Analiz IV : Vize çözümler

① a) $F:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $\qquad F=(M,N,P)$

$$\nabla\times F = \left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z}-\frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$$

$$\nabla\cdot(\nabla\times F) = \frac{\partial}{\partial x}\left[\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right]+\frac{\partial}{\partial y}\left[\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right]+\frac{\partial}{\partial z}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]$$

$$= \frac{\partial^2 P}{\partial x\partial y}-\frac{\partial^2 N}{\partial x\partial z}+\frac{\partial^2 M}{\partial y\partial z}-\frac{\partial^2 P}{\partial y\partial x}+\frac{\partial^2 N}{\partial z\partial x}-\frac{\partial^2 M}{\partial z\partial y}$$

$$=0.$$

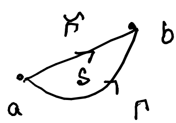
b) $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$

$$\nabla f = \left(\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z}\right)$$

$$\nabla\times\nabla f = \underset{1}{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)}-\underset{1}{\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)}+\overset{2}{\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)}-\overset{2}{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)}+\underset{3}{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}-\underset{3}{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}$$

$$=0.$$

② $F:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $\qquad F=\nabla f$ şeklinde yazılabilir.



$\phi(t)$

$$\tilde{\Gamma}=\{(1-t)a+tb \ ;\ t\in[0,1]\}$$

Γ : herhangi bir yol

Stokes : $$\int_S (\nabla\times F)\cdot n\, d\sigma = \int_{\Gamma} F\cdot T\, ds - \int_{\tilde{\Gamma}} F\cdot T\, ds$$

Ama $\nabla\times\nabla f = 0 \Rightarrow$ Her eğri Γ için

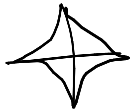
$$\int_{\Gamma} F\cdot T\, ds = \int_0^1 F(\phi(t))\cdot\phi'(t)\, dt$$

$$= \int_0^1 \nabla f\left((1-t)a+tb\right)\cdot(b-a)\, dt$$

$$g(t)=f[(1-t)a+tb] \Rightarrow g'(t)=\nabla f[(1-t)a+tb]\cdot(b-a)$$

$$\Rightarrow \int_{\Gamma} F\cdot T\, ds = \int_0^1 g'(t)\, dt = g(1)-g(0) = f(b)-f(a) \ \square$$

(2) a) $x^{2/3}+y^{2/3}=1$



Parametrization : $\phi(t) = (\cos^3(t), \sin^3(t)) : 0 \leq t \leq 2\pi$

$$\|\phi'(t)\| = (9\cos^4 t \sin^2 t + 9\sin^4 t + \cos^2 t)^{1/2}$$

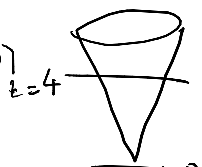
$$= 3\,|\sin t \cos t|\,.$$

Yay uzunluğu $= \displaystyle\int_0^{2\pi} 3\,|\sin t \cos t|\, dt$

$$= 12 \int_0^{\pi/2} \sin t \cos t \, dt$$

$$= 12 \sin^2 t \Big|_0^{\pi/2} = 12 \quad \square$$

b)



$x^2+y^2=4z^2$

Ağırlık $= \displaystyle\iint_S \delta(x,y,z)\, d\sigma$

$$\phi(x,y) = (x, y, 2\sqrt{x^2+y^2})$$

$$\frac{\partial\phi}{\partial x} = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right) \qquad \frac{\partial\phi}{\partial y} = \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\left\| \frac{\partial\phi}{\partial x} \times \frac{\partial\phi}{\partial y} \right\| = \left\| \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right) \right\|$$

$$= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{2}$$

$$\boxed{\pi \cdot \frac{2\sqrt{2} \cdot (4\sqrt{2})^5}{5}}$$

Ağırlık $= \displaystyle\iint_{0 \leq x^2+y^2 \leq 32} x^2 \cdot 2 \cdot \sqrt{x^2+y^2} \cdot \|\phi_x \times \phi_y\|\, dx\, dy$

$$= \iint_{0 \leq x^2+y^2 \leq 32} 2x^2 \sqrt{x^2+y^2} \cdot \sqrt{2}\, dx\, dy = \int_{\theta=0}^{2\pi} \int_{r=0}^{4\sqrt{2}} 2\sqrt{2}\, r^4 \cos^2\theta \, d\theta\, dr$$

c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

$u = \frac{x}{a} \quad v = \frac{y}{b} \quad w = \frac{z}{c}$

$$\underset{E}{\iiint} dx\,dy\,dz = \underset{u^2+v^2+w^2 \leq 1}{\iiint} \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

Ellipsoid

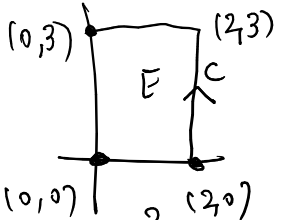
Jacobian $\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \right|$

$$= \left| \det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \right|$$

$$= |abc|$$

$$\underset{E}{\iiint} dx\,dy\,dz = \underset{u^2+v^2+w^2 \leq 1}{\iiint} |abc|\, du\,dv\,dw = \frac{4\pi}{3} \cdot |abc|$$

3) a)



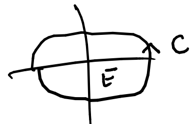
$F(x,y) = (e^y, \ln(x+1))$

$$\int_C F \cdot T\, ds \underset{\text{Green}}{=} \iint_E \left[\frac{\partial}{\partial x} \ln(x+1) - \frac{\partial}{\partial y} e^y \right] dx\,dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{3} \left(\frac{1}{x+1} - e^y \right) dx\,dy$$

$$= 3\ln 3 - 2(e^3 - 1)$$

b) $x^4+y^{20}\leq 1$



$$F(x,y) = (e^x \sin y, -e^x \cos y)$$

$$\int_C F\cdot T\, ds = \iint_E \left[\frac{\partial}{\partial x}\left(-e^x\cos y\right) - \frac{\partial}{\partial y}\left(e^x \sin y\right)\right] dx\,dy$$

$$= \iint_{x^4+y^{20}\leq 1} -2e^x \cos y \; dx\,dy$$

Maalesef bu soruda hata var!

$F(x,y) = (e^x\cos y, -e^x\sin y)$ ya da

$F(x,y) = (e^x\cos y, e^x\sin y)$ olacaktı

Kusura bakmayınız !

---

4 a)

$\Phi(s,t) = (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)$

$0\leq\phi\leq 2\pi$

$0\leq\theta\leq \sin^{-1}\left(\frac{b}{a}\right)$

$$\frac{\partial\Phi}{\partial s}\times\frac{\partial\Phi}{\partial t} = (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)$$

(Dik vektör hesaplamak boşa: Küre olduğu için)

$$\iint_S \omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\sin^{-1}\left(\frac{b}{a}\right)} (xz, 1, z)\cdot(a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)\, d\theta\, d\phi$$

$$= \iint \left(a^3\sin^2\theta\cos\theta\cos^2\phi + a\sin\theta\sin\phi + a^2\cos^2\theta\right)$$

$$= \frac{\pi a^3}{3}\sin^3\left(\sin^{-1}\left(\frac{b}{a}\right)\right) + 2\pi a^2\left[\frac{\sin^{-1}\left(\frac{b}{a}\right)}{2} + \frac{\sin\left(2\sin^{-1}\left(\frac{b}{a}\right)\right)}{4}\right]$$

$\square$

5 b) $x=0 \quad y=0 \quad x+2y+3z=1 \quad z>0$ ile sınırlanan

$$\iint_S (\nabla \times (xy, yz, zx)) \cdot n \, d\sigma$$

$$= \iiint_{Int(S)} \nabla \cdot (\nabla \times (xy, yz, zx)) \, dV \qquad \boxed{\text{Gauss Thm}}$$

$= 0$.

5 a)

$$\int_C F \cdot T \, ds = \iint_E (\nabla \times F) \cdot n \, d\sigma$$

$$= \iint (x+2y, -2x-y, -1) \cdot (x, y, z) \, dx \, dy$$

$$\left\{ \begin{array}{l} x^2+y^2+z^2=1 \\ z>, x^2+y^2 \end{array} \right.$$

kesişim

$z+z^2=1$

$z = \frac{-1 \pm \sqrt{5}}{2}$

-ve olamaz

$z: \frac{\sqrt{5}-1}{2}$

$$= \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} (x^2+2xy-2xy-y^2-z)$$

$$= \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} x^2-y^2-z$$

θ limit : $0 \le \theta \le \cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$

$$= \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} \left[\underbrace{(\cos^2\phi - \sin^2\phi)}_{\int = 0} \sin^2\theta - \cos\theta \right] d\theta \, d\phi$$

$$= -\sin\left[\cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right)\right] \quad \square$$

4b) Gauss Thm

$$\iint_S w = \iiint_V \nabla \cdot (x+y^2+z^2,\ x^2+y+z^2,\ x^2+y^2+z)\, dx\, dy\, dz$$

$$= \iiint 3\, dx\, dy\, dz$$

$$= 3 \times \left\{ x-y \geq 1,\ x^2+y^2+z^2 \leq 4 \right\}$$

nın hacmi

$$= 3 \times \left\{ z \geq \sqrt{2},\ x^2+y^2+z^2 \leq 4 \right\}$$ nın hacmi

(simetriden dolayı)

$$= 3 \times \iint_{x^2+y^2 \leq 2} \left(\sqrt{4-x^2-y^2} - \sqrt{2} \right) dx\, dy$$

$x = 2r\cos\theta$

$y = 2r\sin\theta$

$x^2+y^2 \leq 2$

$\Rightarrow r \leq \frac{1}{\sqrt{2}}$

$$= 3\left[\left(\int_{r=0}^{1/\sqrt{2}} \int_{\theta=0}^{2\pi} 2r \cdot 4r \cdot dr\, d\theta \right) - 2\sqrt{2}\pi \right]$$

$$= (12-6\sqrt{2})\pi \ \square$$