$

  **end**

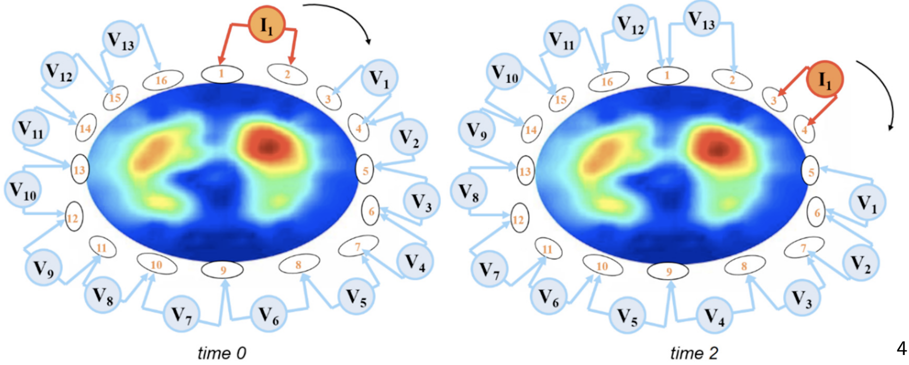
  Update $\rho^{k+1}$ adaptively

**end**

# Electrical Impedance Tomography(EIT)

Electrical Impedance Tomography (EIT) is an imaging modality in which

- An electrical current is introduced on the boundary of an object
- The electric potential is measured on the boundary.
- The potential measurements are used to reconstruct for conductivity.




[4]

[4] https://www.mdpi.com/2077-0383/8/8/1176/htm

# Formulating EIT in the continuous setting

Goal to minimize

$$\frac{1}{q}\sum_{i=1}^{q}\mathcal{L}_i(m)+\mathcal{R}(m),\ \mathcal{L}_i(m)=\frac{1}{2}\int_{\Gamma_i}(u_i-\boldsymbol{d}_i)^2 ds$$

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The regularization used was a combination of TV and $L^2$.
The potential $u_i$ solves the electrostatic Maxwell equation

$$\begin{cases} -\nabla\cdot e^m\nabla u_i = 0 & x\in\Omega \\ \frac{\partial}{\partial\eta}u_i = g_i & x\in\Gamma_N^i \\ u_i = 0 & x\in\Gamma_D^i \end{cases}$$

where $\sigma := e^m$ is the conductivity domain and $u_i$ is the electric potential resulting from introducing the current $g_i$.

# Discretization

For discretization we applied the finite element method(FEM) used in FEniCS

- $\Omega = D^2$
- Coarsest mesh had 8044 degrees of freedom on $\mathcal{M}$ and $\mathcal{U}$
- Parameter updates were accomplished using the INCG algorithm found in hIPPYlib, an extensible software framework for large-scale inverse problems governed by PDEs[5].
- Consensus updates were found using the PETScTAOSolver built into Fenics.

[5] U. Villa, N. Petra, and O. Ghattas,hIPPYlib: An Extensible Software Framework forLarge-Scale Inverse Problems Governed by PDEs; Part I: Deterministic Inversion andLinearized Bayesian Inference, ACM Transactions on Mathematical Software, in print(2021)

# Ground Truth

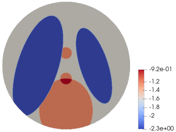



Figure: True parameter

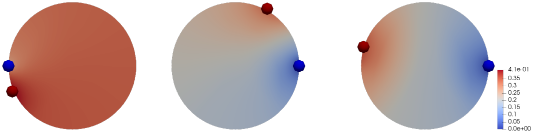



Figure: True states 1,11,16 for EIT problem with $q = 16$

# $H^1$ reconstruction with inexact subproblem solutions

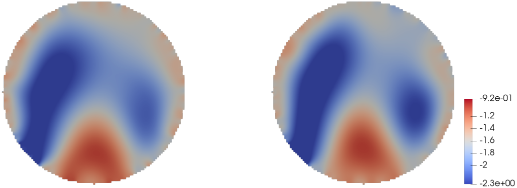
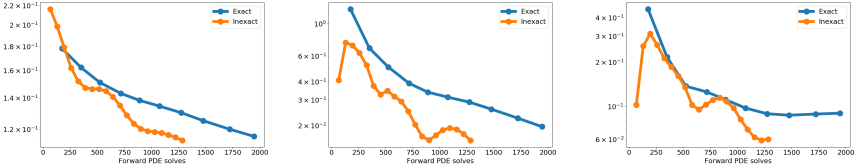


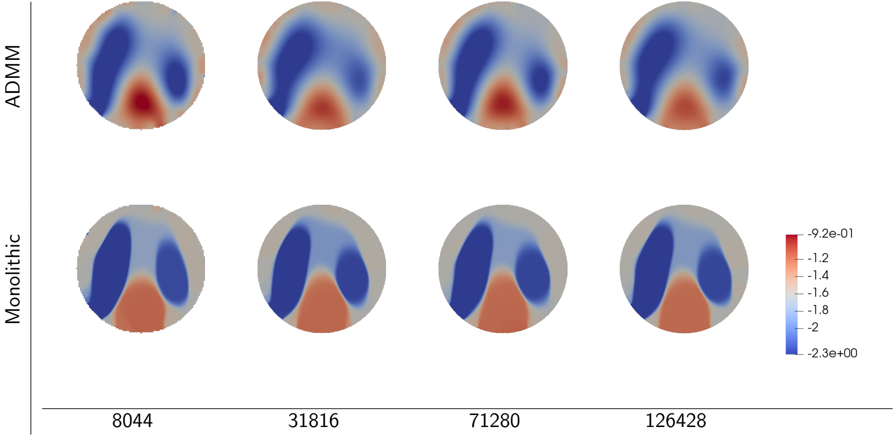

Figure: Inverted consensus for EIT problem using exact and inexact $m$ solves



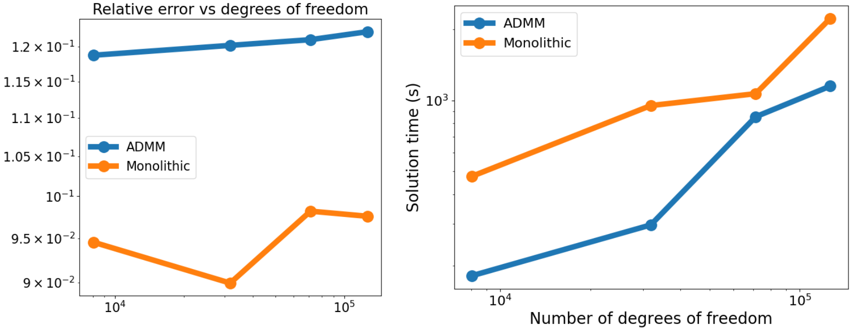

The relative error, primal and dual residuals wrt Foward PDE solves

# Scalability with respect to problem size

Fix $q = 16$ and sequences of uniformly refined meshes

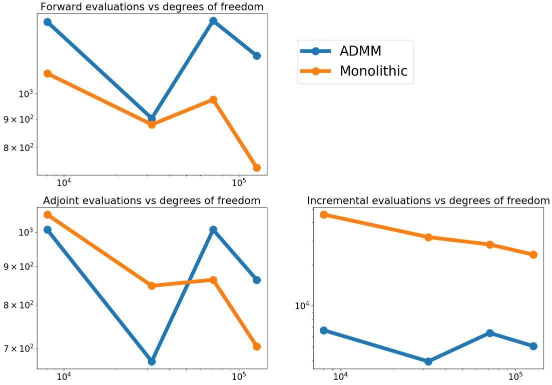


# Accuracy with respect to problem size




Relative error and state misfit for ADMM and monolithic approaches vs number of degrees of freedom

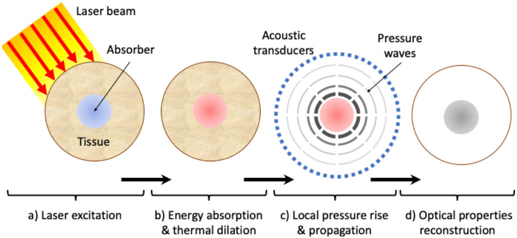
# Scalability with respect to problem size (number of PDE solves)




- Similar results hold for scaling by number of forward models.

# Quantitative photoacoustic tomography(qPACT)

1. A fast laser pulse is sent into an object.
2. Underlying material absorbs this energy generating heat and a local increase pressure distribution.
3. Pressure distribution transitions into acoustic waves and measured on boundary.



# Formulation of the qPACT problem

We focused on reconstructing for optical properties given the initial pressure distribution.

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Diffusion approximation to radiative transport

$$-\nabla \cdot \frac{1}{3(\mu_a+\mu_s')} \nabla \phi + \mu_a \phi = 0 \qquad x \in \Omega$$

with Robin boundary condition

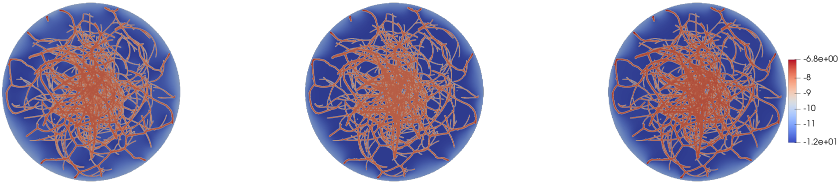
$$\frac{1}{3(\mu_a+\mu_s')} \frac{\partial \phi}{\partial \eta} + \frac{1}{2}\phi = \frac{1}{2}\phi_0 \qquad x \in \partial\Omega$$

Form the data fidelity term

$$\frac{1}{q}\sum_{i=1}^{q} \mathcal{L}_i(s, c_{thb}, \mu_s') = \frac{1}{q}\sum_{i=1}^{q} || \ln(\mu_{a,i}\phi_i) - \ln(d_i)||^2$$

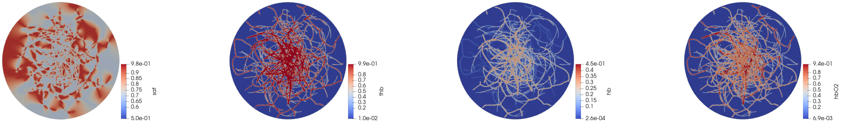
and use regularization with a mixture of Tiknohonv, TV, and L1.

# Forward Results

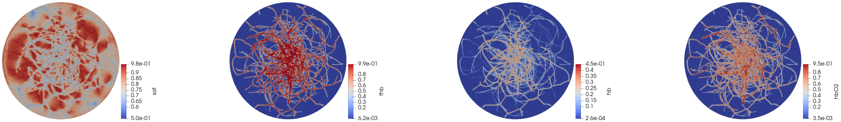



Measurements corresponding to 757, 800, 850 nm

# Results




True $s$, $c_{thb}$, $c_{hb}$, and $c_{hbO_2}$




Reconstructed $s$, $c_{thb}$, $c_{hb}$, and $c_{hbO_2}$

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- Implementing a primal-dual solver for updating the consensus variable.
- Implementing the ADMM process on several processors, with each PDE model being handled by its own set of processors.

The code and EIT example will be included in hIPPYlib.