2D Axisymmetric Heat Flow – Finite Difference Approximation

Diffusion equation general form:

$$\dot{u} = \alpha \nabla^2 u$$

where $u = u(r, \theta, z, t)$, heat in 3 coords. + time,

α = thermal diffusivity

Laplacian in cylindrical coords:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}_{\substack{\text{this term is zero in the} \\ \text{axisymmetric assumption,} \\ \text{derivs. w.r.t. } \theta \text{ are zero}}} + \frac{\partial^2 u}{\partial z^2}$$

$$\begin{aligned}
\nabla^2 u &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} \\
&= \frac{1}{r}\cdot\left[\frac{\partial}{\partial r}(r)\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right] + \frac{\partial^2 u}{\partial z^2} \\
&= \frac{1}{r}\cdot\left[\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right] + \frac{\partial^2 u}{\partial z^2} \\
&= \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2}
\end{aligned}$$

Put this term into the diffusion equation:

$$\boxed{\frac{\partial u}{\partial t} = \alpha\left[\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2}\right]}$$

This is the analytic form of the diff. eq., next is to discretize it w/ the finite difference approx.

I will use forward difference in time, and central difference in space.

$$\frac{\partial u}{\partial t} = \alpha \left[\frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} \right] \tag{1}$$

1st order forward diff. ($\frac{\partial u}{\partial t}$)

1st order central diff. ($\frac{\partial u}{\partial r}$)

2nd order central diff. ($\frac{\partial^2 u}{\partial r^2}$, $\frac{\partial^2 u}{\partial z^2}$)

Indices: $m \rightarrow$ time

$i \rightarrow$ space, r direction

$j \rightarrow$ space, z direction

$U^{m}_{i,j}$ — time coord on top; space coords on bottom

$$\frac{\partial u}{\partial t} \Rightarrow \frac{U^{m+1}_{i,j} - U^{m}_{i,j}}{\Delta t} \qquad \frac{\partial^2 u}{\partial r^2} \Rightarrow \frac{U^{m}_{i-1,j} - 2U^{m}_{i,j} + U^{m}_{i+1,j}}{\Delta r^2}$$

$$\frac{\partial u}{\partial r} \Rightarrow \frac{U^{m}_{i+1,j} - U^{m}_{i-1,j}}{2\Delta r} \qquad \frac{\partial^2 u}{\partial z^2} \Rightarrow \frac{U^{m}_{i,j-1} - 2U^{m}_{i,j} + U^{m}_{i,j+1}}{\Delta z^2}$$

Plug all these approximations into Eqn. (1):

$$\frac{U^{m+1}_{i,j} - U^{m}_{i,j}}{\Delta t} = \alpha \left[\frac{1}{r} \cdot \frac{U^{m}_{i+1,j} - U^{m}_{i-1,j}}{2\Delta r} + \frac{U^{m}_{i-1,j} - 2U^{m}_{i,j} + U^{m}_{i+1,j}}{\Delta r^2} + \frac{U^{m}_{i,j-1} - 2U^{m}_{i,j} + U^{m}_{i,j+1}}{\Delta z^2} \right]$$

Solve for $U^{m+1}_{i,j}$:

$$\boxed{U^{m+1}_{i,j} = U^{m}_{i,j} + \alpha \Delta t \left[\frac{1}{r} \frac{U^{m}_{i+1} - U^{m}_{i-1,j}}{2\Delta r} + \frac{U^{m}_{i-1} - 2U^{m}_{i,j} + U^{m}_{i+1,j}}{\Delta r^2} + \frac{U^{m}_{i,j-1} - 2U^{m}_{i,j} + U^{m}_{i,j+1}}{\Delta z^2} \right]}$$

$\Rightarrow$ Next state of the array (time $\rightarrow m+1$) in terms of the current state of the array (time $\rightarrow m$)