CANONICAL METRICS AND STABILITY OF PROJECTIVE VARIETIES

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This short survey aims to introduce some of the ideas and conjectures relating stability of projective varieties to the existence of canonical metrics (e.g. balanced metrics, or metrics of constant scalar curvature). It was written for the AIM workshop on "Numerical invariants of singularities and higher-dimensional algebraic varieties".

1. Chow stability and balanced metrics

The two most common ways to parameterise subvarieties (or subschemes) of $\mathbb{P}^N$ are through Hilbert points and Chow points. For brevity we shall only consider on the latter which are roughly defined as follows. Suppose $X \subset \mathbb{P}^N$ is smooth of dimension n. Then a general linear subspace of $L \subset \mathbb{P}^N$ of dimension $N-n-1$ will not intersect X. However a general one parameter family of such subspaces clearly does intersect X. Thus those L such that $L \cap X \neq \phi$ form a divisor in the Grassmannian $\mathbb{G}(N, N-n-1)$. This divisor is given by the vanishing of a section of a suitable line bundle, which is unique up to scale, and is called the *Chow point* of X.

The automorphism group of $\mathbb{P}^N$ acts naturally on the space of Chow vectors and thus we have a well defined Geometric Invariant Theory (GIT) problem.

Definition 1.1. We say that a subvariety $X \subset \mathbb{P}^N$ is *Chow (semi)stable* if the Chow point of X is (semi)stable in the sense of GIT. (This is to be understood with respect to the induced SL_{N+1} action coming from the automorphisms of $\mathbb{P}^N$ and the natural linearisation on the hyperplane bundle on the space of Chow points).

Our aim is to discuss analytic characterisations of Chow stability. Given homogeneous coordinates $Z_0, \ldots, Z_N$ on $\mathbb{P}^N$ we define a skew-hermitian matrix by

$$M(X)_{\alpha\beta} = \int_X \frac{Z_\alpha \overline{Z}_\beta}{||Z||^2} d\mu$$

where μ is the volume form coming from the restriction of the Fubini-Study metric to X and $||Z||^2 = \sum_\alpha |Z_\alpha|^2$.

Definition 1.2. We say that $X \subset \mathbb{P}^N$ is *balanced* if $M(X)$ is a scalar multiple of the identity.

Theorem 1.3. *A submanifold $X \subset \mathbb{P}^N$ is Chow stable if and only if there are homogeneous coordinates on $\mathbb{P}^N$ with respect to which it is balanced.*

There are now several proofs of this result [7, 10, 12, 18]. The connection between the balanced condition and Chow stability appears naturally through the framework of moment maps or, what amounts to essentially the same thing the Kempf-Ness function (for the standard connection between stability the reader is suggested to consult Chapter 8 of [9]). Essentially the function $X \mapsto M(X)$ is a moment map for the action of $SU_{N+1} \subset SL_{N+1}$ on the space of Chow-vectors with respect to a suitable symplectic form). This is the point of view taken [3, 18] and we refer the reader to those papers for details.

It will be useful later on not to consider the embedding $X \subset \mathbb{P}^N$ but rather pairs (X, L) where L is a ample line bundle on X. When k is chosen to $L^{\otimes k}$ is very ample we say $(X, L^{\otimes k})$ is balanced if there exists a basis of $H^0(X, L^{\otimes k})$ such that the embedding $X \subset \mathbb{P}(H^0(X, L^{\otimes k}))$ is balanced.

The balanced condition for $(X, L^{\otimes k})$ can be given in terms of the existence of certain "special" Kähler metrics ω on X whose cohomology class is $c_1(L^{\otimes k})$. Given such an ω there exists a Hermitian metric h on the fibres of $L^{\otimes k}$, with curvature form $-i\omega$ (moreover h is unique up to scale). Using ω to define a volume form on X this yields an L^2-inner product on the vector space $H^0(X, L^{\otimes k})$. Choosing an orthonormal basis $\{s_\alpha\}$ we define a function on X by

$$\rho_\omega(x) = \sum_\alpha |s_\alpha(x)|_h^2.$$

One can show that ρ_ω depends only on ω (and not on the choice of orthonormal basis or hermitian metric h). It is not hard to show that $(X, L^{\otimes k})$ is balanced if and only if there is a metric $\omega \in c_1(L^{\otimes k})$ such that ρ_ω is constant.

Definition 1.4. The Kähler metric ω is said to be a *balanced metric* if ρ_ω is a constant function on X. By Theorem 1.3 such a metric exists in $c_1(L^{\otimes k})$ if and only if $X \subset \mathbb{P}(H^0(X, L^{\otimes k})$ is Chow stable.

Remark 1.5. Using the balanced condition gives some geometric understanding of Chow stability that is otherwise hard to obtain. As an illustration suppose that $X \subset \mathbb{P}^N$ and $Y \subset \mathbb{P}^M$ are smooth and Chow stable. Then $X \times Y \subset \mathbb{P}^{(N+1)(M+1)-1}$ is also Chow stable. (This follows as product of the pullbacks of the the balanced metrics ω_X and ω_Y to $X \times Y$ gives a Kähler metric $\omega_{X\times Y}$ which is easily seen to be balanced by Fubini's theorem). However in terms of GIT there appears to be no obvious reason to why this rather basic property should hold. It would be interesting to know if there is a purely algebro-geometric proof (without appealing to the balanced condition) for this result which presumably extends to the case that X and Y have singularities.

2. Asymptotic Stability and constant scalar curvature metrics

Now we look at what happens to these balanced metrics for $(X, L^{\otimes k})$ as k tends to infinity. Again we suppose that X is smooth and L is an ample line bundle on X. The scalar curvature $S(\omega)$ of a Kähler metric ω on X is the function given by the trace (with respect to ω) of the Ricci curvature (see [17] for an introduction to metrics in Kähler geometry).

A Kähler metric is said to have constant scalar curvature (cscK) if $S(\omega)$ is constant. When the canonical bundle of X is a scalar multiple of L these are precisely the Kähler-Einstein metrics. A theorem of Yau guarantees existence of these in two important cases: when L is the canonical bundle, or when the canonical bundle is trivial and L is arbitrary. By contrast when X is Fano there need not exist a Kähler-Einstein metric (already $\mathbb{P}^2$ blown up at a single point provides an example) and it is an important problem in Kähler geometry to find sufficient conditions under which existence is guaranteed.

A striking connection between cscK metrics and asymptotic stability is the following theorem of Donaldson. Assume that X has no infinitesimal automorphisms. Then one can show that any balanced metric in a given Kähler class is unique.

Theorem 2.1 (Donaldson [3])**.**

(1) *Suppose that $(X, L^{\otimes k})$ is balanced for $k \gg 0$ so there exists a balanced metric $\omega_k \in c_1(L^{\otimes k})$. If the sequence $\frac{1}{k}\omega_k \in c_1(L)$ converge in C^∞ to a Kähler metric ω then ω is cscK.*
(2) *Conversely suppose that there is a cscK metric $\omega \in c_1(L)$. Then for $k \gg 0$ the pair $(X, L^{\otimes k})$ is balanced and the balanced metric ω_k converge in C^∞ to ω.*

Roughly one should think of this as saying that a cscK metric as the limit of Fubini-Study metrics coming from a balanced embedding of $X \subset \mathbb{P}(H^0(X, L^{\otimes k})$ as $k \gg 0$. In particular we get that the existence of a cscK metric implies that (X, L) is *asymptotically Chow stable* which means that $X \subset \mathbb{P}(H^0(X, L^{\otimes k}))$ is Chow stable for $k \gg 0$.

3. K-stability

In the previous section we have seen how the existence of a cscK metric implies some notion of stability for (X, L) and an open question that has recently gained much interest is whether there is a converse to this statement. The notion of stability that is most widely expected to be equivalent to the existence of a cscK metric is called K-stability defined in [4]. (See also [1, 14] for a modification of this definition). Here we shall only give some general remarks about K-stability.

(1) The "hard" direction, that K-stability implies the existence of a cscK metric, is very much open in general. It is known for example for Fano-surfaces, and for a large class of ruled surfaces. There are partial results of Donaldson (e.g. [4]) that may end up giving a complete prove in the case of toric surfaces (or even toric manifolds of higher dimension).
(2) The relationship between K-stability and Chow stability can be compared to the relationship between slope stability and Gieseker stability for vector bundles. Precisely there are implications

$$\text{Asymptotic Chow stability} \Rightarrow \text{Asymptotic Chow semistable} \Rightarrow \text{K-semistable}$$

It is thought that one can put K-stability on the left of this sequence, but for technical reasons this is not known.

(3) K-stability is not strictly a GIT notion. That is, it is not defined in terms of the non-vanishing of some invariant section of some line bundle (say on the Hilbert scheme). Instead the definition is made formally in terms of the Hilbert-Mumford criterion: roughly one declares what should be the Mumford "weight" function (or μ-function) is, and K-stability is defined by requiring that for all one-parameter degenerations of (X, L) that this function has the favourable sign.

(4) There does exist a line bundle on the Hilbert Scheme (called the CM line [11, 16]) with respect to which the Hilbert Mumford criterion gives K-stability. However it is not known to what extent the CM line is "positive". Roughly what is currently known is that the CM line is nef on the stable locus (more precisely the asymptotically Hilbert semistable locus) but is not ample in general [6]. At the moment it is not known if in general the CM line is even effective!

(5) As pointed out by Kollár, a fundamental problem when dealing with asymptotic stability is that we get, for each $k \gg 0$, a different GIT problem concerning the different embeddings $X \subset \mathbb{P}^{N_k}$ where $N_k = h^0(L^{\otimes k}) - 1$. One might hope that stability of X does not depend on k, at least for k sufficiently large but this is hard to prove. One redeeming feature of K-stability is the following observation: if $X \subset \mathbb{P}(H^0(X, L^{\otimes k})$ is K-unstable then so is $X \subset \mathbb{P}(H^0(X, L^{\otimes l})$ for $l \gg k$. Even this one direction is not known for Chow or Hilbert stability.

4. Further Reading

A longer introduction to GIT and relation to metrics is given in [15] and an introduction from an more analytic perspective is [5]. Tian's book [17] contains the basics of canonical metrics in Kähler geometry and also has a chapter on stability.

For (Chow) stability of varieties there is Mumford's paper [8]. For the definition of K-stability try [4] and a comparison to other stability notions (e.g. Hilbert stability) is given in [13].

The notion of balanced metrics (resp. cscK metrics) is discussed in [3] (resp. [2]) the problem is put in context in terms of moment maps.

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