THE KÄHLER IDENTITIES

There are several proofs of the Kähler identities, many of which rely ultimately on a calculation on $\mathbb{C}^n$. The following is one I find most palatable, and is adapted only slightly from a similar handout given by Lee.

Proposition 1. *Let $\omega = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z}_j$ be the standard Kähler metric on $\mathbb{C}^n$. Then*

$$[\overline{\partial}^*, L] = i\partial \text{ and } [\bar{\partial}^*, \Lambda] = i\partial.$$

For the proof we start with some notation. Let g be the standard hermitian metric on $\mathbb{C}^n$, so ω is the fundamental form associated to g. Firstly, given an $\alpha \in \mathcal{A}^q(\mathbb{C}^n)$ and $\zeta \in \mathcal{A}^1(\mathbb{C}^n)$ define $\zeta \vee \alpha \in \mathcal{A}^{q-1}(\mathbb{C}^n)$ by requiring

$$g(\zeta \vee \alpha, \beta) = g(\alpha, \overline{\zeta} \wedge \beta) \text{ for all } \beta \in \mathcal{A}^{q-1}(\mathbb{C}^n) \tag{2}$$

We leave as an exercise that this is well-defined. Secondly, if $\alpha = \sum_{IJ} \alpha_{IJ} dz_I \wedge d\overline{z}_J$ is in $\mathcal{A}^{p,q}(\mathbb{C}^n)$ set

$$\partial_j \alpha = \sum_{IJ} \frac{\partial \alpha_{IJ}}{\partial z_j} dz_I \wedge d\overline{z}_J \text{ and } \overline{\partial}_j \alpha = \sum_{IJ} \frac{\partial \alpha_{IJ}}{\partial \overline{z}_j} dz_I \wedge d\overline{z}_J$$

Lemma 3. *For all $\alpha, \beta \in \mathcal{A}^{p,q}(\mathbb{C}^n)$ it holds that*

$$\overline{\partial}\alpha = \sum_j d\overline{z}_j \wedge \overline{\partial}_j \alpha \text{ and } \partial\alpha = \sum_j dz_j \wedge \partial_j \alpha \tag{4}$$

$$\partial_j(g(\alpha,\beta)) = g(\partial_j\alpha, \beta) + g(\alpha, \overline{\partial}_j \beta) \text{ and } \partial_j(dz_k \vee \alpha) = dz_k \vee \partial_j \alpha. \tag{5}$$

Proof. Observe that putting $\beta = 1$ into the defining equation (2) we have $dz_j \vee dz_k = g(dz_j, d\overline{z}_k) = 0$ and $dz_j \vee d\overline{z}_k = g(dz_j, dz_k) = 2\delta_{jk}$. Then $\overline{\partial}\alpha = \sum_{IJ}\sum_k \frac{\partial \alpha_{IJ}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J = \sum_k d\overline{z}_k \vee \overline{\partial}_j \alpha$ from which we deduce the first equation, and the second is similar. For the third use that the coefficients of g are constant, and the fourth is left as an exercise. $\square$

Lemma 6.

$$\overline{\partial}^* \alpha = -\sum_j dz_j \vee \partial_j \alpha$$

Proof. Let $f \in C^\infty_{\mathbb{C}}(\mathbb{C}^n)$ have compact support. Then by an application of Stoke's Theorem, $\int_{\mathbb{C}^n} \partial_j f dvol = 0$ where $dvol$ is the standard volume form on $\mathbb{C}^n$. Now suppose $\alpha \in \mathcal{A}^q(\mathbb{C}^n)$ and $\beta \in \mathcal{A}^{q-1}(\mathbb{C}^n)$ with β having compact support. Then the

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smooth function $g(dz_j \vee \alpha, \beta)$ has compact support, so using the previous Lemma,

$$\begin{aligned} 0 &= \sum_j \int_{\mathbb{C}^n} \partial_j(g(dz_j \vee \alpha, \beta)) dvol = \sum_j \int_{\mathbb{C}^n} g(dz_j \vee \partial_j \alpha, \beta) + g(dz_j \vee \alpha, \overline{\partial}_j \beta) dvol \\ &= \sum_j \int_{\mathbb{C}^n} g(dz_j \vee \partial_j \alpha, \beta) + \int_{\mathbb{C}^n} g(\alpha, d\overline{z}_j \wedge \overline{\partial}_j \beta) = \langle \sum_j dz_j \vee \partial_j \alpha, \beta \rangle + \langle \alpha, \overline{\partial} \beta \rangle \\ &= \langle \sum_j dz_j \vee \partial_j \alpha, \beta \rangle + \langle \overline{\partial}^* \alpha, \beta \rangle \end{aligned}$$

where the last line follows as $\overline{\partial}^*$ is the formal adjoint to $\overline{\partial}$, and β was assumed to have compact support. Since this holds for all such β we get the desired result. □

Proof of Proposition 1.

$$[\overline{\partial}^*, L]\alpha = \overline{\partial}^* L\alpha - L\overline{\partial}^* \alpha = \overline{\partial}^*(\omega \wedge \alpha) - \omega \wedge \overline{\partial}^* \alpha.$$

Now by Lemma 6

$$\overline{\partial}^*(\omega \wedge \alpha) = -\sum_j dz_j \vee \partial_j(\omega \wedge \alpha) = -\sum_j dz_j \vee (\partial_j \omega \wedge \alpha) - \sum_j dz_j \vee (\omega \wedge \partial_j \alpha).$$

Since ω is the standard Kähler metric on $\mathbb{C}^n$, we clearly have $\partial_j \omega = 0$. Hence

$$\overline{\partial}^*(\omega \wedge \alpha) = -\frac{i}{2} \sum_j dz_j \vee \sum_k dz_k \wedge d\overline{z}_k \wedge \partial_j \alpha$$

Hence the above is a sum of three terms, namely

$$-\frac{i}{2} \sum_{jk} (dz_j \vee dz_k) \wedge d\overline{z}_k \wedge \partial_j \alpha,$$

$$\frac{i}{2} \sum_{jk} dz_k \wedge (dz_j \vee d\overline{z}_k) \wedge \partial_j \alpha$$

and

$$-\frac{i}{2} \sum_{jk} dz_k \wedge d\overline{z}_k \wedge (dz_j \vee \partial_j \alpha).$$

The first of these is zero as $dz_j \vee dz_k = 0$, and the third is equal to $-\omega \wedge \sum_j dz_j \vee \partial_j \alpha = \omega \wedge \overline{\partial}^* \alpha$. On the other hand the second is equal to $i\partial\alpha$ by Lemma 3, and putting this all together completes the proof. □

Theorem 7 (The Kähler Identities)**.** *Let ω be a Kähler form on a complex manifold X. Then*

$$[\bar{\partial}^*, L] = i\partial \text{ and } [\partial^*, L] = -i\overline{\partial}$$

$$[\Lambda, \overline{\partial}] = -i\partial^* \text{ and } [\Lambda, \partial] = i\overline{\partial}^*$$

Proof. In local coordinates $z_1, \ldots, z_n$ we know $\omega = \omega_0 + O(|z|^2)$ where ω_0 is the standard Kähler form on $\mathbb{C}^n$. Since the quantitiy $[\overline{\partial}^*, L]$ involves only the first derivative, the calculation for $\mathbb{C}^n$ with the standard Kähler form holds also on X proving the first identity. The second follows from this by conjugation, and the other two by taking adjoints. □

`j.ross@dpmms.cam.ac.uk`