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A NEW APPROACH FOR MULTIPLE OBJECTIVE DECISION MAKING

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Scope and Purpose—Obtaining a compromise solution for multiple objective functions is usually a major topic for solving multiple objective decision-making problems. In the last two decades, goal programming and global criterion methods are some of the most popular approaches. The methods only consider the criterion of the shortest distance from the given goal or the positive ideal solution. However, in practice, we might like to have a decision which not only makes as much profit as possible, but also avoids as much risk as possible. The single criterion of the shortest distance from the positive ideal solution is then not enough to satisfy decision maker(s).

In this study, we propose a new approach, TOPSIS for MODM, for solving multiple objective decision-making problems. The principle of compromise (of TOPSIS) is based on the premise that the chosen solution should be as close to the positive ideal solution as possible and as far away from the negative ideal solution as possible.

Abstract—The proposed TOPSIS for MODM algorithm is developed for solving multiple objective decision-making problems by considering two reference points of the positive ideal solution and the negative ideal solution simultaneously. The basic principle of compromise of TOPSIS for MODM is that the chosen solution should be as close to the positive ideal solution as possible and as far away from the negative ideal solution as possible. Thus, we can reduce a k-objective decision-making problem into an auxiliary bi-objective decision-making problem. That is: instead of k incommensurable and conflicting objective function, we consider two commensurable but conflicting objective functions (distance functions). Then, by using the max–min operator, we can obtain a compromise solution for the auxiliary bi-objective problem and the original k-objective problem. To illustrate the TOPSIS algorithm, a numerical nutrition problem is solved.

1. INTRODUCTION

Most decision-making problems have multiple objectives which cannot be optimized simultaneously due to the inherent incommensurability and conflict between these objectives. Thus making a trade-off between these objectics becomes a major subject to get the "best compromise" solution. A variety of methodologies for solving multiple objective decision making (MODM) problems have been proposed. Typical recent survey works are Hwang and Masud [1,2], Zeleny [3], Yu [4], Steuer [5], Zimmermann [6], Seo and Sakawa [7], and White [8].

Mathematically, MODM problems can be represented as:

$$\max/\min[f_1(x), f_2(x), \ldots, f_k(x)]$$

s.t.

$$x \in X = \{x \mid g_h(x)\{\geqslant, =, \leqslant\}0, h = 1, 2, \ldots, m\}$$

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where

$$\begin{aligned} &f_j(x)\text{: benefit objective for maximization}, j \in J \\ &f_i(x)\text{: cost cobjective for minimization}, i \in I. \end{aligned} \tag{1}$$

The problem consists of n decision variables, m constraints and k objectives. $f_j(x)$, $f_i(x)$ and $g_h(x)$, $\forall i, j, h$, might be linear or nonlinear. To solve equation (1), we develop a new approach—TOPSIS for MODM—in this study.

TOPSIS (technique for order preference by similarity to ideal solution) was first developed by Hwang and Yoon [9] to solve a multiple attribute decision making (MADM) problem. It provides the principle of compromise that the chosen alternative should have "the shortest distance from the positive ideal solution (PIS)" and "the farthest distance from the negative ideal solution (NIS)". Lately, this principle of compromise has also been suggested by Zeleny [3] and Hall [10] for solving MADM problems. In practice, TOPSIS for MADM has been used to solve selection/evaluation problems with a finite number of alternatives [11,12]. However, the TOPSIS concept is never used to develop a methodology for solving MODM (design) problems having an infinite number of alternatives. (The differences between MADM and MODM problems have been systematically discussed in Hwang and Yoon [9].) This is the major purpose of this study.

By using the principle of compromise, TOPSIS for MODM first reduces a k-dimensional objective space to a two-dimensional (the distance from PIS vs the distance from NIS) objective space, and removes the inherent incommensurability between original objectives. Since conflict between the distance from PIS and the distance from NIS is usually existing, we cannot simultaneously obtain a compromise with the shortest distance from PIS and the farthest distance from NIS. And, unlike MADM, MODM always has an infinite number of alternatives. It becomes impossible to find the solution with the shortest distance from PIS and the longest distance from NIS. Thus both criteria of "the shortest distance from PIS" and "the farthest distance from NIS" are substituted by "as close to PIS as possible" and "as far away from NIS as possible". The terms of "as close ... as possible" and "as far ... as possible" are actually fuzzy. To model these fuzzy terms, membership functions of the fuzzy set theory are employed. The problem becomes a fuzzy bi-objective programming problem. We then use Bellman and Zadeh's max–min operator to obtain a compromise solution for the fuzzy bi-objective programming problem. This solution is also chosen as a compromise solution of the original k-objective programming problem.

It should be emphasized that the criteria of "as close to PIS as possible" and "as far away from NIS as possible" are similar to "as much profit as possible" and "as little risk as possible".

In the following section, we will first discuss our new approach—TOPSIS for MODM. To increase readability, we also provide a solution algorithm. In Section 3, a numerical nutrition problem is solved for illustration. Concluding remarks are given in the last section.

TOPSIS FOR MODM

Multiple criteria decision making (MCDM) has been widely studied since the 1970s. Among various methodologies, the first step for solving MCDM problems is usually to define a reference point(s) in the multiple objective/attribute space. With the reference point(s), alternatives are evaluated and a compromise alternative/solution is chosen. For instance, the chosen solution of goal programming is based on the shortest deviation from a reference point or a goal.

The principle of compromise of TOPSIS for MADM is that the chosen alternative should have "the shortest distance from the positive ideal solution (PIS)" and "the farthest distance from the negative ideal solution (NIS)". For MODM problems, this principle is equivalent to *minimise the distance from PIS and maximize the distance from NIS*. Because both criteria are usually conflicting, they can only reach PORTIONS of their individual optima. Thus the principle of compromise of TOPSIS for MODM becomes that *the chosen solution should be "as close to PIS as possible" and "as far away from NIS as possible"*.

To mathematically formulate this principle of compromise, we first define the reference points of PIS and NIS for equation (1) as:

$$\begin{aligned} \mathbf{f}_t^* &= \max_{x \in X} f_j(x) && \text{for } \forall j \in J \\ &= \min_{x \in X} f_i(x) && \text{for } \forall i \in I \end{aligned} \tag{2}$$

$$\begin{aligned} \mathbf{f}_t^- &= \min_{x \in X} f_j(x) && \text{for } \forall j \in J \\ &\max_{x \in X} f_i(x) && \text{for } \forall i \in I. \end{aligned} \tag{3}$$

where $t = 1, 2, \ldots, k$. Let $\mathbf{f}^* = \{\mathbf{f}_1^*, \mathbf{f}_2^*, \ldots, \mathbf{f}_k^*\}$ be the solution vector of equation (2) which consists of individual best feasible solutions for all objectives. $\mathbf{f}^*$ is called the PIS. Similarly, let $\mathbf{f}^- = \{\mathbf{f}_1^-, \mathbf{f}_2^-, \ldots, \mathbf{f}_k^-\}$ be the solution vector of equation (3) which consists of individual worst feasible solutions for all objectives. $\mathbf{f}^-$ is called the NIS. Note that $\mathbf{f}^*$ and $\mathbf{f}^-$ are always out of the feasible region of equation (1).

With PIS and NIS, Minkowski's L_p metric is used for measuring the distance from PIS and the distance from NIS. Because of the incommensurability among objectives, we should first normalize the component distance (from PIS or NIS) for each objective. Thus we obtain the following distance functions:

$$d_p^{\mathrm{PIS}} = \left\{ \sum_{j \in J} w_j^p \left[\frac{\mathbf{f}_j^* - f_j(x)}{\mathbf{f}_j^* - f_j^-} \right]^p + \sum_{i \in I} w_i^p \left[\frac{f_i(x) - \mathbf{f}_i^*}{f_i^- - \mathbf{f}_i^*} \right]^p \right\}^{1/p} \tag{4}$$

and

$$d_p^{\mathrm{NIS}} = \left\{ \sum_{j \in J} w_j^p \left[\frac{f_j(x) - f_j^-}{\mathbf{f}_j^* - f_j^-} \right]^p + \sum_{i \in I} w_i^p \left[\frac{f_i^- - f_i(x)}{f_i^- - \mathbf{f}_i^*} \right]^p \right\}^{1/p} \tag{5}$$

where w_t, $t = 1, 2, \ldots, k$ are the relative importance (weights) of objectives; $p = 1, 2, \ldots, \infty$ is the parameter of distance functions; and d_p^{PIS} (regrets) and d_p^{NIS} (rewards) are the distances from the PIS and NIS, respectively. It should be emphasized that w_t indicates the degree of importance of the tth objective. On the other hand, the property of the distance parameter p is that when p increases, distance d_p decreases, i.e. $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_\infty$ and greater emphasis is given to the largest deviation in forming the total [4,5,13,14]. $p = 1$ implies an equal importance in forming the distance function d_1 for all individual deviations. $p = 2$ implies that more importance is given to a large deviation proportionately. Ultimately, for $p = \infty$, the largest deviation completely dominates the distance determination. Also, d_1 (the Manhatten distance) and d_2 (the Euclidean distance) are the longest and shortest distances in the geometrical sense; and d_∞ (the Tchebycheff distance) is the shortest distance in the numerical sense. Among all p values, we will present the cases of $p = 1$, 2, and ∞ which are operationally and practically important, and even a well-known standard in the fields of control theory and MCDM. Detailed discussions can be found in Yu [4] and Späth [13].

Now, instead of the original k objectives in equation (1), we have the objectives of "minimize the distance from PIS or d_p^{NIS}" and "maximize the distance from NIS or d_p^{NIS}". Thus we have the following bi-objective programming problem:

$$\min d_p^{\mathrm{NIS}}(x)$$
$$\max d_p^{\mathrm{NIS}}(x)$$

s.t.

$$x \in X \tag{6}$$

where $p = 1, 2, \ldots, \infty$. Since these two objectives are usually in conflict with each other, we cannot simultaneously obtain their individual optima. Each objective achieves only a PORTION of its optimum. Thus our objectives become "as close to PIS as possible" and "as far away from NIS as possible" which are essentially fuzzy. To model these fuzzy terms, we can use membership functions of the fuzzy set theory to represent degress of satisfaction/preference of both objectives. In this study, the membership functions $\mu_1(x)$ and $\mu_2(x)$ of two distance objective functions are assumed to be non-increasing/non-decreasing monotonous functions between the extreme points d_p^* and d_p' where:

$$(d_p^{\mathrm{PIS}})^* = \min_{x \in X} d_p^{\mathrm{PIS}}(x) \text{ and the solution is } x^{\mathrm{PIS}} \tag{7}$$

$$(d_p^{\mathrm{NIS}})^* = \max_{x \in X} d_p^{\mathrm{NIS}}(x) \text{ and the solution is } x^{\mathrm{NIS}} \tag{8}$$

$$(d_p^{\mathrm{PIS}})' = d_p^{\mathrm{PIS}}(x^{\mathrm{NIS}}) \tag{9}$$

$$(d_p^{\mathrm{NIS}})' = d_p^{\mathrm{NIS}}(x^{\mathrm{PIS}}) \tag{10}$$

Then, based on the preference concept, we assign a larger degree to the one with the shorter distance from PIS for $\mu_1(x)$ and assign a larger degree to the one with the farther distance from NIS for $\mu_2(x)$. Thus $\mu_1(x)$ and $\mu_2(x)$ can be obtained as in Fig. 1.

For computational efficiency, let us assume $\mu_1(x)$ and $\mu_2(x)$ are the following non-increasing and non-decreasing linear functions, respectively:

$$\mu_1(x) = \begin{cases} 1 & \text{if } d_p^{\mathrm{PIS}}(x) < (d_p^{\mathrm{PIS}})^* \\ \dfrac{(d_p^{\mathrm{PIS}})' - d_p^{\mathrm{PIS}}(x)}{(d_p^{\mathrm{PIS}})' - (d_p^{\mathrm{PIS}})^*} & \text{if } (d_p^{\mathrm{PIS}})^* \leqslant d_p^{\mathrm{PIS}}(x) \leqslant (d_p^{\mathrm{PIS}})' \\ 0 & \text{if } d_p^{\mathrm{PIS}}(x) > (d_p^{\mathrm{PIS}})' \end{cases} \tag{11}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if } d_p^{\mathrm{NIS}}(x) > (d_p^{\mathrm{NIS}})^* \\ \dfrac{d_p^{\mathrm{NIS}}(x) - (d_p^{\mathrm{NIS}})'}{(d_p^{\mathrm{NIS}})^* - (d_p^{\mathrm{NIS}})'} & \text{if } (d_p^{\mathrm{NIS}})' \leqslant d_p^{\mathrm{NIS}}(x) \leqslant (d_p^{\mathrm{NIS}})^* \\ 0 & \text{if } d_p^{\mathrm{NIS}}(x) < (d_p^{\mathrm{NIS}})' \end{cases} \tag{12}$$

(see Fig. 1 also). $\mu_1(x)$ and $\mu_2(x)$ indicate the preference degrees of "as close to PIS as possible" and "as far away from NIS as possible".

To solve equation (6) with the membership functions of equations (11) and (12), we can use the max–min operation proposed by Bellman and Zadeh [15] and applied by Zimmermann [16]. That is: a compromise solution x^* is obtained by solving the following problem:

$$\mu_D(x^*) = \max_{x \in X} \{\min[\mu_1(x), \mu_2(x)]\} \tag{13}$$

(see Fig. 1 also).

Let $\alpha = \min[\mu_1(x), \mu_2(x)]$, or $\mu_1(x) \geqslant \alpha$ and $\mu_2(x) \geqslant \alpha$. Then equation (13) is equivalent to the following programming problem:

$$\max \alpha$$

s.t.

$$\mu_1(x) \geqslant \alpha \quad \text{and} \quad \mu_2(x) \geqslant \alpha$$

$$x \in X \tag{14}$$

where α is the degree of satisfactory level for both criteria of "as close to PIS as possible" and "as

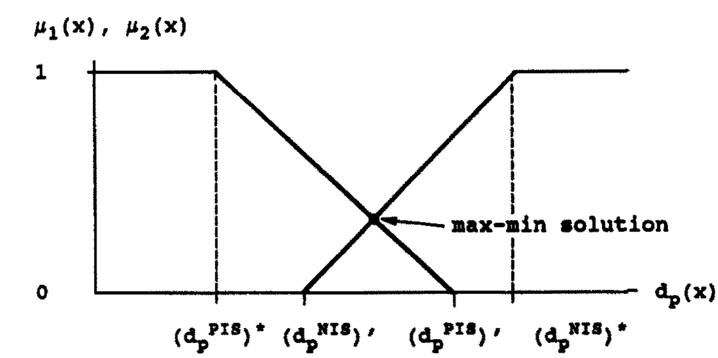

Fig. 1. Membership functions $\mu_1(x)$ and $\mu_2(x)$.

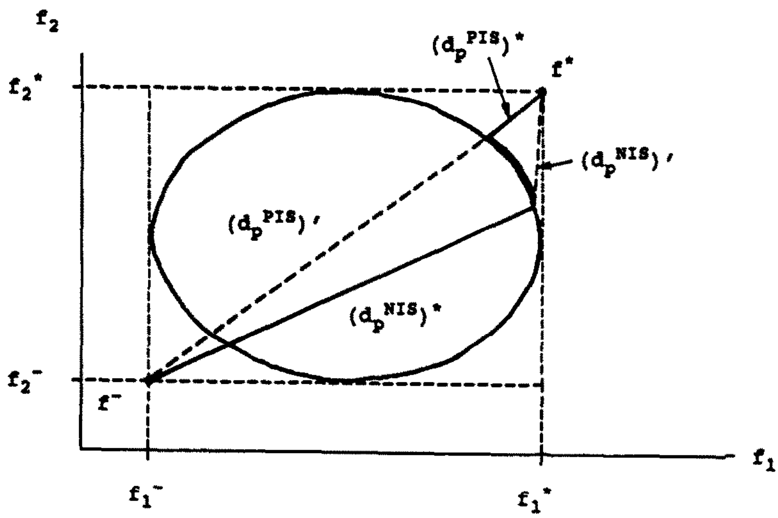

Fig. 2. The compromise solution exists in the dark line when x^{PIS} is not equal to x^{NIS}.

far away from NIS as possible". Note that the compromise solution of equation (14) will be within the dark line as shown in Fig. 2.

Before presenting the solution algorithm, let us first discuss some properties of TOPSIS for the cases of $p = 1$, 2 and 3.

For the special case of $p = \infty$, we will have the following problem, instead of equation (6):

$$\min d_\infty^{\text{PIS}}$$

$$\max d_\infty^{\text{NIS}}$$

s.t.

$$w_j[(\mathbf{f}_j^* - f_j(x))/(\mathbf{f}_j^* - \mathbf{f}_j^-)] \leqslant d_\infty^{\text{PIS}}, \forall j$$
$$w_i[(f_i(x) - \mathbf{f}_i^*)/(\mathbf{f}_i^- - \mathbf{f}_i^*)] \leqslant d_\infty^{\text{PIS}}, \forall i$$
$$w_j[(f_j(x) - \mathbf{f}_j^-)/(\mathbf{f}_j^* - f_j^-)] \geqslant d_\infty^{\text{NIS}}, \forall j$$
$$w_i[(\mathbf{f}_i^- - f_i(x))/(\mathbf{f}_i^- - \mathbf{f}_i^*)] \geqslant d_\infty^{\text{NIS}}, \forall i$$
$$x \in X \tag{6a}$$

where d_∞^{PIS} and d_∞^{NIS} are not real distances, but the largest and smallest components of the k-dimensional distance functions, respectively. The PIS can be obtained by solving the following problems:

$$\min d_\infty^{\text{PIS}}$$

s.t.

$$w_j[(f_j^* - f_j(x))/(f_j^* - f_j^-)] \leqslant d_\infty^{\text{PIS}}, \forall j$$
$$w_i[(f_i(x) - f_i^*(x)/(f_i^- - f_i^*)] \leqslant d_\infty^{\text{PIS}}, \forall i$$
$$x \in X \tag{15}$$

and

$$\max d_\infty^{\text{NIS}}$$

s.t.

$$w_j[(f_j(x) - f_j^-)/(f_j^* - f_j^-)] \geqslant d_\infty^{\text{NIS}}, \forall j$$
$$w_i[(f_i^- - f_i(x))/(f_i^- - f_i^*)] \geqslant d_\infty^{\text{NIS}}, \forall i$$
$$x \in X. \tag{16}$$

Similarly, we can obtain NIS under the same constrains of equations (15) and (16), correspondingly.

As a matter of fact, equation (16) is equivalent to:

$$\min\ -d_{\infty}^{\text{NIS}}$$

s.t.

$$\begin{aligned} &w_j[(f_j^* - f_j(x))/(f_j^* - f_j^-)] \leqslant w_j - d_{\infty}^{\text{NIS}}, \forall j \\ &w_i[(f_i(x) - f_i^*)/(f_i^- - f_i^*)] \leqslant w_i - d_{\infty}^{\text{NIS}}, \forall i \\ &x \in X. \end{aligned} \tag{17}$$

If $w_1 = w_2 = \ldots = w_k = 1/k$, we may set $d_{\infty}^{\text{PIS}} = 1/k - d_{\infty}^{\text{NIS}}$, then $d_{\infty}^{\text{NIS}} = 1/k - d_{\infty}^{\text{PIS}}$. Equation (17) is equivalent to:

$$\min\ d_{\infty}^{\text{PIS}} - 1/k$$

s.t.

$$\begin{aligned} &w_j[(f_j^* - f_j(x))/(f_j^* - f_j^-)] \leqslant d_{\infty}^{\text{PIS}}, \forall j \\ &w_i[(f_i(x) - f_i^*)/(f_i^- - f_i^*)] \leqslant d_{\infty}^{\text{PIS}}, \forall i \\ &x \in X \end{aligned} \tag{18}$$

which will have the same solution as equation (15). Thus equations (15) and (16) are actually the same problem. The compromise procedure of TOPSIS is not necessary for this case. That is, when $p = \infty$ with equal weights of objectives, the compromise solution of TOPSIS can be obtained by solving either equation (15) or equation (16). But, when $p = \infty$ with unequal weights of objectives, equations (15) and (16) will have different solutions. Equation (14) should be solved for the TOPSIS solution.

When $p = 1$, we have $d_1^{\text{PIS}} = 1 - d_1^{\text{NIS}}$. "Min d_1^{PIS}" and "max d_1^{NIS}" are subjected to the same constraints and have the same solution whether the weights of the objectives are the same or not. Thus the compromise solution of TOPSIS can be obtained by either minimizing d_1^{PIS} or maximizing d_1^{NIS}.

For the case of $2 \leqslant p \leqslant \infty$, "min d_p^{PIS}" and "max d_p^{NIS}" usually have different solutions. So we should solve equation (14).

To increase readability, it is necessary to organize the above discussion of TOPSIS for MODM in the following algorithm:

Algorithm

Step 0.1. Determine the distance parameter p. If the decision maker emphasizes the sum of individual distances (regrets for d_p^{PIS} and rewards for d_p^{NIS}), he should choose $p = 1$. On the other hand, if dominating criteria are the maximum of individual regrets and the minimum of individual rewards, $p = \infty$ should be chosen. Beyond both extreme cases, $p = 2$ will be chosen. $p = 2$ is similar to the popular least-square approach, and provides an approximation for the cases $2 < p \ll \infty$. Note that $p = 1$, 2, and ∞ are well-known standard values in the fields of control theory and MCDM [4,13].

Step 0.2. Determine the relative importance w_t of the k objective functions in equation (1). There are various methods for assessing w_t, such as eigenvector, weighted least square, entropy and LINMAP methods (see Hwang and Yoon [9]). In this study, we assume that w_t is given from decision maker(s).

Step 1. Determine PIS (f^*) and NIS (f^-) by solving equations (2) and (3).

Step 2. If $p = 1$ go to Step 3; otherwise go to Step 4.

Step 3. Solve the problem of "$\min_{x \in X} d_1^{\text{PIS}}$". Then go to 7.

Step 4. If $p = \infty$ with $w_1 = w_2 = \ldots = w_k$, solve equation (15); otherwise go to Step 5.

Step 5. If $p = \infty$, go to Step 6. Otherwise, solve equation (6) for $p = 2, 3, \ldots, N$ (a finite number).

Step 5.1. Obtain $(d_p^{\text{PIS}})^*$, $(d_p^{\text{NIS}})^*$, $(d_p^{\text{PIS}})'$ and $(d_p^{\text{NIS}})'$ by solving equations (7)–(10). Go to Step 5.2.
Step 5.2. Obtain membership functions $\mu_1(x)$ and $\mu_2(x)$ by calculating equations (11) and (12). Go to Step 5.3.
Step 5.3. Solve equation (14). Go to Step 7.

Step 6. Solve equation (6a) for $p = \infty$.
Step 6.1. Obtain $(d_p^{\text{PIS}})^*$, $(d_p^{\text{NIS}})^*$, $(d_p^{\text{PIS}})'$ and $(d_p^{\text{NIS}})'$ by solving equations (15), (16), (9) and (10). Go to Step 6.2.
Step 6.2. Obtain membership functions $\mu_1(x)$ and $\mu_2(x)$ by calculating equations (11) and (12). Go to Step 6.3.
Step 6.3. Solve the following equation:

$$\max \alpha$$

s.t.

$$\begin{gathered} \mu_1(d_\infty^{\text{PIS}}) \geqslant \alpha \quad \text{and} \quad \mu_2(d_\infty^{\text{NIS}}) \geqslant \alpha \\ w_j[(f_j^* - f_j(x))/(f_j^* - f_j^-)] \leqslant d_\infty^{\text{PIS}}, \forall j \\ w_i[(f_i(x) - f_i^*)/(f_i^- - f_i^*)] \leqslant d_\infty^{\text{PIS}}, \forall i \\ w_j[(f_j(x) - f_j^-)/(f_j^* - f_j^-)] \geqslant d_\infty^{\text{NIS}}, \forall j \\ w_i[(f_i^- - f_i(x))/(f_i^- - f_i^*)] \geqslant d_\infty^{\text{NIS}}, \forall i \\ x \in X. \end{gathered} \tag{19}$$

Go to Step 7.

Step 7. If the TOPSIS solution is satisfied, stop. However, decision maker(s) may like to change p, w_t, and/or membership functions. Then, we will go back to Step 0.1, Step 0.2, or Step 5.2/Step 6.2. The solution procedure is then repeated.
To illustrate TOPSIS for MODM, let us consider a numerical nutrition problem.

Nutrition problem

The nutrition problem is to find the daily diet requirements of milk, x_1; beef, x_2; eggs, x_3; bread, x_4; lettuce and salad, x_5; and orange juice, x_6 which should be eaten to balance nutritional requirements so as to:

(1) maximize carbohydrate intake
(2) minimize cholesterol intake
(3) minimize cost.

The maximum daily intake of the six foods is: 6 pints of milk, 1 lb of beef, $\frac{1}{4}$ dozen eggs, 10 oz of bread, 10 oz of lettuce and salad, and 4 pints of orange juice. Other information on the six foods is given in Table 1.

Table 1. C_{KI} per HWT LTL shipped from Boston

	Milk (pint)	Beef (lb)	Eggs (dozen)	Bread (oz)	Lettuce salad (oz)	Orange juice for (pint)	Daily intake for adults
Vitamin A (i.u.)	720	107	7080	0	134	1000	5000
Food energy (calories)	344	1460	1040	75	17.4	240	2500
Cholesterol (unit)	10	20	120	0	0	0	
Protein (g)	18	151	78	2.5	0.2	4	63
Carbohydrate (g)	24	27	0	15	1.1	52	
Iron (mg)	0.2	10.1	13.2	0.75	0.15	1.2	12.5
Cost ($)	0.22	2.2	0.8	0.1	0.05	0.26	

The problem becomes to find $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ such that:

$$\max f_1(x) = 24x_1 + 27x_2 + 15x_4 + 1.1x_5 + 52x_6 \qquad \text{(Carbohydrate intake)}$$
$$\min f_2(x) = 10x_1 + 20x_2 + 120x_3 \qquad \text{(Cholesterol intake)}$$
$$\min f_3(x) = 0.22x_1 + 2.2x_2 + 0.8x_3 + 0.1x_4 + 0.05x_5 + 0.26x_6 \qquad \text{(Cost)}$$

s.t.

$$720x_1 + 107x_2 + 7080x_3 + 134x_5 + 1000x_6 \geqslant 5000 \qquad \text{(Vitamin A. i.u.)}$$
$$0.2x_1 + 10.1x_2 + 13.2x_3 + 0.75x_4 + 0.15x_5 + 1.2x_6 \geqslant 12.5 \qquad \text{(Iron mg)}$$
$$344x_1 + 1460x_2 + 1040x_3 + 75x_4 + 17.4x_5 + 240x_6 \geqslant 2500 \qquad \text{(Calories)}$$
$$18x_1 + 151x_2 + 78x_3 + 2.5x_4 + 0.2x_5 + 4.0x_6 \geqslant 63 \qquad \text{(Protein g)}$$
$$x_1 \leqslant 6.0 \qquad \text{(Milk pints)}$$
$$x_2 \leqslant 1.0 \qquad \text{(Beef pounds)}$$
$$x_3 \leqslant 0.25 \qquad \text{(Eggs dozen)}$$
$$x_4 \leqslant 10.0 \qquad \text{(Bread ounces)}$$
$$x_5 \leqslant 10.0 \qquad \text{(Lettuce and salad ounces)}$$
$$x_6 \leqslant 4.0 \qquad \text{(Orange juice pints)}$$
$$x_i \geqslant 0,\ i = 1, 2, \ldots, 6 \qquad (20)$$

where the constraints are also denoted by $x \in X$.

We solve this nutrition problem by our TOPSIS for MODM algorithm as follows:

Step 0.1. Let $p = \infty$.
Step 0.2. Let $w_1 = 0.3$, $w_2 = 0.5$ and $w_3 = 0.2$.
Step 1. By using equations (2) and (3), we can determine PIS (f^*) and NIS (f^-) as shown in Table 2.
Since $p = \infty$ with unequal weights, we then go directly to Step 6.
Step 6. Obtain the TOPSIS solution by the following steps:

Table 2. Payoff table

PIS-Payoff									
	f_1	f_2	f_3	x_1	x_2	x_3	x_4	x_5	x_6
max f_1	540.00*	80.00	6.06	6.00	1.00	0.00	10.00	10.00	4.00
min f_2	382.59	8.44*	3.57	0.00	0.42	0.00	10.00	10.00	4.00
min f_3	281.67	67.83	2.24*	3.78	0.00	0.25	10.00	0.00	0.79
$f^* = (540{,}00, 8.44, 2.24)$									
NIS-Payoff									
	f_1	f_2	f_3	x_1	x_2	x_3	x_4	x_5	x_6
min f_1	93.34⁻	71.53	3.07	2.50	0.83	0.25	0.00	10.00	0.00
max f_2	321.00	110.00⁻	4.72	6.00	1.00	0.25	10.00	0.00	0.00
max f_3	54.0	110.00	6.26⁻	6.00	1.00	0.25	10.00	10.00	4.00
$f^- = (93.34, 110.00, 6.26)$									

Table 3. Solutions for unequal weights ($w_1 = 0.3$, $w_2 = 0.5$, $w_3 = 0.2$)

	f_1	f_2	f_3	x_1	x_2	x_3	x_4	x_5	x_6
f^* (PIS)	540.00	8.44	2.24						
f^- (NIS)	93.34	110.00	6.26						
$p = 1$									
min d_1^{PIS}	400.97	17.91	2.43	1.79	0.00	0.00	10.00	0.00	4.00
(AR)	(68.9%)	(90.7%)	(95.1%)						
max d_1^{NIS}	400.97	17.91	2.43	1.79	0.00	0.00	10.00	0.00	4.00
(AR)	(68.9%)	(90.7%)	(95.1%)						
$p = 2$									
min d_2^{PIS}	412.43	18.70	2.89	1.87	0.00	0.00	10.00	8.68	4.00
(AR)	(71.4%)	(89.9%)	(83.9%)						
max d_2^{NIS}	385.88	14.42	3.20	0.59	0.32	0.02	9.95	6.29	3.98
(AR)	(65.5%)	(94.1%)	(76.2%)						
	$(d_2^{PIS})^* = 0.1046$	$(d_2^{NIS})' = 0.5255$							
	$(d_2^{PIS})' = 0.1177$	$(d_2^{NIS})^* = 0.5322$							
max α	402.91	16.32	3.00	1.38	0.12	0.00	10.00	7.60	4.00
(AR)	(69.3%)	(92.2%)	(81.1%)						
$p = \infty$									
min d_∞^{PIS}	426.44	23.93	3.07	2.39	0.00	0.00	10.00	10.00	4.00
(AR)	(74.6%)	(84.7%)	(79.4%)						
max d_∞^{NIS}	377.34	40.55	2.42	2.49	0.13	0.00	7.31	0.00	4.00
(AR)	(63.6%)	(68.4%)	(95.4%)						
	$(d_\infty^{PIS})^* = 0.0763$	$(d_\infty^{NIS})' = 0.1587$							
	$(d_\infty^{PIS})' = 0.1581$	$(d_\infty^{NIS})^* = 0.1908$							
max α	407.77	24.67	2.52	2.38	0.00	0.01	9.51	0.00	4.00
(AR)	(70.4%)	(84.0%)	(92.9%)						

AR (achieved rate) $= (f_i - f_i^-)/(f_i^* - f_i^-)$; nonlinear programming problems ($p = 2$) are solved by KSU–SUMT.

Step 6.1. Determine $(d_\infty^{PIS})^*$ by solving the following problem [see equation (15)]:

$$\min d_\infty^{PIS}$$

s.t.

$$0.3 * \left[\frac{540.00 - f_1(x)}{540.00 - 03.34}\right] \leqslant d_\infty^{PIS}$$

$$0.5 * \left[\frac{f_2(x) - 8.44}{110.00 - 8.44}\right] \leqslant d_\infty^{PIS}$$

$$0.2 * \left[\frac{f_3(x) - 2.24}{6.26 - 2.24}\right] \leqslant d_\infty^{PIS}$$

$$x \in X$$

whose solution is given in Table 3. $(d_\infty^{PIS})^* = 0.0763$. And, determine $(d_\infty^{NIS})^*$ by solving the following problem [see equation (16)]:

$$\max d_\infty^{NIS}$$

s.t.

$$0.3 * \left[\frac{f_1(x) - 93.34}{540.00 - 93.34}\right] \geqslant d_\infty^{NIS}$$

$$0.5 * \left[\frac{110.00 - f_2(x)}{110.00 - 8.44}\right] \geqslant d_\infty^{NIS}$$

$$0.2 * \left[\frac{6.26 - f_3(x)}{6.26 - 2.24}\right] \geqslant d_\infty^{NIS}$$

$$x \in X$$

whose solution is shown in Table 3. $(d_\infty^{NIS})^* = 0.1908$. Then, we can compute $(d_\infty^{PIS})' = 0.1581$ and $(d_\infty^{NIS})' = 0.1587$ by equations (9) and (10).

Step 6.2. By using equations (11) and (12), we have

$$\mu_1(x) = \begin{cases} 1 & \text{if } d_\infty^{\mathrm{PIS}}(x) < 0.0763 \\ \dfrac{0.1581 - d_\infty^{\mathrm{PIS}}(x)}{0.1581 - 0.0763} & \text{if } 0.0763 \leqslant d_\infty^{\mathrm{PIS}}(x) \leqslant 0.1581 \\ 0 & \text{if } d_p^{\mathrm{PIS}}(x) > 0.1581 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if } d_\infty^{\mathrm{NIS}}(x) > 0.1908 \\ \dfrac{d_\infty^{\mathrm{NIS}}(x) - 0.1587}{0.1908 - 0.1587} & \text{if } 0.1587 \leqslant d_\infty^{\mathrm{NIS}}(x) \leqslant 0.1908 \\ 0 & \text{if } d_p^{\mathrm{NIS}}(x) < 0.1587. \end{cases}$$

Step 6.3. Solve the problem of equation (19). That is:

$$\max \alpha$$

s.t.

$$\begin{aligned} &(0.1581 - d_\infty^{\mathrm{PIS}})/0.0818 \geqslant \alpha \\ &(d_\infty^{\mathrm{NIS}} - 0.1587)/0.0321 \geqslant \alpha \\ &0.3[(540.00 - f_1(x))/446.66] \leqslant d_\infty^{\mathrm{PIS}} \\ &0.5[(f_2(x) - 8.44)/101.56] \leqslant d_\infty^{\mathrm{PIS}} \\ &0.2[(f_3(x) - 2.24)/4.02] \leqslant d_\infty^{\mathrm{PIS}} \\ &0.3[(f_1(x) - 93.34)/446.66] \geqslant d_\infty^{\mathrm{NIS}} \\ &0.5[(110.00 - f_2(x))/101.56] \geqslant d_\infty^{\mathrm{NIS}} \\ &0.2[(6.26 - f_3(c))/4.02] \geqslant d_\infty^{\mathrm{NIS}} \\ &x \in X \end{aligned}$$

where X is the constraint set of equation (20). The compromise solution is shown in Table 3.

With the same weights, we also provide solutions for the cases of $p = 1$ and 2 in Table 3. Table 4 presents the solutions for $p = 1$, 2 and ∞ when $w_1 = w_2 = w_3 = 1/3$.

By referring to Tables 3 and 4, we can see that $p = \infty$ provides more harmonious achieved rates

Table 4. Solutions for equal weights ($w_1 = w_2 = w_3 = 1/3$)

	f_1	f_2	f_3	x_1	x_2	x_3	x_4	x_5	x_6
f^* (PIS)	540.00	8.44	2.24						
f^- (NIS)	93.34	110.00	6.26						
$p = 1$									
min d_1^{PIS}	413.12	22.97	2.54	2.30	0.00	0.00	10.00	0.00	4.00
(AR)	(71.6%)	(85.7%)	(92.3%)						
max d_1^{NIS}	413.12	22.97	2.54	2.30	0.00	0.00	10.00	0.00	4.00
(SR)	(71.6%)	(85.7%)	(92.3%)						
$p = 2$									
min d_2^{PIS}	413.12	22.97	2.54	2.30	0.00	0.00	10.00	0.00	4.00
(AR)	(71.6%)	(85.7%)	(92.3%)						
max d_2^{NIS}	413.12	22.97	2.54	2.30	0.00	0.00	10.00	0.00	4.00
(SR)	(71.6%)	(85.7%)	(92.3%)						
$p = \infty$									
min d_∞^{PIS}	441.16	30.92	3.13	3.09	0.00	0.00	10.00	8.14	4.00
(AR)	(77.9%)	(77.9%)	(77.9%)						
min d_∞^{NIS}	441.16	30.92	3.13	3.09	0.00	0.00	10.00	8.14	4.00
(AR)	(77.9%)	(77.9%)	(77.9%)						

The same solution for min d_2^{PIS} and max d_2^{NIS} is just a special case.

of objectives. In the aspect of group decision making, harmony between decision makers (or objectives) is considered as one of the most important criteria. Thus we recommend $p = \infty$ for solving MODM problems which differs from Yoon's [14] conclusion that the distance function $d_p s$ becomes less specific or less credible as p increases and that $p = 1$ (similar to robust approach in statistics) should be preferred. A comparative study of $p = 1, 2,$ and ∞ can be found in Yoon [14].

CONCLUDING REMARKS

To solve a MODM problem, TOPSIS for MODM proposed that the chosen solution should be as close to PIS as possible and as far away from NIS as possible. The original k objective problem is reduced to a two objective auxiliary problem. To resolve the conflict existing between two distance objective functions, we use membership functions to model the relationship between distance measures and satisfactory/preference levels. The problem becomes how to balance the degrees of satisfaction of the distance from PIS and the distance from NIS. Here, we used the max–min operator to make trade-offs and determine a compromise solution. This solution is then chosen as a compromise solution of the original k objective problem. It cannot be over-emphasized that this chosen solution is "as close to PIS as possible and as far away from NIS as possible" in the original k objective space and that the satisfactory level of this solution is measured by α. The criteria of "as close to PIS as possible" and "as far away from NIS as possible" are similar to "as much profit as possible" and "as little risk as possible".

Indeed the chosen solution will depend upon the chosen p value. Different p values provide decision maker(s) different solutions. Among various p values, $p = 1, 2$ and ∞ are not only practically operationally important, but also well-known standard values in the fields of control theory and MCDM. $p = 1$, 2, and ∞ have their unique merit. $p = 1$ provides better credibility than others in the measuring concept, and emphasizes the sum of individual distances (regrets for d_p^{PIS} and rewards for d_p^{NIS}) in the utility concept. In the distance aspect, the Euclidean distance ($p = 2$) seems to be more acceptable. And, $p = \infty$ emphasizes the maximum of individual regrets and the minimum of individual rewards and provides more harmonious achieved rates for each objective. In the aspect of group decision making (each objective is considered as a decision maker), harmony between objectives is considered as one of the most important criteria. Thus $p = \infty$ is preferred.

Finally, it should be noted that the max–min operator used here is not compensatory. That is: we cannot increase the satisfactory level of "as close to PIS as possible" by decreasing the satisfactory level of "as far away from NIS as possible". This fact may not be consistent with the decision maker's rationality. Thus, future studies should apply compensatory operators (such as min-bounded sum operator, τ-operator, fuzzy "and", weighted mean operator, etc.) to obtain our compromise solution.

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