

Parametrizasyon.
Yüzey.

Örnek 4.5 S^2

$$S^2 = \left\{ (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) : \begin{array}{l} 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

→ $(0, R, 0) + (0, r \cos \theta, r \sin \theta)$

$$\begin{bmatrix} \cos \phi & \sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ R + r \cos \theta \\ r \sin \theta \end{bmatrix}$$



rekt

Tanım 4.1 (Eğriler).

$$\Gamma = \left\{ \gamma(t) : t \in [a, b] \right\}$$

$\gamma: \mathbb{R} \rightarrow \mathbb{R}^n$ türlenebilir fonksiyon.

Γ : Eğri denir

γ : Eğri'nin parametrizasyon.

Örnek 4.2

$$S^1 \subset \mathbb{R}^2$$

$$\Gamma = \left\{ (\cos t, \sin t) : t \in [0, 2\pi] \right\}$$



Not: Aynı eğrinin bir den fazla parametrizasyon olabilir

$$S^1 = \left\{ (\cos t, \sin t) : t \in [0, 2\pi] \right\}$$

$$S^1 = \left\{ (\cos t^3, \sin t^3) : t \in [0, (2\pi)^{1/3}] \right\}$$

Göruntu aynı: Parametrizasyon farklı

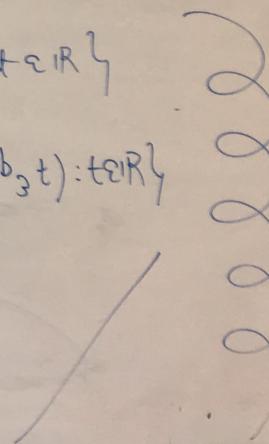
Örnek 4.3

Mehmet

$$H = \left\{ (\cos t, \sin t, t) : t \in \mathbb{R} \right\}$$

Düzenli

$$D = \left\{ (a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t) : t \in \mathbb{R} \right\}$$



emko

Tanım 4.4 (Yüzeyler)

$$S = \{ f(u, v) : (u, v) \in U \subset \mathbb{R}^2 \}$$

$$f(u, v) : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^n$$

f : türevlenebilir.

Parametrizasyon:

Vüzey:

Örnek 4.5 S^2

$$S^2 = \left\{ (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \mid \begin{array}{l} 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

Tanım 4.1 (Eğriler).

$$\gamma = \{ \gamma(t) : t \in [a, b] \}$$

$\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ türevlenebilir
fonksiyon.

Örnek 4.6 3D yüzeye

$$P = \{ (R \cos \phi + r \cos \theta \cos \phi, R \sin \phi + r \cos \theta \sin \phi, r \sin \theta) \mid \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2\pi \end{array} \}$$

$$(r \cos \phi, r \sin \phi, r)$$

$$\begin{bmatrix} \cos \phi & \sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ r \cos \theta \\ r \sin \theta \end{bmatrix}$$



Not: Aynı eğrinin bir den fazla parametrizasyon olabilir.

$$\gamma' = \{ (\cos t, \sin t) : t \in [0, 2\pi] \}$$

$$\gamma' = \{ (\cos t^3, \sin t^3) : t \in [0, (2\pi)^3] \}$$

Görünüş aynı: Parametrizasyon farklı.

Lemma 4.8

$F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$
vektor alan denir.

$\mathcal{I}(a)$: 'a' noktasında bir vektör.

Fizik'tan gelen temel genelik: Sarij, 'a' naktadan 'b' naktaya P üzerinde taraşsak, harcanan enerji

$$\int_a^b F(Y(t)) \cdot Y'(t) dt$$

Not $\gamma_1, \gamma_2 P$ için farklı parametrizeasyon olsun

$$\sigma = \left\{ \begin{array}{l} \gamma(t) = t \in [a, b] \\ \delta(t) = t \in [c, d] \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \gamma(t) = t \in [c, d] \end{array} \right\}$$

O zaman

$$\int_a^b F(\gamma(t)) \cdot \gamma'(t) dt = \int_c^d F(\delta(t)) \cdot \delta'(t) dt$$

$$F(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z))$$

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

$$\mathcal{I}_1 = \int_a^b \left[f(\gamma(t)) \gamma'_1(t) + g(\gamma(t)) \gamma'_2(t) + h(\gamma(t)) \gamma'_3(t) \right] dt$$

$$\mathcal{I}_2 = \int_c^d \left[f(\delta(t)) \delta'_1(t) + g(\delta(t)) \delta'_2(t) + h(\delta(t)) \delta'_3(t) \right] dt \quad (4)$$

Bölüm

$$\int_a^b f(\gamma(t)) \gamma'(t) dt = \int_c^d f(\delta(t)) \delta'(t) dt$$

$$\gamma(a) = \delta(c)$$

$$\gamma(b) = \delta(d)$$

$$t = \gamma^{-1}(\delta(s)) \quad \text{u} \quad \gamma(t) = \delta(s)$$

$$\gamma'(t) dt = \delta'(s) ds$$

$$\Rightarrow \gamma'_1(t) dt = \delta'_1(s) ds$$

$$\int_c^d f(\delta(s)) \delta'_1(s) ds$$

Dolayısıyla, dört eşit.

Örnek 4.7

Zonlu manto

(cone)



$$C = \left\{ (x, y, \sqrt{x^2 + y^2}) : x, y \in \mathbb{R} \right\}$$

Tanım 4.8

$$F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

vektor alan denir.

$F(a)$: 'a' noktada bir vektör.

Not γ_1, γ_2 P içiin farklı parametrizasyon olsun

$$\gamma = \left\{ \gamma(t) : t \in [a, b] \right\}$$

$$= \left\{ \delta(t) : t \in [a, b] \right\}$$

Mantıksal 4.9

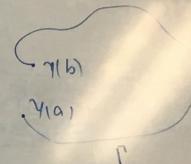
$$E: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

Elektrik alan

q: Bir sayı.

T: Bir eğri.

$$\Gamma = \left\{ \gamma(t) : t \in [a, b] \right\}$$



Fizik'tan gelen temel genelik: 'a' naktadan 'b' noktaya Γ üzerinde taşınan, harcanan enerji

$$\int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$$

$$\delta(t) = (\delta_1(t), \delta_2(t), \delta_3(t))$$

$$\tilde{\gamma}_2 = \int_c^b [f(\delta(t)) \delta'_1(t) + f(\delta(t)) \delta'_2(t) + h(\delta(t)) \delta'_3(t)] dt$$

Toldur

$$\int_a^b f(\gamma(t)) \gamma'(t) dt = \int_c^d f(\delta(t)) \delta'_1(t) dt$$

Özet 4.11

$$\text{Eğri } \Gamma = \left\{ \gamma(t) : t \in [a, b] \right\} \subset \mathbb{R}^n \quad \left. \begin{array}{l} \text{türenlenebilir} \\ \text{vektör alan} : F : \mathbb{R}^n \rightarrow \mathbb{R}^n \end{array} \right\}$$

$$\text{Eğri İntegral} \quad \int_{\Gamma} F \cdot d\Gamma := \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$$

Parametrisasyonla bağımsız.

Örnek 4.12

$$F(x, y) = (xy, y-x)$$

Eğri : (2, 1) ve (3, 9) arasındaki doğru
tanısalı.

$$\int F \cdot d\Gamma \quad \text{hesaplamı 2.}$$

$$\text{aralı } \Gamma = \left\{ \underbrace{(2t+1, 8t+1)}_{\gamma(t)} : t \in [0, 1] \right\}$$

$$\int_0^1 F(\gamma(t)) \cdot \gamma'(t) dt$$

$$= \int_0^1 ((2t+1)(8t+1), 8t+1-2t-1) \cdot (2, 8) dt$$

$$= \int_0^1 \left[2(2t+1)(8t+1) + 8(8t) \right] dt$$

Örnek 4.13

$$\Gamma : \frac{y^2}{4} + 2z^2 = 1 \quad \cap \quad x = -1$$

oluş x-eksen e göre saat yönünün ters yönde

6

$$= \int_0^1 ((2t+1)(8t+1), 8t+1-2t-1) \cdot (2,8) dt$$

6

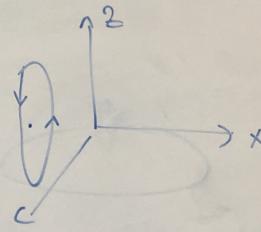
$$= \int_0^1 [2(2t+1)(8t+1) + 8(6t)] dt$$

$$= \int_0^1 (32t^2 + 68t + 2) dt$$

$$= \frac{32}{3} + \frac{68}{2} + 2$$

örnek 4.B
 $P : \frac{y^2}{4^2} + \frac{z^2}{2^2} = 1 \cap x = -1$

olu x-eksen e göre saat yönünün ters yönde.



$$\gamma = \left\{ \gamma(t) = \left(-1, \cos t, \frac{\sin t}{\sqrt{2}} \right) : t \in [0, 2\pi] \right\}$$

$$f(x, y, z) = \left(\sqrt{x^3 + y^3 + 5}, z, x^2 \right)$$

Cevap

$$\gamma'(t) = (0, -\sin t, \frac{\cos t}{\sqrt{2}})$$

$$F(\gamma(t)) = (*, \frac{\sin t}{\sqrt{2}}, 1)$$

$$\int_{\Gamma} F \cdot d\Gamma = \int_0^{2\pi} \left(\frac{-\sin^2 t}{\sqrt{2}} + \frac{\cos t}{\sqrt{2}} \right) dt$$

$$= -\frac{\pi}{\sqrt{2}} \quad \square$$

Örnek 4.14

$$\Gamma: z = x^2 + y^2 \quad \text{ve} \quad x^2 + y^2 + z^2 = 1$$

+z eksen'e göre saat yönünün yönü.

$$F = (1, x+y, x^2+y^2+z^2)$$

$$\Gamma = \left\{ (\lambda \cos t, \lambda \sin t, \lambda^2) : \begin{array}{l} \lambda^4 + \lambda^2 = 1 \\ t \in [0, 2\pi] \end{array} \right\} \quad \lambda^2 = \frac{1 \pm \sqrt{5}}{2}$$

$$\gamma'(t) = (-\lambda \sin t, \lambda \cos t, 0)$$

$$F(\gamma(t)) = (1, \lambda \sin t + \lambda \cos t, *)$$

$$\int_{\Gamma} F \cdot d\Gamma = \int_0^{2\pi} \lambda^2 \sin t \cos t + \lambda^2 \cos^2 t - \lambda \sin t dt$$

$$= \lambda^2 \pi = \pi \left(\frac{\sqrt{5}-1}{2} \right) \quad \square$$

$$\frac{-1-\sqrt{5}}{2} \text{ olmaz}$$

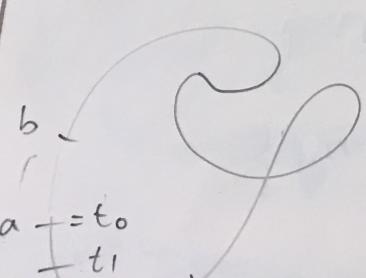
$$\int_0^{2\pi} \sin t dt = 0$$

$$\int_0^{2\pi} \cos^2 t dt = \pi$$

Tanım 4.15 Eğrinin uzunluğu:

$$l = \int \gamma(t) : t \in [a, b]$$

$$\text{uzunluk} (l) \approx \sum |\gamma(t_{i+1}) - \gamma(t_i)|$$



Ortalama değer THM

$$|\gamma(t_{i+1}) - \gamma(t_i)| \approx |\gamma'(t_i)| (t_{i+1} - t_i)$$

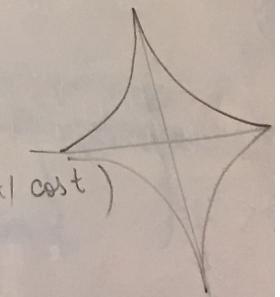
4.16 örneği - Uzunluk hesaplayınız.

$$a), \quad l = \int (\cos^3 t, \sin^3 t) : t \in [0, 2\pi]$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

$$\gamma'(t) = (3 \cos^2 t \sin t, -3 \sin^2 t \cos t)$$

$$|\sin t \cos t|$$



Cevap

$$\gamma'(t) = \left(v, -\sin t, \frac{\cos t}{\sqrt{2}} \right)$$

$$F(\gamma(t)) = \begin{pmatrix} * & \frac{\sin t}{\sqrt{2}} & 1 \end{pmatrix}$$

$$\int_{\Gamma} F \cdot dT = \int_0^{2\pi} \left(-\frac{\sin^2 t}{\sqrt{2}} + \frac{\cos t}{\sqrt{2}} \right) dt \\ = -\frac{\pi}{\sqrt{2}} \quad \square$$

örnek 4.14

$$P: z = x^2 + y^2 \quad \text{ve} \quad z^2 + y^2 + x^2 = 1$$

$+z$ eksen'e göre saat yönünün yönü.

$$F = (1, x+y, x^2+y^2)$$

$$P = \left\{ (x \cos t, x \sin t, x^2) : \begin{array}{l} x^4 + x^2 = 1 \\ t \in [0, 2\pi] \rightarrow 0 \end{array} \right\} \quad \begin{array}{l} x^4 + x^2 = 1 \\ x^2 = -1 \pm \frac{\sqrt{5}}{2} \end{array}$$

$$\gamma(t) = (-x \sin t, x \cos t, 0)$$

$$-\frac{1-\sqrt{5}}{2} \text{ olmaz}$$

$$F(\gamma(t)) = (1, x \sin t + x \cos t, *)$$

$$x^2 = \frac{\sqrt{5}-1}{2} \quad \int_{0}^{2\pi} \sin t dt = 0$$

$$\int_{\Gamma} F \cdot dT = \int_0^{2\pi} x^2 \sin t \cos t + x^2 \cos^2 t - x \sin t dt$$

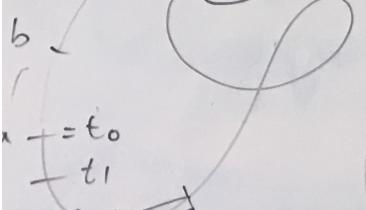
$$x^2 = \frac{\sqrt{5}-1}{2} \quad \int_{0}^{2\pi} \sin^2 t dt = \pi$$

$$= x^2 \pi = \pi \left(\frac{\sqrt{5}-1}{2} \right) \quad \square$$

Tanım 4.15 Eğri nun uzunluğu:

$$L = \{ \gamma(t) : t \in [a, b] \}$$

$$\text{uzunluğu } (L) \approx \sum |\gamma(t_{i+1}) - \gamma(t_i)|$$



Ortalama değer Thm

$$|\gamma(t_{i+1}) - \gamma(t_i)| \approx \gamma'(t_i) (t_{i+1} - t_i)$$

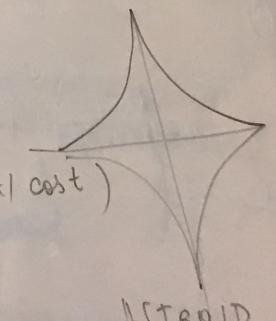
4.16 örneğ = uzunluk hesaplayınız:

$$a), \quad P = \{ (\cos^3 t, \sin^3 t) : t \in [0, 2\pi] \}$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

$$\gamma'(t) = (3 \cos^2 t \sin t, -3 \sin^2 t \cos t)$$

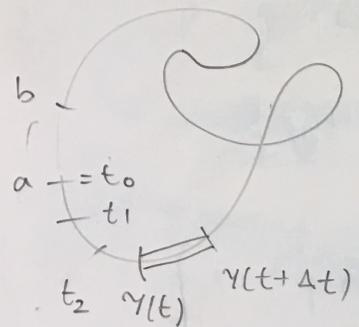
$$|\gamma'(t)| = 9 |\sin t \cos t|$$



$$\begin{aligned} \gamma'(t) &= (1, \alpha \sin t + \alpha \cos t, *) \\ \int F \cdot d\gamma &= \int_0^{2\pi} (\alpha^2 \sin^2 t + \alpha^2 \cos^2 t - \alpha \sin t) dt \\ &= \alpha^2 \pi = \pi \left(\frac{\sqrt{5}-1}{2} \right) \end{aligned}$$

Tanım 4.15 Eğrinin uzunluğu:

$$l = \int |\gamma(t)| : t \in [a, b]$$



$$\text{Uzunluğu } (l) \approx \sum |\gamma(t_{i+1}) - \gamma(t_i)|$$

Ortalama değer THM

$$|\gamma(t_{i+1}) - \gamma(t_i)| \approx |\gamma'(t_i)| (t_{i+1} - t_i)$$

$$\begin{aligned} &\approx \sum_b^a |\gamma'(t_i)| (t_{i+1} - t_i) \\ &\rightarrow \int_a^b |\gamma'(t)| dt \end{aligned}$$

$$\begin{aligned} \text{Tanım } l(\gamma) &:= \int_a^b |\gamma'(t)| dt \\ (\text{Yay Uzunluğu}) \end{aligned}$$

4.16 örneğ Uzunluk hesaplayınız.

$$\text{a)} \quad \gamma = \{(x^3(t), y^3(t)) : t \in [0, 2\pi]\}, \quad \begin{cases} x^{2/3} + y^{2/3} = 1 \end{cases}$$

$$\gamma'(t) = (3 \cos^2 t \sin(t), -3 \sin^2(t) \cos(t))$$

$$|\gamma'(t)| = \sqrt{9 \sin^2 t + 9 \cos^2 t} = 3$$

$$l(\gamma) = \int_0^{2\pi} 3 dt$$

$$= 36$$

ASTROID

Yüzey Integral, Flows Integraller
Tanım 4.17.

$$S = \left\{ \phi(u, v) : (u, v) \in E \subset \mathbb{R}^2 \right\}$$

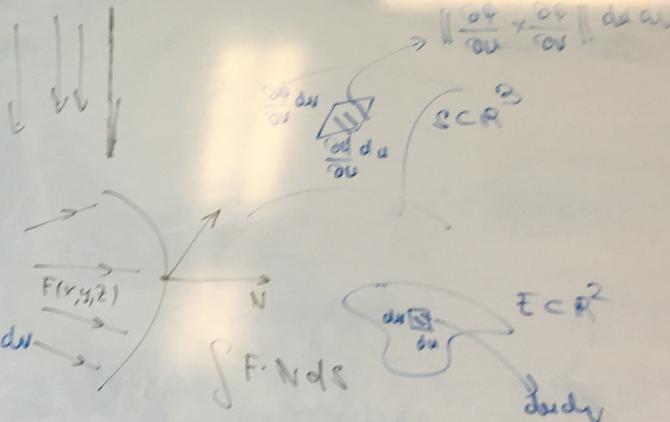
yüzey olsun. (\mathbb{R}^3 ta)

$f : S \rightarrow \mathbb{R}$ sürekli

$$\int_S f := \int f(\phi(u, v)) \left\| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right\| du dv$$

$F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ vektor alam.

$$\int_S F \cdot ds := \int F \cdot \left(\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right) du dv$$



$$P = \{ y(t) : t \in [a, b] \} \subset \mathbb{R}^n$$

özet

$$S = \left\{ \phi(u, v) : u, v \in E \right\} \quad \phi : E \rightarrow \mathbb{R}^3$$

(10)

$$\int_S \int + (\phi(u,v)) \left\| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right\| du dv \rightarrow \int_S F \cdot N \, dS$$

$F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ vektor alan.

$$\int_S F \cdot ds := \int_S F \cdot \left(\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right) du dv$$

$$r = \{ \gamma(t) : t \in [a, b] \} \subset \mathbb{R}^n$$

Eğri

Yay Uzunluğu

$$\int_a^b |\gamma'(t)| dt$$

Eğri Integral
 $f: r \rightarrow \mathbb{R}$
 Sürekli

Yonlu Eğri Integral
 $F: \text{vektor alan}$

$$\int_r f(t) |\gamma'(t)| dt$$

r

$$\int_a^b F \cdot \gamma' dt$$

$$S = \{ \phi(u,v) : u, v \in E \} \quad \phi: E \rightarrow \mathbb{R}^3$$

Yüzey

Yüzey Alan

$$\int \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right|$$

Yüzey

$$\int f \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| \rightarrow \int f(\phi(u,v)) \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| du dv$$

Fluxs

$$\int F \cdot \left(\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right)$$