1. There are N securities which follow a factor structure with 2 factors. The factor returns at time $t+1$ are random variables $f_{A,t+1}$ and $f_{B,t+1}$. For simplicity, assume that the factors have been orthogonalized (and are therefore uncorrelated) and normalized to have a mean of zero. The factor variance covariance matrix is:

$$\begin{pmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_B^2 \end{pmatrix} \tag{1}$$

where σ_A and σ_B are the unconditional standard deviations of factor returns A and B.

Factor structure means that the return of asset i at time $t+1$ is:

$$r_{i,t+1} = \mu_i + \beta_{i,A} f_{A,t+1} + \beta_{i,B} f_{B,t+1} + \sigma_i \epsilon_{i,t+1} \tag{2}$$

where μ_i is a constant equal to the unconditional mean of i's return, $\beta_{i,A}$ and $\beta_{i,B}$ are i's factor loadings, σ_i is the unconditional standard deviation of the idiosyncratic component of i's return, and $\epsilon_{i,t+1}$ is an independent, and identically distributed random variable with mean zero and standard deviation one.

Write down the variance covariance matrix of the N assets as a function of $\beta_{i,A}$, $\beta_{i,B}$, σ_A, σ_B, and σ_i. Compute an analytic expression for what fraction of asset i's variance is *idiosyncratic* (the remainder being *systematic*, that is related to factor variance).

2. Write down analgorithm to solve for the variance covariance matrix as a function of $\beta_{i,A}$, $\beta_{i,B}$, σ_A, σ_B, and σ_i.

Hint: Miller (Mathematics Magazine 1981) provides a way to invert the sum of two matricies. If A^{-1} is known, and if $A+B$ is invertible, and rank of B is 1, then $(A+B)^{-1} = A^{-1} - \frac{1}{1+g} A^{-1} B A^{-1}$ where $g = trace(BA^{-1})$. For your purposes, rank of B is generally $r > 1$, however you can always decompose $B = B_1 + B_2 + ... B_r$ so that B_i has rank 1 for all i. Define $C_0 = A$, and $C_i = A + B_1 + B_2 + ... + B_i$. Then to find $C_r^{-1} = (A+B)^{-1}$ you can iterate the following equation: $C_i^{-1} = C_{i-1}^{-1} - g_i C_{i-1}^{-1} B_i C_{i-1}^{-1}$ where $g_i = trace(C_{i-1}^{-1} B_i)$.

Hint: If there are m factors and $n > m$ assets, then rank of B is m. There are multiple ways to decompose B. The easiest way to decompose B into rank 1 matricies is $B = B_1 + B_2 + ... B_n$ where B_i is equal to B in the i-th row, and zero everywhere else. However, since rank of B is $m < n$ it is also possible to decompose $B = B_1 + B_2 + ... B_m$, doing it this way may be better computationally since typically m is much smaller than n.

3. Recompute 1. and 2. but for the case where the factors are correlated with a variance covariance matrix

$$\begin{pmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{pmatrix} \tag{3}$$

4. Explain step-by-step how you would estimate the necessary inputs ($\beta_{i,A}$, $\beta_{i,B}$, σ_A, σ_B, σ_{AB}, and σ_i) if you were looking to construct a variance covariance matrix for N actual assets.

5. The above questions assume an *unconditional* factor structure. That is, all of the key parameters ($\beta_{i,A}$, $\beta_{i,B}$, σ_A, σ_B, σ_{AB}, and σ_i) are assumed to be constant over time. However, in the real world these parameters might not be constant. For example a factor's volatility might have been expected to be low in 2017 when financial markets were relatively calm, but it might have been expected to be relatively high in 2020 when financial markets were stressed due to the Covid-19 shock. In this case, a *conditional* factor model is more appropriate. Specifically, the equation for return at $t+1$ changes to:

$$r_{i,t+1} = \mu_{i,t} + \beta_{i,A,t} f_{A,t+1} + \beta_{i,B,t} f_{B,t+1} + \sigma_{i,t}\epsilon_{i,t+1} \tag{4}$$

where $\sigma_{i,t}$ is the *conditional* idiosyncratic standard deviation of i's return (which may change over time), $\beta_{i,A,t}$ is the *conditional* factor loading of i's return on factor A (which may change over time), etc.

Assume that even though the model's parameters ($\beta_{i,A,t}$, $\beta_{i,B,t}$, $\sigma_{A,t}$, $\sigma_{B,t}$, $\sigma_{AB,t}$, and $\sigma_{i,t}$) may be changing over time, they are changing slowly. Thus, recent realizations of the parameters should be good estimates of future realizations. Explain step-by-step how you would estimate the necessary inputs if you were looking to construct a *conditional* variance covariance matrix for N actual assets.

6. Suppose you constructed a conditional variance covariance matrix as in 5. This implies that for each input, you have a time series of values. For simplicity, lets focus on just one: $\sigma_{A,t}$. For example, you have $\sigma_{A,2010:1}$, $\sigma_{A,2010:2}$, ... $\sigma_{A,2022:12}$. How would you test whether your conditional model is doing a reasonable job?