Implied Covariance Matrix Calculation

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Introduction

Suppose we have N_S stocks, N_F factors, with $N_S \gg N_F$.

The $N_S \times N_S$ **implied variance covariance matrix** Σ can be written as

$$\Sigma = B + \Psi,$$

where Ψ is a $N_S \times N_S$ diagonal **matrix of idiosyncratic returns** and B is the factor matrix,

$$B = F^{-1}AF,$$

where F is an $N_F \times N_S$ **matrix of factor-stock loadings** and A is an $N_F \times N_F$ **factor variance covariance matrix**.

Example

```
STOCKS:      GME     MSFT        NVDA        SBUX       TSLA
Date
2021-01-04   4.3125  211.996597  130.851624  97.033089  243.256668
2021-01-05   4.3425  212.201141  133.757812  97.353081  245.036667
2021-01-06   4.5900  206.698914  125.872383  98.058937  251.993332
2021-01-07   4.5200  212.580917  133.151642  97.268364  272.013336
2021-01-08   4.4225  213.876144  132.480576  99.451843  293.339996

FACTORS:     QQQ         SPY
Date
2021-01-04   304.244324  354.197357
2021-01-05   306.752563  356.636871
2021-01-06   302.503387  358.769043
2021-01-07   309.821533  364.099487
2021-01-08   313.805145  366.173981

Calculated F with shape (2, 5)
Calculated A with shape (2, 2)
Calculated F^-1 with shape (5, 2)
Calculated B with shape (5, 5)

Calculated PSI with shape (5, 5)

Implied Variance-Covariance Matrix:
[[ 1.28411350e-01  2.21528468e-06  5.10564157e-06  8.84244400e-07  5.38855554e-06]
 [ 1.12731732e-04  1.84688390e-02  2.42270650e-04  1.08987187e-04  2.05739261e-04]
 [ 8.17390748e-05  9.73680229e-05  3.48392259e-02  7.67323950e-05  1.49727232e-04]
 [ 1.80624187e-04  2.19347272e-04  3.88921784e-04  1.89742793e-02  3.29225943e-04]
 [-3.77996246e-05 -4.81475066e-05 -8.29466915e-05 -4.05592148e-05  3.84230376e-02]]
```

Inverse of Implied Covariance Matrix

$$\Sigma = B + \Psi.$$

How do we compute Σ^{-1}? If $N_S \gg 1$ this becomes very expensive. Use a lemma from Miller 1981:

Knowing Ψ^{-1} (cheap: $\Psi_{ii} \to 1/\Psi_{ii}$), and if B_i is **rank 1**, we have:

$$(B_i + \Psi)^{-1} = \Psi^{-1} - \frac{1}{1+g} \Psi^{-1} B_i \Psi^{-1} \quad \text{where} \quad g = \mathrm{Tr}\left(B\Psi^{-1}\right)$$

So, rewrite B as the **sum of matrices** B_i each of **rank 1**:

$$B = \sum_{i=1}^{N_S} B_i$$

Inverse of Implied Covariance Matrix

So, rewrite B as the **sum of matrices** B_i each of **rank 1**:

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And build up the inverse of Σ from these:

$$\Sigma^{-1} = (B + \Psi)^{-1} = C_{N_S}^{-1} \quad \text{where} \quad C_i^{-1} = C_{i-1}^{-1} - \frac{1}{1+g_i} C_{i-1}^{-1} B_i C_{i-1}^{-1},$$

$$\text{and} \quad g_i = \mathrm{Tr}\left(C_{i-1}^{-1} B_i\right),\ C_0 = \Psi.$$

Note: B will have rank of **at most** N_F, so it is possible to write B as a sum of N_F rank 1 matrices, further speeding up computation.

Example

```
print(np.linalg.inv(SIGMA))

>>  [[ 7.78747575e+00 -9.23359207e-04 -1.13026653e-03 -3.52683837e-04 -1.07733194e-03]
     [-4.69478343e-02  5.41501739e+01 -3.73768708e-01 -3.10132223e-01 -2.85830195e-01]
     [-1.80107789e-02 -1.50253936e-01  2.87053449e+01 -1.15460760e-01 -1.10066649e-01]
     [-7.33496003e-02 -6.24058299e-01 -5.85104837e-01  5.27079334e+01 -4.45991405e-01]
     [ 7.48576611e-03  6.68696503e-02  6.08800699e-02  5.49990215e-02  2.60249841e+01]]

C = PSI.copy()
C_inv = np.diag(1/np.diagonal(C))
for i in range(0,N_S):
    B_i = np.zeros_like(B)
    B_i[i,:] = B[i,:]
    # They have a typo in the worksheet, the line below is false:
    #                  vvvvv this is not just the trace
    # C_inv = C_inv - np.trace(B_i @ C_inv) * C_inv @ B_i @ C_inv
    C_inv = C_inv - 1/(1+np.trace(B_i @ C_inv)) * C_inv @ B_i @ C_inv

print(C_inv)

>>  [[ 7.78747575e+00 -9.23359207e-04 -1.13026653e-03 -3.52683837e-04 -1.07733194e-03]
     [-4.69478343e-02  5.41501739e+01 -3.73768708e-01 -3.10132223e-01 -2.85830195e-01]
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     [ 7.48576611e-03  6.68696503e-02  6.08800699e-02  5.49990215e-02  2.60249841e+01]]
```

It's a match!