EXISTENCE OF PARTITIONS OF UNITY
PART III DIFFERENTIAL GEOMETRY

(The following is a technical tool, that for this course is non-examinable in the non-compact case)

Theorem 1. *Let M be a manifold. Then given any open cover $\{V_\beta\}$ there exists a partition of unity $\{\phi_i\}$ subordinate to $\{V_\beta\}$.*

Proof. Recall that a topological space is said to be *locally compact* if every point is contained in an open set whose closure is compact. Clearly Euclidean space is locally compact, and as M is locally homeomorphic to Euclidean space and it is also locally compact.

Now we use the assumption that M is second countable, so there is a countable basis of open sets $\{B_j : j \in \mathbb{N}\}$. By local compactness, any $x \in M$ is contained in some open U_x whose closure is compact. Thus there is a $j(x)$ such that $x \in B_{j(x)} \subset U_x$, and in particular $\overline{B_{j(x)}} \subset \overline{U_x}$. Now a closed subset of a compact set is compact, so $B_{j(x)}$ has compact closure. Hence by shrinking our countable basis if necessary, we may assume that each B_j has compact closure.

Now define $W_1 = B_1$. Then by compactness

$$\overline{W_1} \subset \bigcup_{j=1}^{m} B_j$$

for some m. Next define W_2 to be the union $W_2 = \cup_{j=1}^m B_j \cup B_2$. Then $\overline{W_2}$ is a union of open sets which have compact closure, and thus is also open with compact closure. Repeating the above we get an open cover W_j of sets which each have compact closure and such that $\overline{W_j} \subset W_{j+1}$. In particular then

$$\overline{W_j}/W_{j-1} \subset W_{j+1}/\overline{W_{j-2}}. \tag{2}$$

[see the remark below for an explaination of what is going on]

Now fix $p \in M$ and let j be the largest natural number so $p \in M/\overline{W}_j$. Then $p \in V_\beta \cap (W_{j+2}/\overline{W_{j-1}})$ for some β. Take a chart U_p contained this open set and let f be a bump function which is identically 1 on an neighbourhood N_p of p and whose support is within this chart. Now as p ranges over $W_{j+2}/\overline{W_{j-1}}$, the N_p cover $\overline{W_{j+1}}/W_j$ so by compactness we can take a finite subcover. Thus there exist a finite number of bump functions f_k with the property that at least one of them does not vanish at any given point in $\overline{W_{j+1}}/W_j$ and such that each f_k has support contained in $V_{\beta(k)} \cap (W_{j+2}/\overline{W_{j-1}})$ for some $\beta(k)$. Repeating this for all j we are left with a countable collection of bump function $\{\phi_i\}$ whose supports are locally finite and such that at every point at least one does not vanish. Thus $\psi_i := \phi_i / \sum_u \phi_u$ is a well-defined partition of unity subordinate to $\{V_\beta\}$.

□

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Remark. Some of you may have heard of Urysohn's metrization theorem which says that any regular second countable topological space is metrizable. Applying this to our manifold M we obtain a metric d that defines the topology (I know of no other place where this metric is in the least bit useful for the study of manifolds; in fact for metric properties one usually assumes there is a Riemannian metric which will appear later in the course). However given d one can define the open sets W_j more easily by fixing a base point $p \in M$ and letting $W_j := \{x \in M : d(x,p) < j\}$. Clearly this is open and has compact closure, and (2) is obvious.

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