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Part 1: Fundamentals & Classical Statistical Methods

Time Series Analysis in Python

Setup and Imports

Before we begin, install the required packages:

```
pip install pandas numpy matplotlib statsmodels scipy scikit-learn
```

In [27]:

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from statsmodels.tsa.seasonal import seasonal_decompose
from statsmodels.graphics.tsaplots import plot_acf, plot_pacf
from statsmodels.tsa.stattools import adfuller, kpss
from statsmodels.tsa.arima.model import ARIMA
from statsmodels.tsa.statespace.sarimax import SARIMAX
from statsmodels.stats.diagnostic import acorr_ljungbox
from scipy import stats
from sklearn.metrics import mean_absolute_error, mean_squared_error
import warnings
warnings.filterwarnings('ignore')

plt.style.use('seaborn-v0_8-darkgrid')
```

1. What Is a Time Series?

A **time series** is a sequence of observations collected **over time**, usually at regular intervals.

Examples:

- Daily stock prices
- Hourly electricity consumption
- Monthly sales revenue
- Sensor measurements every second

Why Time Series Are Different

- Observations are **dependent**

- The order of data points **cannot be shuffled**
- Past values influence future values

This dependency structure is the core challenge of time series analysis.

Key Components

Time series typically contain four components:

1. **Trend (T)**: Long-term increase or decrease in the data
2. **Seasonality (S)**: Regular, predictable patterns that repeat over fixed periods (e.g., yearly, monthly)
3. **Cyclicality (C)**: Patterns that repeat but not at fixed intervals (e.g., economic cycles)
4. **Noise/Irregular (I)**: Random variation that cannot be attributed to trend, seasonality, or cyclicality

Mathematical Representation

- **Additive Model**: $Y(t) = T(t) + S(t) + C(t) + I(t)$
  - Use when seasonal variation is roughly constant over time
- **Multiplicative Model**: $Y(t) = T(t) \times S(t) \times C(t) \times I(t)$
  - Use when seasonal variation increases with the level of the series

Creating a Sample Time Series

The code below makes fake daily data that looks like a real-world time series.

More specifically:

- It creates **daily dates from 2020 to 2023**.
- It makes the values **slowly increase over time**.
- It adds a **repeating yearly up-and-down pattern**.
- It adds a bit of **random randomness** so it is not perfectly smooth.
- It combines all of that into one dataset and shows a small sample.

In short:

> **It simulates realistic daily data with a trend, seasonality, and noise.**

In [28]:
```
# Set random seed for reproducibility
np.random.seed(42)

# Generate date range
```

```
date_range = pd.date_range(start='2020-01-01', end='2023-12-31', freq='D')
n = len(date_range)

# Components
trend = np.linspace(100, 150, n)  # Linear trend from 100 to 150
seasonality = 10 * np.sin(2 * np.pi * np.arange(n) / 365.25)  # Yearly seasonality
noise = np.random.normal(0, 5, n)  # Random noise

# Combine components (additive model)
ts_data = trend + seasonality + noise

# Create pandas Series with datetime index
ts = pd.Series(ts_data, index=date_range, name='Value')

print("Sample Time Series:")
print(ts.head(10))
print(f"\nShape: {ts.shape}")
print(f"Period: {ts.index.min()} to {ts.index.max()}")
```

```
Sample Time Series:
2020-01-01    102.483571
2020-01-02     99.514941
2020-01-03    103.650916
2020-01-04    108.233733
2020-01-05     99.653774
2020-01-06     99.859609
2020-01-07    109.131857
2020-01-08    105.278161
2020-01-09     99.298455
2020-01-10    104.563060
Freq: D, Name: Value, dtype: float64

Shape: (1461,)
Period: 2020-01-01 00:00:00 to 2023-12-31 00:00:00
```

Visualizing the Time Series

The code below draws a line chart of the time-series data.

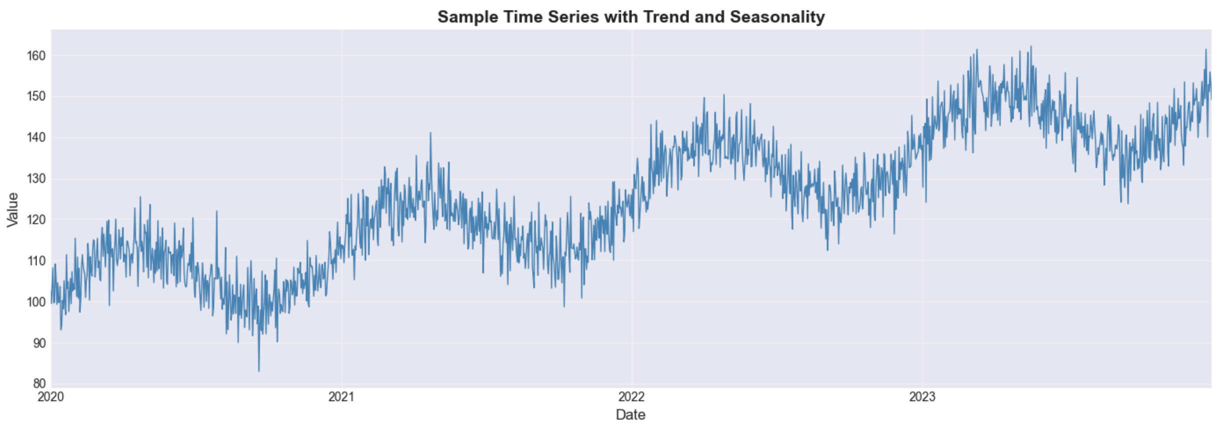
More specifically:

- It creates a **plotting area** of a fixed size.
- It **plots the time series as a line** on that area.
- It adds a **title** and **labels** for the x- and y-axes.
- It turns on a **light grid** to make the chart easier to read.
- It adjusts spacing so labels are not cut off.
- It **displays the chart**.

In short:

> **It visualizes the time series so you can see the trend and seasonal pattern over time.**

In [29]:
```
#subplot function returns a tuple (figure, axes), figure is the entire figure, axes
# Axes object is what we plot on
# figsize is in inches
fig, ax = plt.subplots(figsize=(14, 5))
ts.plot(ax=ax, linewidth=1, color='steelblue')
ax.set_title('Sample Time Series with Trend and Seasonality', fontsize=14, fontweig
ax.set_xlabel('Date', fontsize=12)
ax.set_ylabel('Value', fontsize=12)
ax.grid(True, alpha=0.3)
plt.tight_layout() # Adjust layout to prevent clipping of labels
plt.show()
```



2. Time Series Decomposition

Theory

Decomposition is the process of separating a time series into its constituent components. This helps us:

- Understand the underlying patterns
- Remove seasonality for better modeling
- Identify anomalies
- Choose appropriate forecasting methods

Classical Decomposition Methods

The most common method is **moving average decomposition**:

1. Estimate trend using moving average
2. Detrend the series
3. Estimate seasonal component by averaging detrended values for each season
4. Calculate residuals: Residual = Original - Trend - Seasonal

Performing Decomposition

The code below breaks the time series into its main parts.

More specifically:

- It tells Python the data has a **yearly pattern** (365 days).
- It **separates the data** into:
  - a **trend** (long-term direction),
  - a **seasonal pattern** (repeating yearly ups and downs),
  - and **random noise** (what's left over).
- It stores each part separately.
- It prints the size of each component to confirm they match the original data.

In short:

> **It decomposes the time series into trend, seasonality, and noise.**

In [30]:
```
# Perform additive decomposition
# Period = 365 because we have daily data with yearly seasonality
decomposition = seasonal_decompose(ts, model='additive', period=365)

# Extract components
trend_component = decomposition.trend
seasonal_component = decomposition.seasonal
residual_component = decomposition.resid

print("Decomposition Components:")
print(f"Trend:    {trend_component.shape}")
print(f"Seasonal: {seasonal_component.shape}")
print(f"Residual: {residual_component.shape}")
```

```
Decomposition Components:
Trend:    (1461,)
Seasonal: (1461,)
Residual: (1461,)
```

Visualizing Decomposition

The code below draws four stacked charts to show the decomposition results.

More specifically:

- It creates **four plots arranged vertically**.
- The first plot shows the **original time series**.

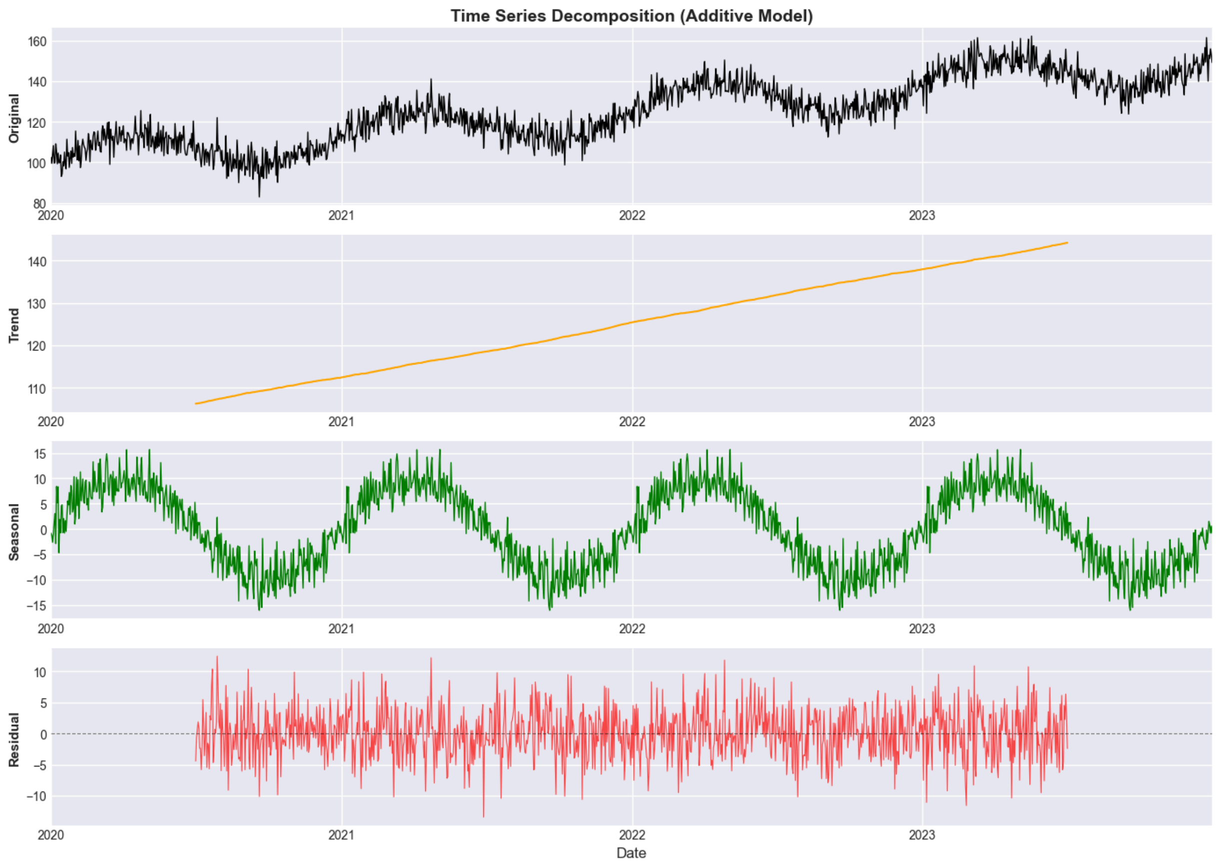
- The second shows the **trend** (long-term movement).
- The third shows the **seasonal pattern** (repeating yearly cycle).
- The fourth shows the **residuals** (random noise left over).
- It labels each plot so you can clearly see what each line represents.
- It draws a zero line on the residual plot to make deviations easy to spot.
- Finally, it displays the figure.

In short:

> **It visually explains how the original data is split into trend, seasonality, and noise.**

In [31]:

```
fig, axes = plt.subplots(4, 1, figsize=(14, 10))

# Original
ts.plot(ax=axes[0], linewidth=1, color='black')
axes[0].set_ylabel('Original', fontsize=11, fontweight='bold')
axes[0].set_title('Time Series Decomposition (Additive Model)', fontsize=14, fontwe

# Trend
trend_component.plot(ax=axes[1], linewidth=1.5, color='orange')
axes[1].set_ylabel('Trend', fontsize=11, fontweight='bold')

# Seasonal
seasonal_component.plot(ax=axes[2], linewidth=1, color='green')
axes[2].set_ylabel('Seasonal', fontsize=11, fontweight='bold')

# Residual
residual_component.plot(ax=axes[3], linewidth=0.8, color='red', alpha=0.7)
axes[3].set_ylabel('Residual', fontsize=11, fontweight='bold')
axes[3].set_xlabel('Date', fontsize=12)
axes[3].axhline(y=0, color='black', linestyle='--', linewidth=0.8, alpha=0.5)

plt.tight_layout()
plt.show()
```



3. Stationarity

What is Stationarity?

A time series is **stationary** if its statistical properties (mean, variance, autocorrelation) do not change over time.

Why Does Stationarity Matter?

Most classical time series models (ARIMA, SARIMA) assume stationarity because:

- Statistical properties are easier to model when constant
- Predictions are more reliable
- Mathematical theory is simpler

Types of Stationarity

1. **Strict Stationarity**: Joint distribution is time-invariant (very restrictive)
2. **Weak/Covariance Stationarity**: Only mean, variance, and autocorrelation are constant (commonly used)

Common Non-Stationary Patterns

- **Trend**: Mean changes over time
- **Seasonality**: Pattern repeats at regular intervals
- **Heteroscedasticity**: Variance changes over time

Augmented Dickey-Fuller (ADF) Test

The ADF test checks the null hypothesis that the series has a unit root (non-stationary).

- **H_0**: Series has a unit root (non-stationary)
- **H_1**: Series is stationary

Decision Rule: If p-value < 0.05, reject H_0 (series is stationary)

In [32]:
```
def check_stationarity(timeseries, name='Series'):
    """
    Perform Augmented Dickey-Fuller test
    """
    print(f"\n{'='*60}")
    print(f"Stationarity Test: {name}")
    print('='*60)

    # Remove NaN values
    ts_clean = timeseries.dropna()

    # Perform ADF test
    result = adfuller(ts_clean, autolag='AIC')

    print(f'ADF Statistic:      {result[0]:.6f}')
    print(f'p-value:            {result[1]:.6f}')
    print(f'# Lags Used:        {result[2]}')
    print(f'# Observations:     {result[3]}')
    print('\nCritical Values:')
    for key, value in result[4].items():
        print(f'  {key}: {value:.3f}')

    # Interpretation
    print('\n' + '-'*60)
    if result[1] <= 0.05:
        print(f"✓ STATIONARY (p={result[1]:.4f} < 0.05)")
        print("  → Reject null hypothesis")
    else:
        print(f"✗ NON-STATIONARY (p={result[1]:.4f} > 0.05)")
        print("  → Fail to reject null hypothesis")
    print('-'*60)

    return result[1] <= 0.05
```

Testing Our Series

In [33]:
```
# Test original series
is_stationary = check_stationarity(ts, 'Original Time Series')
```

```
============================================================
Stationarity Test: Original Time Series
============================================================
ADF Statistic:     -0.835398
p-value:           0.808502
# Lags Used:       12
# Observations:    1448

Critical Values:
  1%: -3.435
  5%: -2.864
  10%: -2.568

------------------------------------------------------------
✗ NON-STATIONARY (p=0.8085 > 0.05)
  → Fail to reject null hypothesis
------------------------------------------------------------
```

Making a Series Stationary using the Differencing Method

Differencing removes trends and can stabilize the mean:

First Difference: Y'(t) = Y(t) - Y(t-1)

In [34]:
```
# Apply first differencing
ts_diff = ts.diff().dropna()

# Test differenced series
is_stationary_diff = check_stationarity(ts_diff, 'First Differenced Series')
```

```
============================================================
Stationarity Test: First Differenced Series
============================================================
ADF Statistic:     -17.823692
p-value:           0.000000
# Lags Used:       11
# Observations:    1448

Critical Values:
  1%: -3.435
  5%: -2.864
  10%: -2.568

------------------------------------------------------------
✓ STATIONARY (p=0.0000 < 0.05)
  → Reject null hypothesis
------------------------------------------------------------
```

Visualizing the Transformation

The code below compares the original data with a transformed version that removes trend.

More specifically:

- The top plot shows the **original time series**, which changes over time (non-stationary).
- The bottom plot shows the **first-differenced series**, where each value is replaced by the change from the previous one.
- The differenced series fluctuates around zero, making it **more stable (stationary)**.
- The horizontal zero line helps you see this stability.
- Both plots share the same time axis for easy comparison.

In short:

> **It shows how differencing turns a trending series into a stationary one.**

In [35]:
```
fig, axes = plt.subplots(2, 1, figsize=(14, 8), sharex=True)

# Original
ts.plot(ax=axes[0], linewidth=1.2, color='steelblue')
axes[0].set_title('Original Series (Non-Stationary)', fontsize=12, fontweight='bold
axes[0].set_ylabel('Value', fontsize=11)
axes[0].grid(True, alpha=0.3)

# Differenced
ts_diff.plot(ax=axes[1], linewidth=1, color='orange')
axes[1].set_title('First Differenced Series (Stationary)', fontsize=12, fontweight=
axes[1].set_ylabel('Differenced Value', fontsize=11)
axes[1].axhline(y=0, color='red', linestyle='--', linewidth=1, alpha=0.7)
axes[1].set_xlabel('Date', fontsize=12)
axes[1].grid(True, alpha=0.3)

plt.tight_layout()
plt.show()
```

4. Autocorrelation Analysis

ACF: Autocorrelation Function

The ACF measures the correlation between observations at different time lags:

$ACF(k) = Corr(Y_t, Y_{t-k})$

- Lag 1: correlation between consecutive observations
- Lag 2: correlation between observations 2 time periods apart
- And so on...

PACF: Partial Autocorrelation Function

The PACF measures the correlation between Y_t and Y_{t-k} after removing the effect of intermediate lags.

Why Are ACF and PACF Important?

They help identify the order of AR and MA components:

Pattern	Model Suggestion
PACF cuts off after lag p, ACF decays	AR(p)

Pattern	Model Suggestion
ACF cuts off after lag q, PACF decays	MA(q)
Both decay gradually	ARMA(p,q)

Plotting ACF and PACF

The code below looks at how the data is related to its past values.

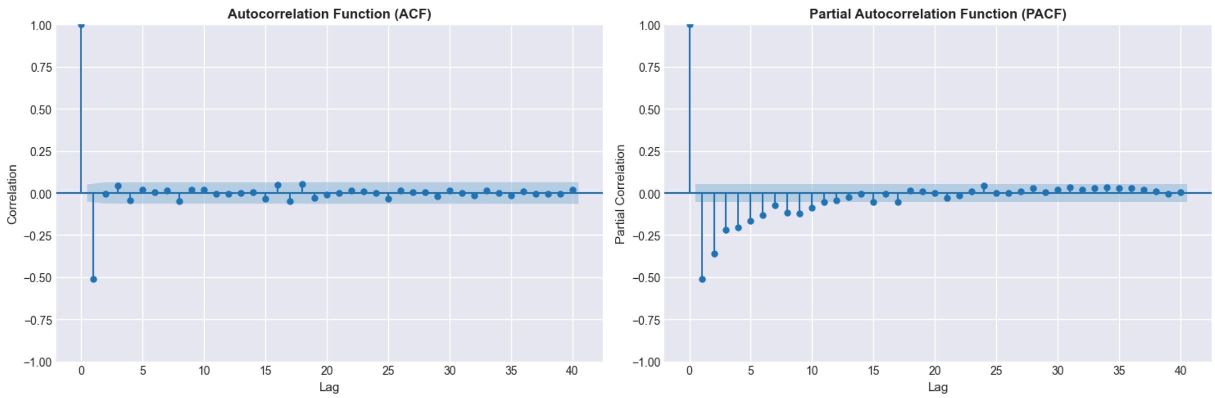
More specifically:

- It uses the **stationary (differenced) series**.
- The **ACF plot** shows how today's value is correlated with previous days.
- The **PACF plot** shows the *direct* effect of past days, removing indirect effects.
- The plots help decide how many past values a time-series model (like ARIMA) should use.

In short:

> **It helps you choose the right ARIMA model by showing which past values matter.**

In [36]:
```
# Use differenced series (should be stationary)
fig, axes = plt.subplots(1, 2, figsize=(15, 5))

# ACF
plot_acf(ts_diff, lags=40, ax=axes[0])
axes[0].set_title('Autocorrelation Function (ACF)', fontsize=12, fontweight='bold')
axes[0].set_xlabel('Lag', fontsize=11)
axes[0].set_ylabel('Correlation', fontsize=11)

# PACF
plot_pacf(ts_diff, lags=40, ax=axes[1])
axes[1].set_title('Partial Autocorrelation Function (PACF)', fontsize=12, fontweigh
axes[1].set_xlabel('Lag', fontsize=11)
axes[1].set_ylabel('Partial Correlation', fontsize=11)

plt.tight_layout()
plt.show()
```



Interpretation:

- Blue shaded area represents the confidence interval
- Bars outside this area are statistically significant
- Look for where bars drop inside the confidence interval (cutoff point)

The ACF plot shows a single strong and statistically significant spike at lag 1, followed by correlations that lie within the confidence bounds at higher lags. This indicates a clear cutoff in the autocorrelation function after the first lag.

The PACF plot does not exhibit a sharp cutoff. Instead, the partial autocorrelations decay gradually toward zero over several lags, indicating a tailing-off pattern.

This combination—ACF cutting off at lag 1 and PACF tailing off—is characteristic of a moving-average process of order 1. When applied to a first-differenced series, this pattern is consistent with an ARIMA(0,1,1) model.

5. Classical Time Series Models

Model	Full Name	Equation	When to Use
AR(p)	Autoregressive	$Y_t = c + \varphi_1 Y_{t-1} + \ldots + \varphi_p Y_{t-p} + \varepsilon_t$	PACF cuts off
MA(q)	Moving Average	$Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_\varphi \varepsilon_{t-\varphi}$	ACF cuts off
ARMA(p,q)	AR + MA	Combines both	Both ACF & PACF decay
ARIMA(p,d,q)	Integrated ARMA	ARMA on d-times differenced data	Non-stationary series
SARIMA(p,d,q)(P,D,Q,s)	Seasonal ARIMA	Adds seasonal components	Seasonal patterns

Preparing Data for Modeling

Let's create a simpler dataset to demonstrate the models:

The code below **creates a simple autoregressive time series and splits it into training and test data**.

More specifically:

- It generates **300 days of fake daily data**.
- Each day's value depends on **70% of the previous day's value**, plus some random noise. This is an **AR(1) process**.
- The result is a series where values are **correlated with their immediate past**.
- The data is then split into:
  - **80% for training** (used to fit a model),
  - **20% for testing** (used to evaluate predictions).
- Finally, it prints how many observations are in each part.

In short:

The code simulates a simple AR(1) time series and prepares it for model training and evaluation.

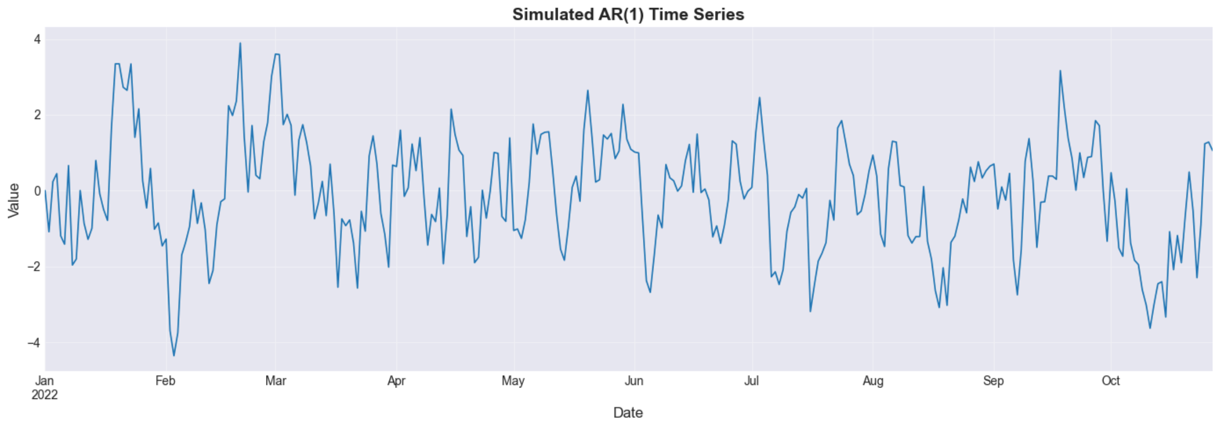
In [37]:
```
# Generate AR(1) process
np.random.seed(123)
n = 300
dates = pd.date_range(start='2022-01-01', periods=n, freq='D')

# AR(1): Y(t) = 0.7 * Y(t-1) + noise
ar_coef = 0.7
ar_data = [0]
for i in range(1, n):
    ar_data.append(ar_coef * ar_data[i-1] + np.random.normal(0, 1))

ts_simple = pd.Series(ar_data, index=dates, name='Value')

# Train-test split (80-20)
train_size = int(len(ts_simple) * 0.8)
train = ts_simple[:train_size]
test = ts_simple[train_size:]

print(f"Train: {len(train)} observations")
print(f"Test:  {len(test)} observations")
```

```
Train: 240 observations
Test:  60 observations
```

Plot the dataset

In [38]:
```
import matplotlib.pyplot as plt

# Plot the AR(1) time series
```

```
plt.figure(figsize=(14, 5))
ts_simple.plot(linewidth=1.2)
plt.title('Simulated AR(1) Time Series', fontsize=14, fontweight='bold')
plt.xlabel('Date', fontsize=12)
plt.ylabel('Value', fontsize=12)
plt.grid(True, alpha=0.3)
plt.tight_layout()
plt.show()
```



Model 1: AR (Autoregressive)

Theory

An AR(p) model predicts the current value using p past values:

$$\mathbf{Y_t = c + \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \ldots + \varphi_p Y_{t-p} + \varepsilon_t}$$

Where:

- c is a constant
- $\varphi_1, \varphi_2, \ldots, \varphi_p$ are coefficients
- ε_t is white noise error

Implementation

The code below **fits an AR(1) model to the training data and compares what the model learns to the true value used to generate the data**.

More specifically:

- It fits an **ARIMA(1,0,0)** model, which is an **AR(1)** model (one autoregressive term, no differencing, no moving average).
- The model is trained using the **training portion** of the simulated series.
- It prints a **summary** showing estimated parameters and diagnostics.
- It extracts the **estimated AR(1) coefficient** from the fitted model.

- It compares that estimate to the **true coefficient (0.7)** used to generate the data.

In short:

The code checks whether an AR(1) model can correctly recover the underlying dependency in the simulated time series.

In [39]:
```
# Fit AR(1) model
# ARIMA(p,d,q) with p=1, d=0, q=0
ar_model = ARIMA(train, order=(1, 0, 0))
ar_fit = ar_model.fit()

print("AR(1) Model Summary:")
print(ar_fit.summary())
print(f"\nEstimated coefficient: {ar_fit.params[1]:.4f}")
print(f"True coefficient: {ar_coef}")
```

```
AR(1) Model Summary:
                               SARIMAX Results
==============================================================================
Dep. Variable:                  Value   No. Observations:                  240
Model:                 ARIMA(1, 0, 0)   Log Likelihood                -345.782
Date:                Tue, 30 Dec 2025   AIC                            697.563
Time:                        08:17:58   BIC                            708.005
Sample:                    01-01-2022   HQIC                           701.771
                         - 08-28-2022
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.0308      0.229     -0.135      0.893      -0.479       0.418
ar.L1          0.7119      0.047     15.074      0.000       0.619       0.805
sigma2         1.0415      0.094     11.072      0.000       0.857       1.226
===================================================================================
Ljung-Box (L1) (Q):                   0.15   Jarque-Bera (JB):                 0.10
Prob(Q):                              0.70   Prob(JB):                         0.95
Heteroskedasticity (H):               0.60   Skew:                            -0.04
Prob(H) (two-sided):                  0.03   Kurtosis:                         3.05
===================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (complex-ste
p).

Estimated coefficient: 0.7119
True coefficient: 0.7
```

Interpreting AR(1) Model Output - OPTIONAL

What this really is (in one sentence)

AR(1) is just linear regression where the predictor is yesterday's value.

Connect it to regular regression (intuition first)

In ordinary regression you write:

> $y = \beta_0 + \beta_1 x_1 + \text{error}$

In AR(1), you write:

> $y_t = \text{constant} + \beta_1 \times y_{t-1} + \text{error}$

Same idea. The "feature" is just the lagged value of y itself.

Translate the output into regression language

1. The coefficients

```
const     = -0.0308
ar.L1     =  0.7119
sigma2    =  1.0415
```

Think of this as a regression table:

Feature	Coefficient	What it means
Intercept	-0.031	Baseline (when $y_{t-1} = 0$)
y_{t-1}	0.7119	Today = 71% of yesterday + noise
Error variance	1.042	Residual spread

The prediction equation is:

> $y_t = -0.031 + 0.7119 \times y_{t-1} + \varepsilon_t$

This is identical to:

> sales_today = -0.031 + 0.7119 × sales_yesterday + noise

2. Standard errors and p-values

Exactly the same as regression:

```
              coef    std err    p-value
const        -0.031   0.229      0.893
ar.L1         0.712   0.047      0.000
```

const: $p = 0.893$ (high)

Not significant. Could drop it.

ar.L1: $p = 0.000$ (very low)

Highly significant. Yesterday strongly predicts today.

Same interpretation as:

> feature_1: p = 0.893 → not useful feature_2: p = 0.000 → very useful

The model found **strong autocorrelation** in the data.

3. Model quality metrics

```
AIC = 697.6
BIC = 708.0
Log Likelihood = -345.8
```

These are **goodness-of-fit measures**, not accuracy metrics.

AIC and BIC: Lower is better

Use these to compare models:

- AR(1): AIC = 697.6
- AR(2): AIC = 695.3 → better
- MA(1): AIC = 701.2 → worse

Think of AIC like:

> penalized training error

It balances fit with model complexity.

Log Likelihood: Higher is better

Similar to minimizing loss in ML. Maximum likelihood estimation finds coefficients that make the observed data most probable.

4. Diagnostic tests (are residuals well-behaved?)

```
Ljung-Box (Q):           0.15    p = 0.70
Jarque-Bera (JB):        0.10    p = 0.95
Heteroskedasticity (H):  0.60    p = 0.03
```

Think of these as **residual plots in test form**.

Ljung-Box test

Question: "Do residuals have patterns left?"

- p > 0.05 → No patterns (good)
- p < 0.05 → Patterns remain (bad)

Here: p = 0.70 → residuals look random

This is like checking:

> plot(residuals) shows no trend

Jarque-Bera test

Question: "Are residuals normally distributed?"

- $p > 0.05$ → Yes (good)
- $p < 0.05$ → No (may need transformation)

Here: p = 0.95 → very normal

This is like checking:

> histogram(residuals) looks bell-shaped

Heteroskedasticity test

Question: "Is variance constant over time?"

- $p > 0.05$ → Yes (good)
- $p < 0.05$ → No (variance changes)

Here: p = 0.03 → some heteroskedasticity detected

This is like seeing:

> residual spread increases over time

5. Model validation

```
Estimated coefficient: 0.7119
True coefficient:      0.7000
```

The model recovered the true data-generating process.

In ML terms:

> The model learned the correct function

This would be like:

- You generate data: y = 2x + noise
- Your model learns: y = 1.98x
- Close match → model works

Why it feels alien

1. No feature matrix visible

The "feature" (y_{t-1}) is created internally from the time series.

2. Maximum likelihood instead of MSE

Same goal (fit the data), different math. Likelihood is more general than squared error.

3. Heavy focus on diagnostics

Econometrics cares deeply about **why** the model works, not just **that** it works.

Statistical inference > predictive accuracy

4. Different vocabulary

- ML says: "R^2 = 0.85"
- Econometrics says: "Log Likelihood = -345, AIC = 697"

Same information, different packaging.

How to read this output (decision rules)

Check coefficients:

- $p < 0.05$ → significant → keep
- $p > 0.05$ → not significant → consider dropping

Check diagnostics:

- Ljung-Box $p > 0.05$ → good (no autocorrelation left)
- Ljung-Box $p < 0.05$ → bad (model incomplete)

Compare models:

- Lower AIC → better model
- Lower BIC → better model (penalizes complexity more)

Validate:

- Residuals should look random
- Residuals should be roughly normal
- Variance should be constant (ideally)

One grounding sentence you can remember

> AR(1) is linear regression where X is yesterday's y, and the output tells you both fit quality and residual behavior in statistical language instead of ML metrics.

Quick interpretation of this specific output

Model found: $y_t = 0.71 \times y_{t-1}$ + noise

Constant term: Not significant ($p = 0.89$) → probably zero in reality

Autocorrelation: Strong (coefficient = 0.71, $p < 0.001$) → yesterday matters a lot

Residuals: Clean (Ljung-Box $p = 0.70$, JB $p = 0.95$) → no patterns left

Minor issue: Slight heteroskedasticity ($p = 0.03$) → variance not perfectly constant

Conclusion: This is a good model. The AR(1) structure correctly captures the data-generating process.

Model 2: MA (Moving Average)

Theory

An MA(q) model predicts the current value using past forecast errors:

$$\mathbf{Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \ldots + \theta_\varphi\varepsilon_{t-\varphi}}$$

Where:

- μ is the mean
- $\theta_1, \theta_2, \ldots, \theta_\varphi$ are coefficients
- $\varepsilon_t, \varepsilon_{t-1}, \ldots$ are error terms

Implementation

The code below **fits a moving-average model of order 1 to the training data**.

More specifically:

- It fits an **ARIMA(0,0,1)** model, which is an **MA(1)** model.
- The model assumes the series depends on **one past shock (error term)** rather than past values.
- It estimates the MA coefficient using the **training data**.
- It prints a **model summary** with parameter estimates and diagnostics.

In short:

The code fits an MA(1) model so it can be compared with the AR(1) model on the same data.

In [40]:
```
# Fit MA(1) model
ma_model = ARIMA(train, order=(0, 0, 1))
ma_fit = ma_model.fit()
```

```
print("MA(1) Model Summary:")
print(ma_fit.summary())
```

```
MA(1) Model Summary:
                               SARIMAX Results
==============================================================================
Dep. Variable:                  Value   No. Observations:                  240
Model:                 ARIMA(0, 0, 1)   Log Likelihood                -370.431
Date:                Tue, 30 Dec 2025   AIC                            746.863
Time:                        08:17:58   BIC                            757.305
Sample:                    01-01-2022   HQIC                           751.070
                         - 08-28-2022
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.0349      0.120     -0.290      0.771      -0.270       0.200
ma.L1          0.6287      0.051     12.246      0.000       0.528       0.729
sigma2         1.2801      0.119     10.766      0.000       1.047       1.513
===================================================================================
Ljung-Box (L1) (Q):                  10.99   Jarque-Bera (JB):                 0.40
Prob(Q):                              0.00   Prob(JB):                         0.82
Heteroskedasticity (H):               0.60   Skew:                            -0.10
Prob(H) (two-sided):                  0.03   Kurtosis:                         2.94
===================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (complex-ste
p).
```

Model 3: ARMA

Theory

ARMA(p,q) combines AR(p) and MA(q):

$$\mathbf{Y_t = c + \varphi_1 Y_{t-1} + \ldots + \varphi_p Y_{t-p} + \theta_1 \varepsilon_{t-1} + \ldots + \theta_\varphi \varepsilon_{t-\varphi} + \varepsilon_t}$$

Important: ARMA requires stationary data.

Implementation

The code below **fits a combined autoregressive and moving-average model of order (1,1)** to the training data.

More specifically:

- It fits an **ARIMA(1,0,1)** model, also known as **ARMA(1,1)**.
- The model allows the series to depend on:

  - **one past value** (AR part), and
  - **one past shock or error** (MA part).
- The model is trained on the **training dataset**.
- It prints a **summary** showing parameter estimates and diagnostics.

In short:

The code fits an ARMA(1,1) model so its performance and parameters can be compared against the simpler AR(1) and MA(1) models.

In [41]:
```
# Fit ARMA(1,1) model
arma_model = ARIMA(train, order=(1, 0, 1))
arma_fit = arma_model.fit()

print("ARMA(1,1) Model Summary:")
print(arma_fit.summary())
```

```
ARMA(1,1) Model Summary:
                               SARIMAX Results
==============================================================================
Dep. Variable:                  Value   No. Observations:                  240
Model:                 ARIMA(1, 0, 1)   Log Likelihood                -345.671
Date:                Tue, 30 Dec 2025   AIC                            699.341
Time:                        08:17:58   BIC                            713.264
Sample:                    01-01-2022   HQIC                           704.951
                         - 08-28-2022
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.0308      0.223     -0.138      0.890      -0.468       0.407
ar.L1          0.6912      0.072      9.667      0.000       0.551       0.831
ma.L1          0.0424      0.096      0.441      0.659      -0.146       0.231
sigma2         1.0405      0.094     11.083      0.000       0.857       1.225
===================================================================================
Ljung-Box (L1) (Q):                   0.00   Jarque-Bera (JB):                 0.13
Prob(Q):                              0.96   Prob(JB):                         0.94
Heteroskedasticity (H):               0.60   Skew:                            -0.05
Prob(H) (two-sided):                  0.02   Kurtosis:                         3.05
===================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (complex-ste
p).
```

Model 4: ARIMA

Theory

ARIMA(p,d,q) is ARMA applied to d-times differenced data:

- **p**: Order of autoregressive part
- **d**: Degree of differencing
- **q**: Order of moving average part

Steps:

1. Difference the series d times to make it stationary
2. Apply ARMA(p,q) to the differenced series

Implementation

The code below **fits a full ARIMA model, evaluates it, and uses it to make forecasts**.

More specifically:

- It fits an **ARIMA(1,1,1)** model:
  - one autoregressive term,
  - one differencing step to remove trend,
  - one moving-average term.
- The model is trained on the **training data**.
- It prints a **model summary** with estimated parameters.
- It reports **AIC and BIC**, which measure model quality while penalizing complexity (lower is better).
- It then **predicts future values** for the same number of time steps as the test set.

In short:

The code builds a trend-aware ARIMA model and uses it to forecast the unseen portion of the time series.

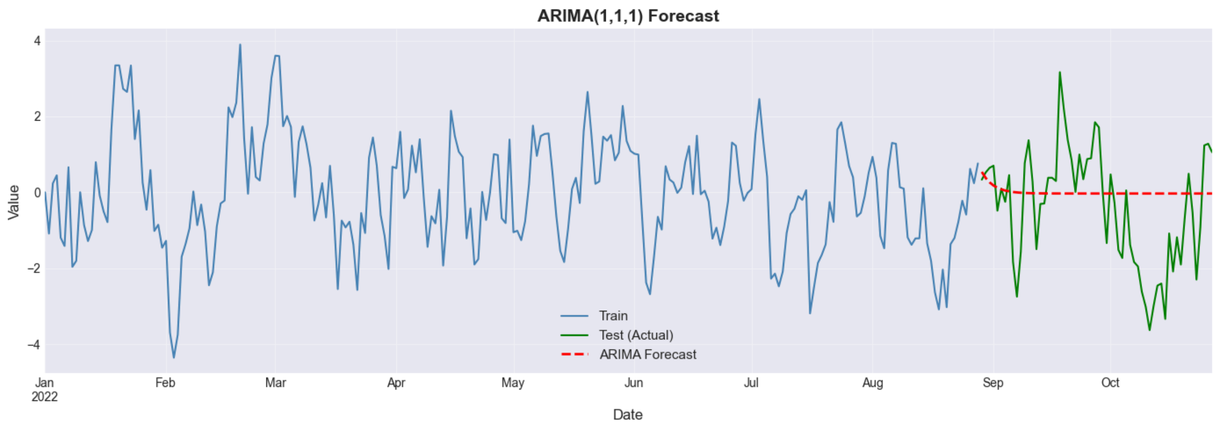
In [42]:
```
# Fit ARIMA(1,1,1)
arima_model = ARIMA(train, order=(1, 1, 1))
arima_fit = arima_model.fit()

print("ARIMA(1,1,1) Model Summary:")
print(arima_fit.summary())
print(f"\nAIC: {arima_fit.aic:.2f}")
print(f"BIC: {arima_fit.bic:.2f}")

# Forecast
forecast_steps = len(test)
arima_forecast = arima_fit.forecast(steps=forecast_steps)
```

```
ARIMA(1,1,1) Model Summary:
                               SARIMAX Results
==============================================================================
Dep. Variable:                  Value   No. Observations:                  240
Model:                 ARIMA(1, 1, 1)   Log Likelihood                -346.335
Date:                Tue, 30 Dec 2025   AIC                            698.670
Time:                        08:17:58   BIC                            709.100
Sample:                    01-01-2022   HQIC                           702.873
                         - 08-28-2022
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
ar.L1          0.7190      0.052     13.919      0.000       0.618       0.820
ma.L1         -0.9999      2.479     -0.403      0.687      -5.860       3.860
sigma2         1.0460      2.583      0.405      0.685      -4.016       6.108
===================================================================================
Ljung-Box (L1) (Q):                   0.08   Jarque-Bera (JB):                 0.08
Prob(Q):                              0.78   Prob(JB):                         0.96
Heteroskedasticity (H):               0.59   Skew:                            -0.03
Prob(H) (two-sided):                  0.02   Kurtosis:                         3.06
===================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (complex-ste
p).

AIC: 698.67
BIC: 709.10
```

Visualizing Forecast

In [43]:

```python
fig, ax = plt.subplots(figsize=(14, 5))

train.plot(ax=ax, label='Train', linewidth=1.5, color='steelblue')
test.plot(ax=ax, label='Test (Actual)', linewidth=1.5, color='green')
arima_forecast.plot(ax=ax, label='ARIMA Forecast', linewidth=2,
                    linestyle='--', color='red')

ax.set_title('ARIMA(1,1,1) Forecast', fontsize=14, fontweight='bold')
ax.set_xlabel('Date', fontsize=12)
ax.set_ylabel('Value', fontsize=12)
ax.legend(fontsize=11)
ax.grid(True, alpha=0.3)
plt.tight_layout()
plt.show()
```



Model 5: SARIMA (Seasonal ARIMA)

Theory

SARIMA(p,d,q)(P,D,Q,s) extends ARIMA to handle seasonality:

Non-seasonal part: (p,d,q) **Seasonal part**: (P,D,Q,s)

- P: Seasonal AR order
- D: Seasonal differencing order
- Q: Seasonal MA order
- s: Seasonal period (e.g., 12 for monthly data with yearly seasonality)

Creating Seasonal Data

The code below **creates fake monthly data that looks like long-term seasonal sales and splits it into training and test sets**.

More specifically:

- It creates **20 years of monthly dates**.
- It builds data with:
  - a **slow upward trend** over time,
  - a **repeating yearly pattern** (seasonality every 12 months),
  - and some **random noise**.
- These components are added together to look like realistic sales data.
- The data is then split into:
  - **80% for training**,
  - **20% for testing**.
- Finally, it prints the size of the full series and each split.

In short:

The code simulates a realistic monthly time series with trend and seasonality and prepares it for seasonal time-series modeling.

In [44]:
```
# Generate monthly data with seasonality
n_months = 240  # 20 years
dates_monthly = pd.date_range(start='2004-01-01', periods=n_months, freq='MS')

# Components
trend = np.linspace(50, 100, n_months)
seasonal = 15 * np.sin(2 * np.pi * np.arange(n_months) / 12)
noise = np.random.normal(0, 3, n_months)

ts_seasonal = pd.Series(trend + seasonal + noise,
                        index=dates_monthly,
                        name='Sales')

# Split
train_seas = ts_seasonal[:int(len(ts_seasonal) * 0.8)]
test_seas = ts_seasonal[int(len(ts_seasonal) * 0.8):]

print(f"Seasonal series shape: {ts_seasonal.shape}")
print(f"Train: {len(train_seas)}, Test: {len(test_seas)}")
```

```
Seasonal series shape: (240,)
Train: 192, Test: 48
```

Fitting SARIMA

The code below **fits a seasonal ARIMA model and uses it to make forecasts**.

More specifically:

- It fits a **SARIMA(1,1,1)(1,1,1,12)** model, which means:
  - one autoregressive term, one differencing, and one moving-average term for the **non-seasonal part**,
  - one autoregressive term, one differencing, and one moving-average term for the **seasonal part**,
  - with a **12-month seasonal cycle**.
- The model is trained on the **seasonal training data**.
- It prints a **summary** showing estimated parameters and diagnostics.
- It then **forecasts future values** for the length of the test period.

In short:

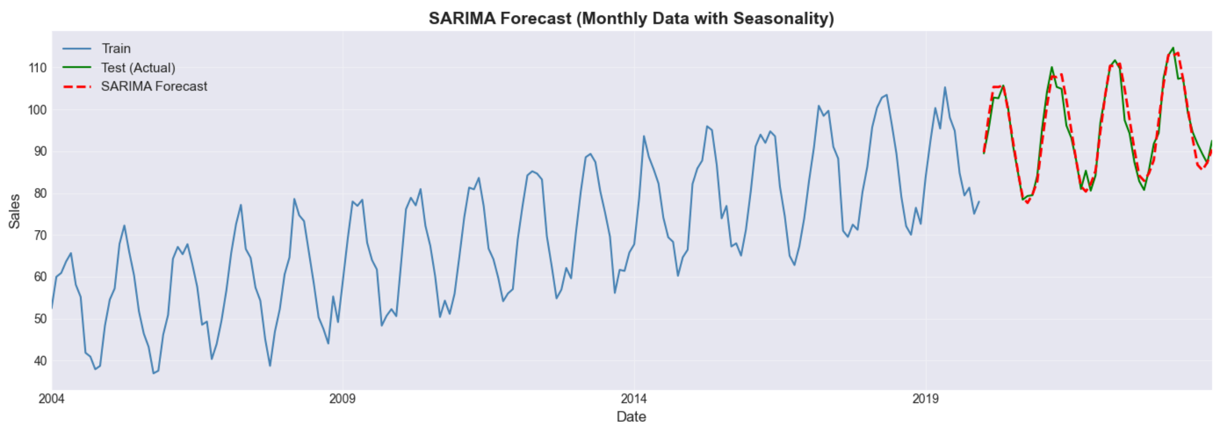
The code models both long-term trends and repeating yearly patterns, then uses that model to predict future seasonal behavior.

In [45]:
```python
# Fit SARIMA(1,1,1)(1,1,1,12)
# Seasonal period = 12 months
sarima_model = SARIMAX(train_seas,
                       order=(1, 1, 1),
                       seasonal_order=(1, 1, 1, 12))
sarima_fit = sarima_model.fit(disp=False)

print("SARIMA(1,1,1)(1,1,1,12) Model Summary:")
print(sarima_fit.summary())

# Forecast
sarima_forecast = sarima_fit.forecast(steps=len(test_seas))
```

```
SARIMA(1,1,1)(1,1,1,12) Model Summary:
                                      SARIMAX Results                                      
====================================================================================
======
Dep. Variable:                             Sales   No. Observations:                
192
Model:             SARIMAX(1, 1, 1)x(1, 1, 1, 12)   Log Likelihood                -4
63.423
Date:                           Tue, 30 Dec 2025   AIC                            9
36.845
Time:                                   08:17:59   BIC                            9
52.782
Sample:                               01-01-2004   HQIC                           9
43.307
                                    - 12-01-2019                                    
Covariance Type:                             opg                                    
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
ar.L1         -0.0128      0.084     -0.151      0.880      -0.178       0.152
ma.L1         -0.9801      0.040    -24.634      0.000      -1.058      -0.902
ar.S.L12      -0.0238      0.104     -0.230      0.818      -0.227       0.179
ma.S.L12      -0.9156      0.140     -6.530      0.000      -1.190      -0.641
sigma2         8.9146      1.336      6.671      0.000       6.295      11.534
===================================================================================
Ljung-Box (L1) (Q):                   0.03   Jarque-Bera (JB):                 1.26
Prob(Q):                              0.87   Prob(JB):                         0.53
Heteroskedasticity (H):               1.30   Skew:                            -0.07
Prob(H) (two-sided):                  0.31   Kurtosis:                         2.61
===================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (complex-ste
p).
```

Visualizing Seasonal Forecast

In [46]:
```
fig, ax = plt.subplots(figsize=(14, 5))

train_seas.plot(ax=ax, label='Train', linewidth=1.5, color='steelblue')
test_seas.plot(ax=ax, label='Test (Actual)', linewidth=1.5, color='green')
sarima_forecast.plot(ax=ax, label='SARIMA Forecast', linewidth=2,
                     linestyle='--', color='red')

ax.set_title('SARIMA Forecast (Monthly Data with Seasonality)',
             fontsize=14, fontweight='bold')
ax.set_xlabel('Date', fontsize=12)
ax.set_ylabel('Sales', fontsize=12)
ax.legend(fontsize=11)
ax.grid(True, alpha=0.3)
plt.tight_layout()
plt.show()
```



6. Model Selection

Information Criteria

AIC (Akaike Information Criterion)

AIC = 2k - 2ln(L)

Where:

- k = number of parameters
- L = maximum likelihood

BIC (Bayesian Information Criterion)

BIC = k·ln(n) - 2ln(L)

Where:

- n = number of observations

Rule: Lower AIC/BIC = better model (balances fit and complexity)

Grid Search for Best Model

In [47]:
```
def evaluate_arima_models(data, p_range, d_range, q_range):
    """
    Evaluate different ARIMA configurations
    """
    results = []

    for p in p_range:
        for d in d_range:
            for q in q_range:
                try:
                    model = ARIMA(data, order=(p, d, q))
                    fitted = model.fit()

                    results.append({
                        'order': (p, d, q),
                        'AIC': fitted.aic,
                        'BIC': fitted.bic
                    })
                except:
                    continue

    return pd.DataFrame(results)

# Search for best model
print("Searching for best ARIMA model...")
results_df = evaluate_arima_models(
    train,
    p_range=range(0, 3),
    d_range=range(0, 2),
    q_range=range(0, 3)
)

# Sort and display
results_df = results_df.sort_values('AIC')
print("\nTop 5 Models by AIC:")
print(results_df.head())

best_order = results_df.iloc[0]['order']
print(f"\nBest Model: ARIMA{best_order}")
```

```
Searching for best ARIMA model...

Top 5 Models by AIC:
        order         AIC         BIC
6   (1, 0, 0)  697.563191  708.005108
13  (2, 0, 1)  698.028252  715.431446
10  (1, 1, 1)  698.670168  709.099558
12  (2, 0, 0)  699.335230  713.257786
7   (1, 0, 1)  699.341125  713.263681

Best Model: ARIMA(1, 0, 0)
```

7. Model Diagnostics

Why Check Diagnostics?

After fitting a model, we need to verify that:

1. **Residuals are white noise** (random, no pattern)
2. **Residuals are normally distributed**
3. **No autocorrelation in residuals**
4. **Model assumptions are satisfied**

Theory: Good Residuals

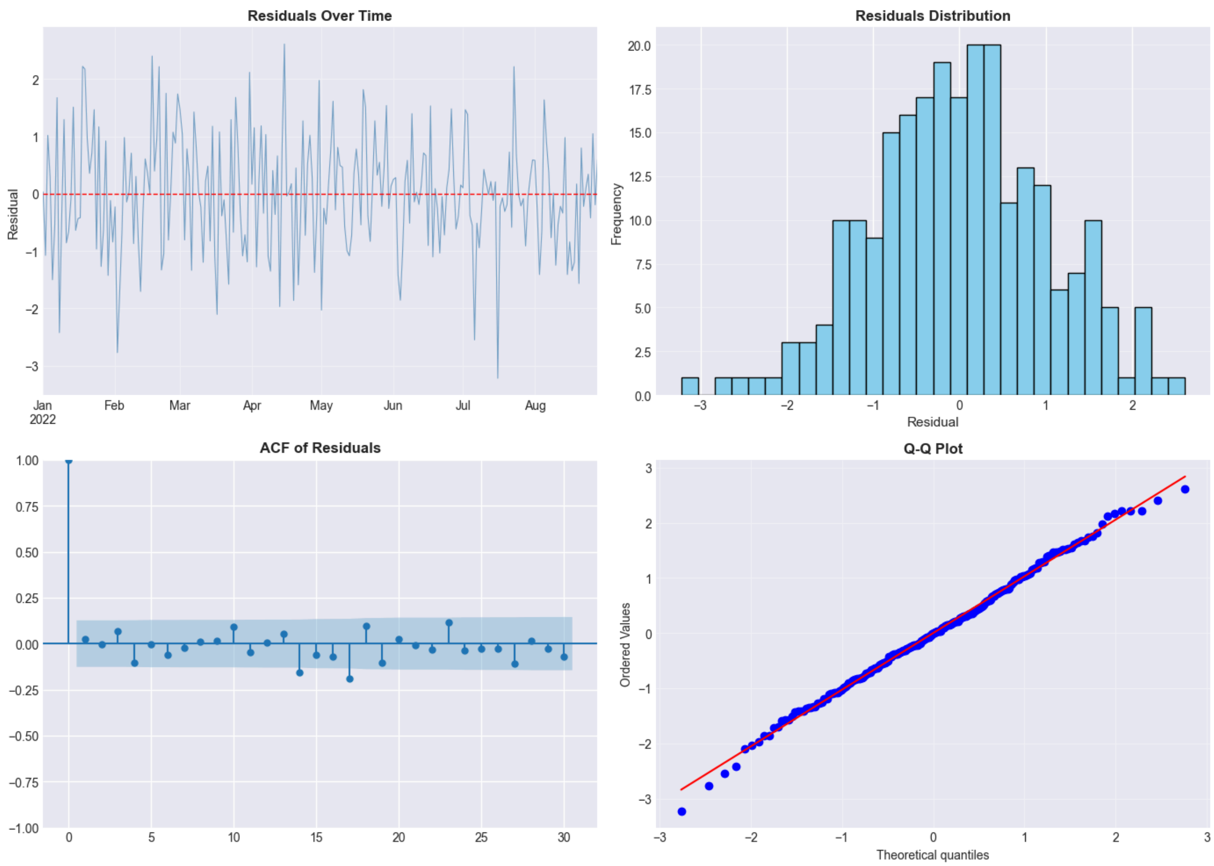
If the model is appropriate, residuals should have:

- Mean ≈ 0
- Constant variance (homoscedastic)
- No autocorrelation
- Normal distribution

Diagnostic Plots

In [48]:
```python
# Get residuals from best model
best_model = ARIMA(train, order=best_order).fit()
residuals = best_model.resid

# Create diagnostic plots
fig, axes = plt.subplots(2, 2, figsize=(14, 10))

# 1. Residuals over time
residuals.plot(ax=axes[0, 0], linewidth=0.8, color='steelblue', alpha=0.7)
axes[0, 0].axhline(y=0, color='red', linestyle='--', linewidth=1)
axes[0, 0].set_title('Residuals Over Time', fontsize=12, fontweight='bold')
axes[0, 0].set_ylabel('Residual', fontsize=11)
axes[0, 0].grid(True, alpha=0.3)
```

```
# 2. Histogram
residuals.hist(ax=axes[0, 1], bins=30, edgecolor='black', color='skyblue')
axes[0, 1].set_title('Residuals Distribution', fontsize=12, fontweight='bold')
axes[0, 1].set_xlabel('Residual', fontsize=11)
axes[0, 1].set_ylabel('Frequency', fontsize=11)
axes[0, 1].grid(True, alpha=0.3, axis='y')

# 3. ACF of residuals
plot_acf(residuals, lags=30, ax=axes[1, 0])
axes[1, 0].set_title('ACF of Residuals', fontsize=12, fontweight='bold')

# 4. Q-Q plot
stats.probplot(residuals, dist="norm", plot=axes[1, 1])
axes[1, 1].set_title('Q-Q Plot', fontsize=12, fontweight='bold')
axes[1, 1].grid(True, alpha=0.3)

plt.tight_layout()
plt.show()
```



Ljung-Box Test

Tests the null hypothesis that residuals are independently distributed (no autocorrelation).

- **H_0**: Residuals are white noise (no autocorrelation)
- **H_1**: Residuals have autocorrelation

Decision: p-value > 0.05 → residuals are white noise (good!)

In [49]:
```python
# Perform Ljung-Box test
lb_test = acorr_ljungbox(residuals, lags=[10, 20, 30], return_df=True)

print("\nLjung-Box Test Results:")
print(lb_test)
print("\nInterpretation:")
print("If p-value > 0.05: Residuals are white noise ✓")
print("If p-value < 0.05: Residuals have autocorrelation ✗")
```

```
Ljung-Box Test Results:
      lb_stat  lb_pvalue
10   7.147022   0.711496
20  31.838671   0.045048
30  41.570079   0.077860

Interpretation:
If p-value > 0.05: Residuals are white noise ✓
If p-value < 0.05: Residuals have autocorrelation ✗
```

Residual Statistics

In [50]:
```python
print("\nResidual Summary Statistics:")
print("="*50)
print(f"Mean:      {residuals.mean():>10.6f}  (should be ≈ 0)")
print(f"Std Dev:   {residuals.std():>10.6f}")
print(f"Min:       {residuals.min():>10.6f}")
print(f"Max:       {residuals.max():>10.6f}")
print(f"Skewness:  {residuals.skew():>10.6f}  (should be ≈ 0)")
print(f"Kurtosis:  {residuals.kurtosis():>10.6f}  (should be ≈ 0)")
```

```
Residual Summary Statistics:
==================================================
Mean:       -0.000090  (should be ≈ 0)
Std Dev:     1.022687
Min:        -3.222888
Max:         2.614799
Skewness:   -0.045184  (should be ≈ 0)
Kurtosis:    0.071627  (should be ≈ 0)
```

8. Model Evaluation

Common Forecasting Metrics

1. MAE (Mean Absolute Error)

MAE = (1/n) Σ|y_i - $\hat{y}_i$|

- Average absolute difference
- Same units as original data

- Easy to interpret

2. RMSE (Root Mean Squared Error)

RMSE = $\sqrt{[(1/n)\ \Sigma(y_i - \hat{y}_i)^2]}$

- Penalizes large errors more
- Same units as original data
- More sensitive to outliers than MAE

3. MAPE (Mean Absolute Percentage Error)

MAPE = $(100/n)\ \Sigma|((y_i - \hat{y}_i)/y_i)|$

- Expressed as percentage
- Scale-independent
- Undefined when $y_i = 0$

Computing Metrics

In [51]:
```python
def calculate_metrics(actual, predicted):
    """Calculate forecasting metrics"""
    mae = mean_absolute_error(actual, predicted)
    rmse = np.sqrt(mean_squared_error(actual, predicted))
    mape = np.mean(np.abs((actual - predicted) / actual)) * 100

    return {'MAE': mae, 'RMSE': rmse, 'MAPE': mape}

# Evaluate forecast
forecast = best_model.forecast(steps=len(test))
metrics = calculate_metrics(test, forecast)

print("\nForecast Evaluation Metrics:")
print("="*50)
print(f"MAE:  {metrics['MAE']:.4f}")
print(f"RMSE: {metrics['RMSE']:.4f}")
print(f"MAPE: {metrics['MAPE']:.2f}%")
```

```
Forecast Evaluation Metrics:
==================================================
MAE:  1.2387
RMSE: 1.5522
MAPE: 102.35%
```

Summary and Key Takeaways

Workflow for Time Series Modeling

1. **Explore Data**
   - Plot the series
   - Identify trend, seasonality, noise
   - Perform decomposition
2. **Check Stationarity**
   - Use ADF test
   - Apply differencing if needed
   - Verify stationarity after transformation
3. **Identify Model Orders**
   - Examine ACF plot → suggests MA order (q)
   - Examine PACF plot → suggests AR order (p)
   - Consider seasonal patterns
4. **Fit Candidate Models**
   - Start with simple models (AR, MA)
   - Try ARIMA for non-stationary data
   - Use SARIMA for seasonal data
5. **Select Best Model**
   - Compare AIC/BIC values
   - Lower is better
   - Balance complexity and fit
6. **Validate Model**
   - Check residual plots
   - Perform Ljung-Box test
   - Ensure residuals are white noise
7. **Forecast and Evaluate**
   - Generate predictions
   - Calculate MAE, RMSE, MAPE
   - Compare with test data

Model Quick Reference

If you see...	Consider...
Clear trend	Differencing (d > 0) or ARIMA
Seasonality	SARIMA with appropriate period
PACF cuts off at lag p	AR(p)
ACF cuts off at lag q	MA(q)

If you see...	Consider...
Both decay slowly	ARMA or ARIMA
Non-constant variance	Log transformation or multiplicative model

Python Libraries

- **pandas**: Data manipulation and time series structures
- **statsmodels**: ARIMA, SARIMA, statistical tests, diagnostics
- **matplotlib**: Visualization
- **scipy**: Statistical functions
- **sklearn**: Evaluation metrics

Next Steps

In **Part 2**, we'll cover:

- Advanced pandas techniques for time series
- Resampling and frequency conversion
- Rolling windows and moving averages
- Time-based indexing and slicing
- Practical data preparation workflows

In **Part 3**, we'll explore:

- Feature engineering for ML models
- Machine learning approaches (XGBoost, Random Forest)
- Deep learning (LSTM, GRU)
- Prophet and modern forecasting tools

End of Part 1