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Part 1: Fundamentals & Classical Statistical Methods

Time Series Analysis in Python

Setup and Imports

Before we begin, install the required packages:

```
pip install pandas numpy matplotlib statsmodels scipy scikit-learn
```

In [2]:
```python
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from statsmodels.tsa.seasonal import seasonal_decompose
from statsmodels.graphics.tsaplots import plot_acf, plot_pacf
from statsmodels.tsa.stattools import adfuller, kpss
from statsmodels.tsa.arima.model import ARIMA
from statsmodels.tsa.statespace.sarimax import SARIMAX
from statsmodels.stats.diagnostic import acorr_ljungbox
from scipy import stats
from sklearn.metrics import mean_absolute_error, mean_squared_error
import warnings
warnings.filterwarnings('ignore')

plt.style.use('seaborn-v0_8-darkgrid')
```

1. What Is a Time Series?

A **time series** is a sequence of observations collected **over time**, usually at regular intervals.

Examples:

- Daily stock prices
- Hourly electricity consumption
- Monthly sales revenue
- Sensor measurements every second

Why Time Series Are Different

- Observations are **dependent**

- The order of data points **cannot be shuffled**
- Past values influence future values

This dependency structure is the core challenge of time series analysis.

Key Components

Time series typically contain four components:

1. **Trend (T)**: Long-term increase or decrease in the data
2. **Seasonality (S)**: Regular, predictable patterns that repeat over fixed periods (e.g., yearly, monthly)
3. **Cyclicality (C)**: Patterns that repeat but not at fixed intervals (e.g., economic cycles)
4. **Noise/Irregular (I)**: Random variation that cannot be attributed to trend, seasonality, or cyclicality

Mathematical Representation

- **Additive Model**: $Y(t) = T(t) + S(t) + C(t) + I(t)$
  - Use when seasonal variation is roughly constant over time
- **Multiplicative Model**: $Y(t) = T(t) \times S(t) \times C(t) \times I(t)$
  - Use when seasonal variation increases with the level of the series

Creating a Sample Time Series

Let's create a synthetic time series that contains trend, seasonality, and noise:

In [3]:
```
# Set random seed for reproducibility
np.random.seed(42)

# Generate date range
date_range = pd.date_range(start='2020-01-01', end='2023-12-31', freq='D')
n = len(date_range)

# Components
trend = np.linspace(100, 150, n)  # Linear trend from 100 to 150
seasonality = 10 * np.sin(2 * np.pi * np.arange(n) / 365.25)  # Yearly seasonality
noise = np.random.normal(0, 5, n)  # Random noise

# Combine components (additive model)
ts_data = trend + seasonality + noise

# Create pandas Series with datetime index
ts = pd.Series(ts_data, index=date_range, name='Value')

print("Sample Time Series:")
```

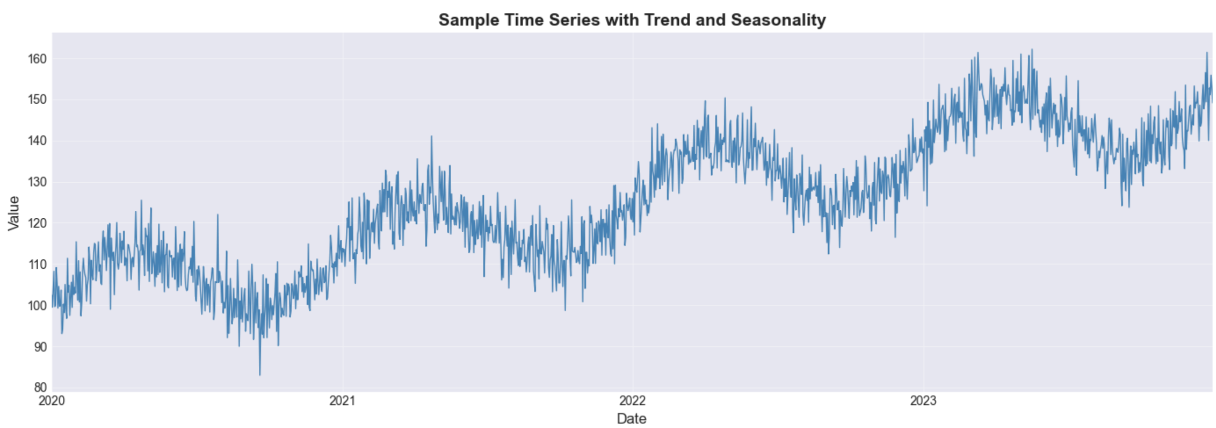
```python
print(ts.head(10))
print(f"\nShape: {ts.shape}")
print(f"Period: {ts.index.min()} to {ts.index.max()}")
```

```
Sample Time Series:
2020-01-01    102.483571
2020-01-02     99.514941
2020-01-03    103.650916
2020-01-04    108.233733
2020-01-05     99.653774
2020-01-06     99.859609
2020-01-07    109.131857
2020-01-08    105.278161
2020-01-09     99.298455
2020-01-10    104.563060
Freq: D, Name: Value, dtype: float64

Shape: (1461,)
Period: 2020-01-01 00:00:00 to 2023-12-31 00:00:00
```

Visualizing the Time Series

Let's create a synthetic time series that contains trend, seasonality, and noise:

In [4]:
```python
fig, ax = plt.subplots(figsize=(14, 5))
ts.plot(ax=ax, linewidth=1, color='steelblue')
ax.set_title('Sample Time Series with Trend and Seasonality', fontsize=14, fontweig
ax.set_xlabel('Date', fontsize=12)
ax.set_ylabel('Value', fontsize=12)
ax.grid(True, alpha=0.3)
plt.tight_layout()
plt.show()
```



2. Time Series Decomposition

Theory

Decomposition is the process of separating a time series into its constituent components. This helps us:

- Understand the underlying patterns
- Remove seasonality for better modeling
- Identify anomalies
- Choose appropriate forecasting methods

Classical Decomposition Methods

The most common method is **moving average decomposition**:

1. Estimate trend using moving average
2. Detrend the series
3. Estimate seasonal component by averaging detrended values for each season
4. Calculate residuals: Residual = Original - Trend - Seasonal

Performing Decomposition

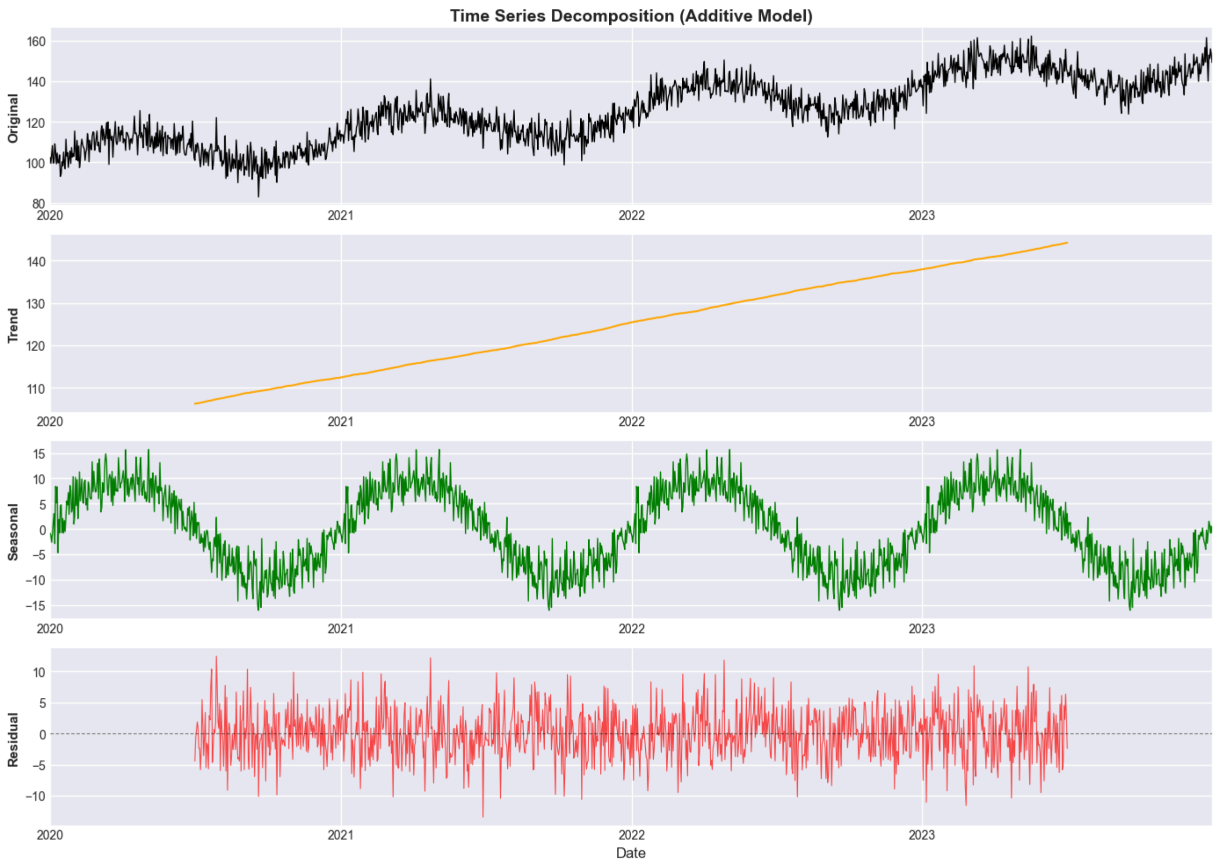
In [5]:
```
# Perform additive decomposition
# Period = 365 because we have daily data with yearly seasonality
decomposition = seasonal_decompose(ts, model='additive', period=365)

# Extract components
trend_component = decomposition.trend
seasonal_component = decomposition.seasonal
residual_component = decomposition.resid

print("Decomposition Components:")
print(f"Trend:    {trend_component.shape}")
print(f"Seasonal: {seasonal_component.shape}")
print(f"Residual: {residual_component.shape}")
```

```
Decomposition Components:
Trend:    (1461,)
Seasonal: (1461,)
Residual: (1461,)
```

Visualizing Decomposition

In [6]:
```
fig, axes = plt.subplots(4, 1, figsize=(14, 10))

# Original
ts.plot(ax=axes[0], linewidth=1, color='black')
axes[0].set_ylabel('Original', fontsize=11, fontweight='bold')
axes[0].set_title('Time Series Decomposition (Additive Model)', fontsize=14, fontwe

# Trend
```

```python
trend_component.plot(ax=axes[1], linewidth=1.5, color='orange')
axes[1].set_ylabel('Trend', fontsize=11, fontweight='bold')

# Seasonal
seasonal_component.plot(ax=axes[2], linewidth=1, color='green')
axes[2].set_ylabel('Seasonal', fontsize=11, fontweight='bold')

# Residual
residual_component.plot(ax=axes[3], linewidth=0.8, color='red', alpha=0.7)
axes[3].set_ylabel('Residual', fontsize=11, fontweight='bold')
axes[3].set_xlabel('Date', fontsize=12)
axes[3].axhline(y=0, color='black', linestyle='--', linewidth=0.8, alpha=0.5)

plt.tight_layout()
plt.show()
```



3. Stationarity

What is Stationarity?

A time series is **stationary** if its statistical properties (mean, variance, autocorrelation) do not change over time.

Why Does Stationarity Matter?

Most classical time series models (ARIMA, SARIMA) assume stationarity because:

- Statistical properties are easier to model when constant
- Predictions are more reliable
- Mathematical theory is simpler

Types of Stationarity

1. **Strict Stationarity**: Joint distribution is time-invariant (very restrictive)
2. **Weak/Covariance Stationarity**: Only mean, variance, and autocorrelation are constant (commonly used)

Common Non-Stationary Patterns

- **Trend**: Mean changes over time
- **Seasonality**: Pattern repeats at regular intervals
- **Heteroscedasticity**: Variance changes over time

Augmented Dickey-Fuller (ADF) Test

The ADF test checks the null hypothesis that the series has a unit root (non-stationary).

- **H_0**: Series has a unit root (non-stationary)
- **H_1**: Series is stationary

Decision Rule: If p-value < 0.05, reject H_0 (series is stationary)

In [7]:
```
def check_stationarity(timeseries, name='Series'):
    """
    Perform Augmented Dickey-Fuller test
    """
    print(f"\n{'='*60}")
    print(f"Stationarity Test: {name}")
    print('='*60)

    # Remove NaN values
    ts_clean = timeseries.dropna()

    # Perform ADF test
    result = adfuller(ts_clean, autolag='AIC')

    print(f'ADF Statistic:     {result[0]:.6f}')
    print(f'p-value:           {result[1]:.6f}')
    print(f'# Lags Used:       {result[2]}')
    print(f'# Observations:    {result[3]}')
    print('\nCritical Values:')
    for key, value in result[4].items():
        print(f'  {key}: {value:.3f}')
```

```python
    # Interpretation
    print('\n' + '-'*60)
    if result[1] <= 0.05:
        print(f"✓ STATIONARY (p={result[1]:.4f} < 0.05)")
        print("  → Reject null hypothesis")
    else:
        print(f"✗ NON-STATIONARY (p={result[1]:.4f} > 0.05)")
        print("  → Fail to reject null hypothesis")
    print('-'*60)

    return result[1] <= 0.05
```

Testing Our Series

In [8]:
```python
# Test original series
is_stationary = check_stationarity(ts, 'Original Time Series')
```

```
============================================================
Stationarity Test: Original Time Series
============================================================
ADF Statistic:     -0.835398
p-value:           0.808502
# Lags Used:       12
# Observations:    1448

Critical Values:
  1%: -3.435
  5%: -2.864
  10%: -2.568

------------------------------------------------------------
✗ NON-STATIONARY (p=0.8085 > 0.05)
  → Fail to reject null hypothesis
------------------------------------------------------------
```

Making a Series Stationary

Method 1: Differencing

Differencing removes trends and can stabilize the mean:

First Difference: Y'(t) = Y(t) - Y(t-1)

In [9]:
```python
# Apply first differencing
ts_diff = ts.diff().dropna()

# Test differenced series
is_stationary_diff = check_stationarity(ts_diff, 'First Differenced Series')
```

```
============================================================
Stationarity Test: First Differenced Series
============================================================
ADF Statistic:      -17.823692
p-value:            0.000000
# Lags Used:        11
# Observations:     1448

Critical Values:
  1%: -3.435
  5%: -2.864
  10%: -2.568

------------------------------------------------------------
✓ STATIONARY (p=0.0000 < 0.05)
  → Reject null hypothesis
------------------------------------------------------------
```

Visualizing the Transformation

In [10]:

```
fig, axes = plt.subplots(2, 1, figsize=(14, 8), sharex=True)

# Original
ts.plot(ax=axes[0], linewidth=1.2, color='steelblue')
axes[0].set_title('Original Series (Non-Stationary)', fontsize=12, fontweight='bold
axes[0].set_ylabel('Value', fontsize=11)
axes[0].grid(True, alpha=0.3)

# Differenced
ts_diff.plot(ax=axes[1], linewidth=1, color='orange')
axes[1].set_title('First Differenced Series (Stationary)', fontsize=12, fontweight=
axes[1].set_ylabel('Differenced Value', fontsize=11)
axes[1].axhline(y=0, color='red', linestyle='--', linewidth=1, alpha=0.7)
axes[1].set_xlabel('Date', fontsize=12)
axes[1].grid(True, alpha=0.3)

plt.tight_layout()
plt.show()
```

4. Autocorrelation Analysis

ACF: Autocorrelation Function

The ACF measures the correlation between observations at different time lags:

$ACF(k) = Corr(Y_t, Y_{t-k})$

- Lag 1: correlation between consecutive observations
- Lag 2: correlation between observations 2 time periods apart
- And so on...

PACF: Partial Autocorrelation Function

The PACF measures the correlation between Y_t and Y_{t-k} after removing the effect of intermediate lags.

Why Are ACF and PACF Important?

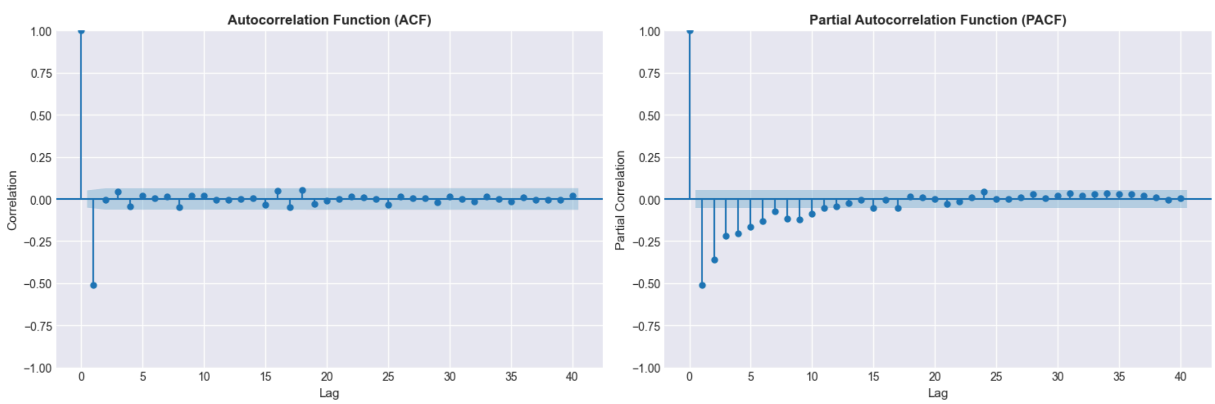
They help identify the order of AR and MA components:

Pattern	Model Suggestion
PACF cuts off after lag p, ACF decays	AR(p)

Pattern	Model Suggestion
ACF cuts off after lag q, PACF decays	MA(q)
Both decay gradually	ARMA(p,q)

Plotting ACF and PACF

In [11]:
```
# Use differenced series (should be stationary)
fig, axes = plt.subplots(1, 2, figsize=(15, 5))

# ACF
plot_acf(ts_diff, lags=40, ax=axes[0])
axes[0].set_title('Autocorrelation Function (ACF)', fontsize=12, fontweight='bold')
axes[0].set_xlabel('Lag', fontsize=11)
axes[0].set_ylabel('Correlation', fontsize=11)

# PACF
plot_pacf(ts_diff, lags=40, ax=axes[1])
axes[1].set_title('Partial Autocorrelation Function (PACF)', fontsize=12, fontweigh
axes[1].set_xlabel('Lag', fontsize=11)
axes[1].set_ylabel('Partial Correlation', fontsize=11)

plt.tight_layout()
plt.show()
```



Interpretation:

- Blue shaded area represents the confidence interval
- Bars outside this area are statistically significant
- Look for where bars drop inside the confidence interval (cutoff point)

5. Classical Time Series Models

Model	Full Name	Equation	When to Use
AR(p)	Autoregressive	$Y_t = c + \varphi_1 Y_{t-1} + \ldots + \varphi_p Y_{t-p} + \varepsilon_t$	PACF cuts off

Model	Full Name	Equation	When to Use
MA(q)	Moving Average	$Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + ... + \theta_\varphi\varepsilon_{t-\varphi}$	ACF cuts off
ARMA(p,q)	AR + MA	Combines both	Both ACF & PACF decay
ARIMA(p,d,q)	Integrated ARMA	ARMA on d-times differenced data	Non-stationary series
SARIMA(p,d,q)(P,D,Q,s)	Seasonal ARIMA	Adds seasonal components	Seasonal patterns

Preparing Data for Modeling

Let's create a simpler dataset to demonstrate the models:

In [12]:
```
# Generate AR(1) process
np.random.seed(123)
n = 300
dates = pd.date_range(start='2022-01-01', periods=n, freq='D')

# AR(1): Y(t) = 0.7 * Y(t-1) + noise
ar_coef = 0.7
ar_data = [0]
for i in range(1, n):
    ar_data.append(ar_coef * ar_data[i-1] + np.random.normal(0, 1))

ts_simple = pd.Series(ar_data, index=dates, name='Value')

# Train-test split (80-20)
train_size = int(len(ts_simple) * 0.8)
train = ts_simple[:train_size]
test = ts_simple[train_size:]

print(f"Train: {len(train)} observations")
print(f"Test:  {len(test)} observations")
```

```
Train: 240 observations
Test:  60 observations
```

Model 1: AR (Autoregressive)

Theory

An AR(p) model predicts the current value using p past values:

$$\mathbf{Y_t = c + \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + ... + \varphi_p Y_{t-p} + \varepsilon_t}$$

Where:

- c is a constant

- $\varphi_1, \varphi_2, \ldots, \varphi_p$ are coefficients
- ε_t is white noise error

Implementation

In [13]:
```
# Fit AR(1) model
# ARIMA(p,d,q) with p=1, d=0, q=0
ar_model = ARIMA(train, order=(1, 0, 0))
ar_fit = ar_model.fit()

print("AR(1) Model Summary:")
print(ar_fit.summary())
print(f"\nEstimated coefficient: {ar_fit.params[1]:.4f}")
print(f"True coefficient: {ar_coef}")
```

```
AR(1) Model Summary:
                               SARIMAX Results
==============================================================================
Dep. Variable:                  Value   No. Observations:                  240
Model:                 ARIMA(1, 0, 0)   Log Likelihood                -345.782
Date:                Tue, 23 Dec 2025   AIC                            697.563
Time:                        17:33:48   BIC                            708.005
Sample:                    01-01-2022   HQIC                           701.771
                         - 08-28-2022
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.0308      0.229     -0.135      0.893      -0.479       0.418
ar.L1          0.7119      0.047     15.074      0.000       0.619       0.805
sigma2         1.0415      0.094     11.072      0.000       0.857       1.226
===================================================================================
Ljung-Box (L1) (Q):                   0.15   Jarque-Bera (JB):                 0.10
Prob(Q):                              0.70   Prob(JB):                         0.95
Heteroskedasticity (H):               0.60   Skew:                            -0.04
Prob(H) (two-sided):                  0.03   Kurtosis:                         3.05
===================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (complex-ste
p).

Estimated coefficient: 0.7119
True coefficient: 0.7
```

Interpreting AR(1) Model Output - OPTIONAL

What this really is (in one sentence)

AR(1) is just linear regression where the predictor is yesterday's value.

Connect it to regular regression (intuition first)

In ordinary regression you write:

> $y = \beta_0 + \beta_1 x_1 + \text{error}$

In AR(1), you write:

> $y_t = \text{constant} + \beta_1 \times y_{t-1} + \text{error}$

Same idea. The "feature" is just the lagged value of y itself.

Translate the output into regression language

1. The coefficients

```
const     = -0.0308
ar.L1     =  0.7119
sigma2    =  1.0415
```

Think of this as a regression table:

Feature	Coefficient	What it means
Intercept	-0.031	Baseline (when $y_{t-1} = 0$)
y_{t-1}	0.7119	Today = 71% of yesterday + noise
Error variance	1.042	Residual spread

The prediction equation is:

> $y_t = -0.031 + 0.7119 \times y_{t-1} + \varepsilon_t$

This is identical to:

> sales_today = -0.031 + 0.7119 × sales_yesterday + noise

2. Standard errors and p-values

Exactly the same as regression:

```
              coef    std err    p-value
const        -0.031   0.229      0.893
ar.L1         0.712   0.047      0.000
```

const: p = 0.893 (high)

Not significant. Could drop it.

ar.L1: p = 0.000 (very low)

Highly significant. Yesterday strongly predicts today.

Same interpretation as:

> feature_1: p = 0.893 → not useful feature_2: p = 0.000 → very useful

The model found **strong autocorrelation** in the data.

3. Model quality metrics

```
AIC = 697.6
BIC = 708.0
Log Likelihood = -345.8
```

These are **goodness-of-fit measures**, not accuracy metrics.

AIC and BIC: Lower is better

Use these to compare models:

- AR(1): AIC = 697.6
- AR(2): AIC = 695.3 → better
- MA(1): AIC = 701.2 → worse

Think of AIC like:

> penalized training error

It balances fit with model complexity.

Log Likelihood: Higher is better

Similar to minimizing loss in ML. Maximum likelihood estimation finds coefficients that make the observed data most probable.

4. Diagnostic tests (are residuals well-behaved?)

```
Ljung-Box (Q):           0.15    p = 0.70
Jarque-Bera (JB):        0.10    p = 0.95
Heteroskedasticity (H):  0.60    p = 0.03
```

Think of these as **residual plots in test form**.

Ljung-Box test

Question: "Do residuals have patterns left?"

- $p > 0.05$ → No patterns (good)
- $p < 0.05$ → Patterns remain (bad)

Here: $p = 0.70$ → residuals look random

This is like checking:

> plot(residuals) shows no trend

Jarque-Bera test

Question: "Are residuals normally distributed?"

- $p > 0.05$ → Yes (good)
- $p < 0.05$ → No (may need transformation)

Here: $p = 0.95$ → very normal

This is like checking:

> histogram(residuals) looks bell-shaped

Heteroskedasticity test

Question: "Is variance constant over time?"

- $p > 0.05$ → Yes (good)
- $p < 0.05$ → No (variance changes)

Here: $p = 0.03$ → some heteroskedasticity detected

This is like seeing:

> residual spread increases over time

5. Model validation

```
Estimated coefficient: 0.7119
True coefficient:      0.7000
```

The model recovered the true data-generating process.

In ML terms:

> The model learned the correct function

This would be like:

- You generate data: $y = 2x + \text{noise}$
- Your model learns: $y = 1.98x$
- Close match → model works

Why it feels alien

1. No feature matrix visible

The "feature" (y_{t-1}) is created internally from the time series.

2. Maximum likelihood instead of MSE

Same goal (fit the data), different math. Likelihood is more general than squared error.

3. Heavy focus on diagnostics

Econometrics cares deeply about **why** the model works, not just **that** it works.

Statistical inference > predictive accuracy

4. Different vocabulary

- ML says: "$R^2 = 0.85$"
- Econometrics says: "Log Likelihood = -345, AIC = 697"

Same information, different packaging.

How to read this output (decision rules)

Check coefficients:

- $p < 0.05$ → significant → keep
- $p > 0.05$ → not significant → consider dropping

Check diagnostics:

- Ljung-Box $p > 0.05$ → good (no autocorrelation left)
- Ljung-Box $p < 0.05$ → bad (model incomplete)

Compare models:

- Lower AIC → better model
- Lower BIC → better model (penalizes complexity more)

Validate:

- Residuals should look random
- Residuals should be roughly normal
- Variance should be constant (ideally)

One grounding sentence you can remember

> AR(1) is linear regression where X is yesterday's y, and the output tells you both fit quality and residual behavior in statistical language instead of ML metrics.

Quick interpretation of this specific output

Model found: $y_t = 0.71 \times y_{t-1}$ + noise

Constant term: Not significant ($p = 0.89$) → probably zero in reality

Autocorrelation: Strong (coefficient = 0.71, $p < 0.001$) → yesterday matters a lot

Residuals: Clean (Ljung-Box $p = 0.70$, JB $p = 0.95$) → no patterns left

Minor issue: Slight heteroskedasticity ($p = 0.03$) → variance not perfectly constant

Conclusion: This is a good model. The AR(1) structure correctly captures the data-generating process.

Model 2: MA (Moving Average)

Theory

An MA(q) model predicts the current value using past forecast errors:

$$\mathbf{Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \ldots + \theta_\varphi\varepsilon_{t-\varphi}}$$

Where:

- μ is the mean
- $\theta_1, \theta_2, \ldots, \theta_\varphi$ are coefficients
- $\varepsilon_t, \varepsilon_{t-1}, \ldots$ are error terms

Implementation

In [14]:
```
# Fit MA(1) model
ma_model = ARIMA(train, order=(0, 0, 1))
ma_fit = ma_model.fit()

print("MA(1) Model Summary:")
print(ma_fit.summary())
```

```
MA(1) Model Summary:
                               SARIMAX Results
==============================================================================
Dep. Variable:                  Value   No. Observations:                  240
Model:                 ARIMA(0, 0, 1)   Log Likelihood                -370.431
Date:                Tue, 23 Dec 2025   AIC                            746.863
Time:                        17:33:48   BIC                            757.305
Sample:                    01-01-2022   HQIC                           751.070
                         - 08-28-2022
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.0349      0.120     -0.290      0.771      -0.270       0.200
ma.L1          0.6287      0.051     12.246      0.000       0.528       0.729
sigma2         1.2801      0.119     10.766      0.000       1.047       1.513
===================================================================================
Ljung-Box (L1) (Q):                  10.99   Jarque-Bera (JB):                 0.40
Prob(Q):                              0.00   Prob(JB):                         0.82
Heteroskedasticity (H):               0.60   Skew:                            -0.10
Prob(H) (two-sided):                  0.03   Kurtosis:                         2.94
===================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (complex-ste
p).
```

Model 3: ARMA

Theory

ARMA(p,q) combines AR(p) and MA(q):

$$\mathbf{Y_t = c + \varphi_1 Y_{t-1} + \ldots + \varphi_p Y_{t-p} + \theta_1 \varepsilon_{t-1} + \ldots + \theta_\varphi \varepsilon_{t-\varphi} + \varepsilon_t}$$

Important: ARMA requires stationary data.

Implementation

In [15]:
```python
# Fit ARMA(1,1) model
arma_model = ARIMA(train, order=(1, 0, 1))
arma_fit = arma_model.fit()

print("ARMA(1,1) Model Summary:")
print(arma_fit.summary())
```

```
ARMA(1,1) Model Summary:
                               SARIMAX Results
==============================================================================
Dep. Variable:                  Value   No. Observations:                  240
Model:                 ARIMA(1, 0, 1)   Log Likelihood                -345.671
Date:                Tue, 23 Dec 2025   AIC                            699.341
Time:                        17:33:49   BIC                            713.264
Sample:                    01-01-2022   HQIC                           704.951
                         - 08-28-2022
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.0308      0.223     -0.138      0.890      -0.468       0.407
ar.L1          0.6912      0.072      9.667      0.000       0.551       0.831
ma.L1          0.0424      0.096      0.441      0.659      -0.146       0.231
sigma2         1.0405      0.094     11.083      0.000       0.857       1.225
===================================================================================
Ljung-Box (L1) (Q):                   0.00   Jarque-Bera (JB):                 0.13
Prob(Q):                              0.96   Prob(JB):                         0.94
Heteroskedasticity (H):               0.60   Skew:                            -0.05
Prob(H) (two-sided):                  0.02   Kurtosis:                         3.05
===================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (complex-ste
p).
```

Model 4: ARIMA

Theory

ARIMA(p,d,q) is ARMA applied to d-times differenced data:

- **p**: Order of autoregressive part
- **d**: Degree of differencing
- **q**: Order of moving average part

Steps:

1. Difference the series d times to make it stationary
2. Apply ARMA(p,q) to the differenced series

Implementation

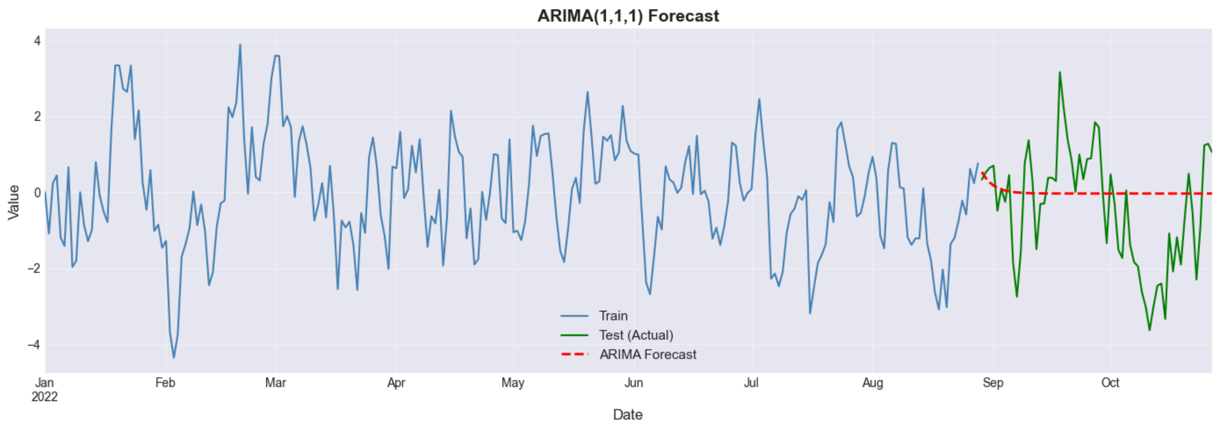
In [16]:
```python
# Fit ARIMA(1,1,1)
arima_model = ARIMA(train, order=(1, 1, 1))
arima_fit = arima_model.fit()

print("ARIMA(1,1,1) Model Summary:")
print(arima_fit.summary())
```

```python
print(f"\nAIC: {arima_fit.aic:.2f}")
print(f"BIC: {arima_fit.bic:.2f}")

# Forecast
forecast_steps = len(test)
arima_forecast = arima_fit.forecast(steps=forecast_steps)
```

```
ARIMA(1,1,1) Model Summary:
                               SARIMAX Results
==============================================================================
Dep. Variable:                  Value   No. Observations:                  240
Model:                 ARIMA(1, 1, 1)   Log Likelihood                -346.335
Date:                Tue, 23 Dec 2025   AIC                            698.670
Time:                        17:33:49   BIC                            709.100
Sample:                    01-01-2022   HQIC                           702.873
                         - 08-28-2022
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
ar.L1          0.7190      0.052     13.919      0.000       0.618       0.820
ma.L1         -0.9999      2.479     -0.403      0.687      -5.860       3.860
sigma2         1.0460      2.583      0.405      0.685      -4.016       6.108
===================================================================================
Ljung-Box (L1) (Q):                   0.08   Jarque-Bera (JB):                 0.08
Prob(Q):                              0.78   Prob(JB):                         0.96
Heteroskedasticity (H):               0.59   Skew:                            -0.03
Prob(H) (two-sided):                  0.02   Kurtosis:                         3.06
===================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (complex-ste
p).

AIC: 698.67
BIC: 709.10
```

Visualizing Forecast

In [17]:
```python
fig, ax = plt.subplots(figsize=(14, 5))

train.plot(ax=ax, label='Train', linewidth=1.5, color='steelblue')
test.plot(ax=ax, label='Test (Actual)', linewidth=1.5, color='green')
arima_forecast.plot(ax=ax, label='ARIMA Forecast', linewidth=2,
                    linestyle='--', color='red')

ax.set_title('ARIMA(1,1,1) Forecast', fontsize=14, fontweight='bold')
ax.set_xlabel('Date', fontsize=12)
ax.set_ylabel('Value', fontsize=12)
ax.legend(fontsize=11)
ax.grid(True, alpha=0.3)
plt.tight_layout()
plt.show()
```



Model 5: SARIMA (Seasonal ARIMA)

Theory

SARIMA(p,d,q)(P,D,Q,s) extends ARIMA to handle seasonality:

Non-seasonal part: (p,d,q) **Seasonal part**: (P,D,Q,s)

- P: Seasonal AR order
- D: Seasonal differencing order
- Q: Seasonal MA order
- s: Seasonal period (e.g., 12 for monthly data with yearly seasonality)

Creating Seasonal Data

In [18]:
```
# Generate monthly data with seasonality
n_months = 240  # 20 years
dates_monthly = pd.date_range(start='2004-01-01', periods=n_months, freq='MS')

# Components
trend = np.linspace(50, 100, n_months)
seasonal = 15 * np.sin(2 * np.pi * np.arange(n_months) / 12)
noise = np.random.normal(0, 3, n_months)

ts_seasonal = pd.Series(trend + seasonal + noise,
                        index=dates_monthly,
                        name='Sales')

# Split
train_seas = ts_seasonal[:int(len(ts_seasonal) * 0.8)]
test_seas = ts_seasonal[int(len(ts_seasonal) * 0.8):]

print(f"Seasonal series shape: {ts_seasonal.shape}")
print(f"Train: {len(train_seas)}, Test: {len(test_seas)}")
```

```
Seasonal series shape: (240,)
Train: 192, Test: 48
```

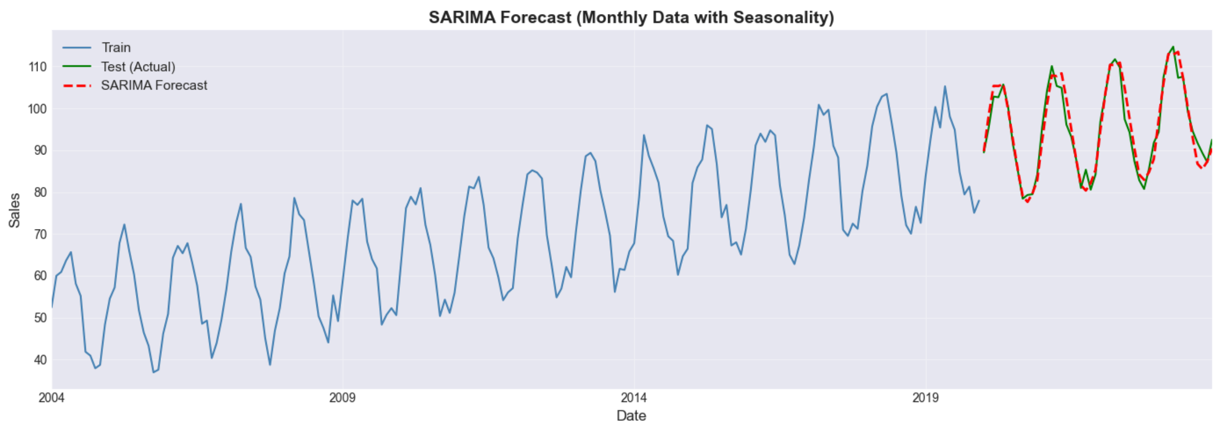
Fitting SARIMA

In [19]:
```
# Fit SARIMA(1,1,1)(1,1,1,12)
# Seasonal period = 12 months
sarima_model = SARIMAX(train_seas,
                       order=(1, 1, 1),
                       seasonal_order=(1, 1, 1, 12))
sarima_fit = sarima_model.fit(disp=False)

print("SARIMA(1,1,1)(1,1,1,12) Model Summary:")
print(sarima_fit.summary())

# Forecast
sarima_forecast = sarima_fit.forecast(steps=len(test_seas))
```

```
SARIMA(1,1,1)(1,1,1,12) Model Summary:
                                     SARIMAX Results
==========================================================================================
======
Dep. Variable:                                Sales   No. Observations:
192
Model:             SARIMAX(1, 1, 1)x(1, 1, 1, 12)   Log Likelihood                -4
63.423
Date:                              Tue, 23 Dec 2025   AIC                            9
36.845
Time:                                      17:33:49   BIC                            9
52.782
Sample:                                  01-01-2004   HQIC                           9
43.307
                                       - 12-01-2019
Covariance Type:                                opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
ar.L1         -0.0128      0.084     -0.151      0.880      -0.178       0.152
ma.L1         -0.9801      0.040    -24.634      0.000      -1.058      -0.902
ar.S.L12      -0.0238      0.104     -0.230      0.818      -0.227       0.179
ma.S.L12      -0.9156      0.140     -6.530      0.000      -1.190      -0.641
sigma2         8.9146      1.336      6.671      0.000       6.295      11.534
===================================================================================
Ljung-Box (L1) (Q):                   0.03   Jarque-Bera (JB):                 1.26
Prob(Q):                              0.87   Prob(JB):                         0.53
Heteroskedasticity (H):               1.30   Skew:                            -0.07
Prob(H) (two-sided):                  0.31   Kurtosis:                         2.61
===================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (complex-ste
p).
```

Visualizing Seasonal Forecast

In [20]:
```
fig, ax = plt.subplots(figsize=(14, 5))

train_seas.plot(ax=ax, label='Train', linewidth=1.5, color='steelblue')
test_seas.plot(ax=ax, label='Test (Actual)', linewidth=1.5, color='green')
sarima_forecast.plot(ax=ax, label='SARIMA Forecast', linewidth=2,
                     linestyle='--', color='red')

ax.set_title('SARIMA Forecast (Monthly Data with Seasonality)',
             fontsize=14, fontweight='bold')
ax.set_xlabel('Date', fontsize=12)
ax.set_ylabel('Sales', fontsize=12)
ax.legend(fontsize=11)
ax.grid(True, alpha=0.3)
plt.tight_layout()
plt.show()
```



6. Model Selection

Information Criteria

AIC (Akaike Information Criterion)

AIC = 2k - 2ln(L)

Where:

- k = number of parameters
- L = maximum likelihood

BIC (Bayesian Information Criterion)

BIC = k·ln(n) - 2ln(L)

Where:

- n = number of observations

Rule: Lower AIC/BIC = better model (balances fit and complexity)

Grid Search for Best Model

In [21]:
```
def evaluate_arima_models(data, p_range, d_range, q_range):
    """
    Evaluate different ARIMA configurations
    """
    results = []

    for p in p_range:
        for d in d_range:
            for q in q_range:
                try:
                    model = ARIMA(data, order=(p, d, q))
                    fitted = model.fit()

                    results.append({
                        'order': (p, d, q),
                        'AIC': fitted.aic,
                        'BIC': fitted.bic
                    })
                except:
                    continue

    return pd.DataFrame(results)

# Search for best model
print("Searching for best ARIMA model...")
results_df = evaluate_arima_models(
    train,
    p_range=range(0, 3),
    d_range=range(0, 2),
    q_range=range(0, 3)
)

# Sort and display
results_df = results_df.sort_values('AIC')
print("\nTop 5 Models by AIC:")
print(results_df.head())

best_order = results_df.iloc[0]['order']
print(f"\nBest Model: ARIMA{best_order}")
```

```
Searching for best ARIMA model...

Top 5 Models by AIC:
        order         AIC         BIC
6   (1, 0, 0)  697.563191  708.005108
13  (2, 0, 1)  698.028252  715.431446
10  (1, 1, 1)  698.670168  709.099558
12  (2, 0, 0)  699.335230  713.257786
7   (1, 0, 1)  699.341125  713.263681

Best Model: ARIMA(1, 0, 0)
```

7. Model Diagnostics

Why Check Diagnostics?

After fitting a model, we need to verify that:

1. **Residuals are white noise** (random, no pattern)
2. **Residuals are normally distributed**
3. **No autocorrelation in residuals**
4. **Model assumptions are satisfied**

Theory: Good Residuals

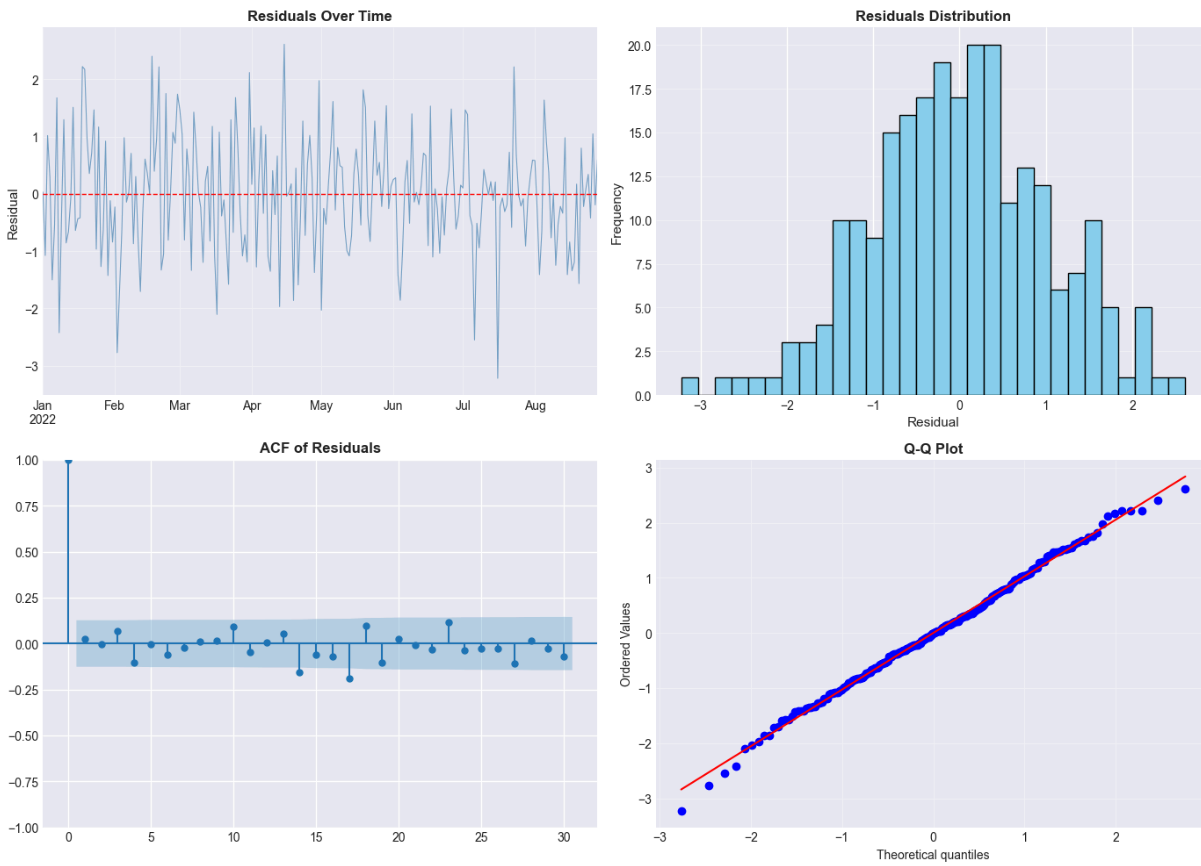
If the model is appropriate, residuals should have:

- Mean ≈ 0
- Constant variance (homoscedastic)
- No autocorrelation
- Normal distribution

Diagnostic Plots

In [22]:
```python
# Get residuals from best model
best_model = ARIMA(train, order=best_order).fit()
residuals = best_model.resid

# Create diagnostic plots
fig, axes = plt.subplots(2, 2, figsize=(14, 10))

# 1. Residuals over time
residuals.plot(ax=axes[0, 0], linewidth=0.8, color='steelblue', alpha=0.7)
axes[0, 0].axhline(y=0, color='red', linestyle='--', linewidth=1)
axes[0, 0].set_title('Residuals Over Time', fontsize=12, fontweight='bold')
axes[0, 0].set_ylabel('Residual', fontsize=11)
axes[0, 0].grid(True, alpha=0.3)
```

```
# 2. Histogram
residuals.hist(ax=axes[0, 1], bins=30, edgecolor='black', color='skyblue')
axes[0, 1].set_title('Residuals Distribution', fontsize=12, fontweight='bold')
axes[0, 1].set_xlabel('Residual', fontsize=11)
axes[0, 1].set_ylabel('Frequency', fontsize=11)
axes[0, 1].grid(True, alpha=0.3, axis='y')

# 3. ACF of residuals
plot_acf(residuals, lags=30, ax=axes[1, 0])
axes[1, 0].set_title('ACF of Residuals', fontsize=12, fontweight='bold')

# 4. Q-Q plot
stats.probplot(residuals, dist="norm", plot=axes[1, 1])
axes[1, 1].set_title('Q-Q Plot', fontsize=12, fontweight='bold')
axes[1, 1].grid(True, alpha=0.3)

plt.tight_layout()
plt.show()
```



Ljung-Box Test

Tests the null hypothesis that residuals are independently distributed (no autocorrelation).

- **H_0**: Residuals are white noise (no autocorrelation)
- **H_1**: Residuals have autocorrelation

Decision: p-value > 0.05 → residuals are white noise (good!)

In [23]:
```
# Perform Ljung-Box test
lb_test = acorr_ljungbox(residuals, lags=[10, 20, 30], return_df=True)

print("\nLjung-Box Test Results:")
print(lb_test)
print("\nInterpretation:")
print("If p-value > 0.05: Residuals are white noise ✓")
print("If p-value < 0.05: Residuals have autocorrelation ✗")
```

```
Ljung-Box Test Results:
      lb_stat  lb_pvalue
10   7.147022   0.711496
20  31.838671   0.045048
30  41.570079   0.077860

Interpretation:
If p-value > 0.05: Residuals are white noise ✓
If p-value < 0.05: Residuals have autocorrelation ✗
```

Residual Statistics

In [24]:
```
print("\nResidual Summary Statistics:")
print("="*50)
print(f"Mean:      {residuals.mean():>10.6f}  (should be ≈ 0)")
print(f"Std Dev:   {residuals.std():>10.6f}")
print(f"Min:       {residuals.min():>10.6f}")
print(f"Max:       {residuals.max():>10.6f}")
print(f"Skewness:  {residuals.skew():>10.6f}  (should be ≈ 0)")
print(f"Kurtosis:  {residuals.kurtosis():>10.6f}  (should be ≈ 0)")
```

```
Residual Summary Statistics:
==================================================
Mean:       -0.000090  (should be ≈ 0)
Std Dev:     1.022687
Min:        -3.222888
Max:         2.614799
Skewness:   -0.045184  (should be ≈ 0)
Kurtosis:    0.071627  (should be ≈ 0)
```

8. Model Evaluation

Common Forecasting Metrics

1. MAE (Mean Absolute Error)

MAE = $(1/n)\ \Sigma|y_i - \hat{y}_i|$

- Average absolute difference
- Same units as original data

- Easy to interpret

2. RMSE (Root Mean Squared Error)

RMSE = $\sqrt{[(1/n) \Sigma(y_i - \hat{y}_i)^2]}$

- Penalizes large errors more
- Same units as original data
- More sensitive to outliers than MAE

3. MAPE (Mean Absolute Percentage Error)

MAPE = $(100/n) \Sigma|((y_i - \hat{y}_i)/y_i)|$

- Expressed as percentage
- Scale-independent
- Undefined when $y_i = 0$

Computing Metrics

In [25]:

```
def calculate_metrics(actual, predicted):
    """Calculate forecasting metrics"""
    mae = mean_absolute_error(actual, predicted)
    rmse = np.sqrt(mean_squared_error(actual, predicted))
    mape = np.mean(np.abs((actual - predicted) / actual)) * 100

    return {'MAE': mae, 'RMSE': rmse, 'MAPE': mape}

# Evaluate forecast
forecast = best_model.forecast(steps=len(test))
metrics = calculate_metrics(test, forecast)

print("\nForecast Evaluation Metrics:")
print("="*50)
print(f"MAE:  {metrics['MAE']:.4f}")
print(f"RMSE: {metrics['RMSE']:.4f}")
print(f"MAPE: {metrics['MAPE']:.2f}%")
```

```
Forecast Evaluation Metrics:
==================================================
MAE:  1.2387
RMSE: 1.5522
MAPE: 102.35%
```

Summary and Key Takeaways

Workflow for Time Series Modeling

1. **Explore Data**
    - Plot the series
    - Identify trend, seasonality, noise
    - Perform decomposition
2. **Check Stationarity**
    - Use ADF test
    - Apply differencing if needed
    - Verify stationarity after transformation
3. **Identify Model Orders**
    - Examine ACF plot → suggests MA order (q)
    - Examine PACF plot → suggests AR order (p)
    - Consider seasonal patterns
4. **Fit Candidate Models**
    - Start with simple models (AR, MA)
    - Try ARIMA for non-stationary data
    - Use SARIMA for seasonal data
5. **Select Best Model**
    - Compare AIC/BIC values
    - Lower is better
    - Balance complexity and fit
6. **Validate Model**
    - Check residual plots
    - Perform Ljung-Box test
    - Ensure residuals are white noise
7. **Forecast and Evaluate**
    - Generate predictions
    - Calculate MAE, RMSE, MAPE
    - Compare with test data

Model Quick Reference

If you see...	Consider...
Clear trend	Differencing (d > 0) or ARIMA
Seasonality	SARIMA with appropriate period
PACF cuts off at lag p	AR(p)
ACF cuts off at lag q	MA(q)

If you see...	Consider...
Both decay slowly	ARMA or ARIMA
Non-constant variance	Log transformation or multiplicative model

Python Libraries

- **pandas**: Data manipulation and time series structures
- **statsmodels**: ARIMA, SARIMA, statistical tests, diagnostics
- **matplotlib**: Visualization
- **scipy**: Statistical functions
- **sklearn**: Evaluation metrics

Next Steps

In **Part 2**, we'll cover:

- Advanced pandas techniques for time series
- Resampling and frequency conversion
- Rolling windows and moving averages
- Time-based indexing and slicing
- Practical data preparation workflows

In **Part 3**, we'll explore:

- Feature engineering for ML models
- Machine learning approaches (XGBoost, Random Forest)
- Deep learning (LSTM, GRU)
- Prophet and modern forecasting tools

End of Part 1