Tanım 5.1

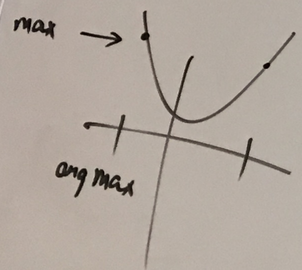
Analiz III

(1)

$f: E \subset \mathbb{R}^n \longrightarrow \mathbb{R}$

optimizasyon: $\max_{x \in E} f(x) \;/\; \min_{x \in E} f(x)$

$$\arg\max_{x \in E} f(x) = a \;\Big|\; f(a) = \max_{x \in E} f(x)$$



örnek: Logistic Regression

$$f(x_1, \dots, x_n) = -\sum y_i \log f(x_i) - \sum (1-y_i) \log(1 - f(x_i))$$

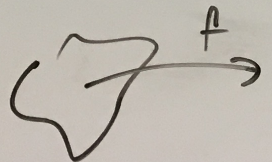
$$f(x_i) = \frac{e^{x_i}}{\sum_{i=1}^{n} e^{x_i}}$$

Amaç: $\min_{\mathbb{R}^n} f(x_1, \dots, x_n)$

$\arg\min_{\mathbb{R}^n} f(x_1, \dots, x_n)$

...ırlatma

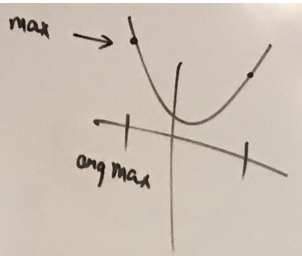
...me 4.1 (Taylor formül)



... açık: $a \in U$

soru a) $\sin(0.57)$ 100 mertebe'ye kadar hesaplayınız.

b) π 100000 mertebe'ye



$\arg\max\limits_{x\in E} f(x) = a \quad \Big| \quad f(a) = \max\limits_{x\in E} f(x)$

örnek : Logistic Regression

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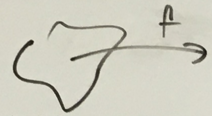
$$f(x_i) = \frac{e^{x_i}}{\sum_{i=1}^{n} e^{x_i}}$$

Amaç: $\min\limits_{\mathbb{R}^n} f(x_1,\dots,y_n)$

$\arg\min\limits_{\mathbb{R}^n} f(x_1,\dots,y_n)$

Hatırlatma

Önerme 4.1 (Taylor formül)



1-d $\quad f: U \longrightarrow \mathbb{R}$, U açık, $a \in U$

f $(n+1)$ defa türevlenebilir olsun. O zaman

$$f(a+h) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}$$

sağlayan $\xi \in [a, a+h]$ vardır.

2

Soru a) $\sin(0.57)$ 100 mertebe'ye kadar hesaplayınız.

b) π 100000 mertebe'ye kadar hesaplayınız.

Cevap

$$\sin(0.57) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}(0.57)^{2k+1} + R_{n+1}$$

(Formül'de $a=0$, $h=0.57$)

$$|R_{n+1}| \le \frac{(0.57)^{2n+3}}{(2n+3)!}$$

n – ne olmak lazım ki $|R_{n+1}| < 10^{-100}$

(1)

$$\frac{(0.57)^{2n+3}}{(2n+3)!} \leq 10^{-100} \qquad n! \geq \left(\frac{n}{e}\right)^n$$

$$\frac{(0.57)^{2n+3}}{(2n+3)!} \leq \frac{(0.57)^{2n+3}}{\left(\frac{2n+3}{e}\right)^{2n+3}} = \left[\frac{e\,(0.57)}{2n+3}\right]^{2n+3}$$

$$\Uparrow$$

$$(2n+3)\log_{10}\left[\frac{e\cdot(0.57)}{2n+3}\right] \leq -100$$

$$(2n+3)\log(2n+3) - (2n+3)(0.2) \geq 100$$

$n = 50$ ise olur.

sin (0.57) 'nin ilk 100 mertebe hesaplamak için (2)

$$\sum_{k=0}^{50} \frac{(-1)^{2k+1}}{(2k+1)!} (0.57)^{2k+1}$$

$\square$

$$\arcsin(1) = \frac{\pi}{2}$$

$\hookrightarrow f$

$$f(1) = \sum^{n} \frac{f^{(k)}(0)}{\ } 1^k + R_{n+1}$$

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$

$f(x) = x^t$ (4)

$$\checkmark \quad (1+y)^t = \sum^{\infty} \binom{t}{k} y^k \qquad |y| < 1$$

$$(2n+3)\log_{10}\left[\frac{e\cdot(0.57)}{2n+3}\right] \leq -100$$

$$(2n+3)\log(2n+3) - (2n+3)(0.2) \geq 100$$

$n = 50$ ise olur.

$$\arcsin(1) = \frac{\pi}{2}$$

$\hookrightarrow f$

$$f(1) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} 1^k + R_{n+1}$$

$$\boxed{\begin{array}{l} |R_{n+1}| = \dfrac{t(t-1)\cdots(t-k)}{(k+1)!} |\xi|^{n+1} \\ \xi \in [0,y] \\ |R_{n+1}| \xrightarrow[n\to\infty]{} 0 \end{array}}$$

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$

$f(x) = x^t$ (t)

$\checkmark$ $$(1+y)^t = \sum_{k=0}^{\infty} \binom{t}{k} y^k \qquad |y| < 1$$

<u>İspat</u>

$$f(1+y) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!} y^k + R_{n+1}$$

$$f(x) = x^t$$

$$f^{(k)}(x) = t(t-1)\cdots(t-k+1)\, x^{t-k}$$

$$f^{(k)}(1) = t(t-1)\cdots(t-k+1)$$

$$\Rightarrow (1+y)^t = \sum_{k=0}^{n} \binom{t}{k} y^k + R_{n+1}$$

Analiz III

(5)

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{2k} \binom{-1/2}{k}$$

$|x| < 1 \Rightarrow$ yakınsar

$$\arcsin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \binom{-1/2}{k}$$

Arcsin(x)'nın Taylor seri.

$\boxed{\text{Neden?}}$

$$\arcsin^{(2k+1)}(0) = \frac{(-1)^k \binom{-1/2}{k} k!}{2k+1}$$

$$\arcsin^{(2k)}(0) = 0$$

Taylor Formül ⇒

$$\arcsin(1) = \sum_{k=0}^{n} \frac{(-1)^k \binom{-1/2}{k}}{(2k+1)} + R_{n+1}$$

$$|R_{n+1}| \leq \max_{0 \leq \xi \leq 1} \frac{\binom{-1/2}{n+1} |\xi|^{2n+3}}{(2n+3)}$$

$$= \frac{\binom{-1/2}{n+1}}{(2n+3)} \approx \frac{1}{2n+3}$$

Eğer $|R_{n+1}| < 10^{-100000}$

$\Rightarrow n \gtrsim 10^{100000}$

$\arcsin(1) = \frac{\pi}{2}$ 'nin 100,000 mertebe için $10^{100,000}$ terim toplamak lazım.

(6)

Daha İyi (Pratik) yöntem?

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$e^{-N^2} \leq 10^{-100000}$$

$$\Rightarrow N^2 \geq 100000 \log_e 10 \quad (\approx 2.3)$$

$$\Rightarrow N \geq 500$$

$$e^{-x^2} \leq x e^{-x^2}$$

Sonuç, $\sqrt{\pi}$ hesaplamak için $\int^{500} e^{-x^2} dx$

$\infty \quad 2 \quad \infty \quad 2$

$k=0$ $\quad 2k+1$

$$\arcsin^{(2k+1)}(0) = \frac{(-1)^k \binom{-1/2}{k} k!}{2k+1}$$

$$\arcsin^{(2k)}(0) = 0$$

seri.

$\boxed{\text{Neden?}}$

$(2n+3)$

Eğer $|R_{n+1}| < 10^{-100000}$

$\Rightarrow n \gtrsim 10^{100000}$

$\arcsin(1) = \frac{\pi}{2}$'nin 100,000 mertebe için $10^{100,000}$ terim toplamak lazım.

Daha İyi (Pratik) yöntem?

6

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$$\int_N^\infty e^{-x^2}\,dx \le \int_N^\infty x e^{-x^2}\,dx \qquad e^{-x^2} \le x e^{-x^2}, \; x \ge 1$$

$$= \frac{e^{-N^2}}{2} \le e^{-N^2}$$

$$\left| \int_0^N e^{-x^2}\,dx - \frac{\sqrt{\pi}}{2} \right| = \left| \int_N^\infty e^{-x^2}\,dx \right| \le e^{-N^2}$$

$$e^{-N^2} \le 10^{-100000}$$

$$\Rightarrow N^2 \ge 100000 \log_e 10 \quad (\to \approx 2.3)$$

$$\Rightarrow N \ge 500$$

Sonuç: $\frac{\sqrt{\pi}}{2}$ hesaplamak için (ilk 100,000 mertebe) $\int_0^{500} e^{-x^2}\,dx$ yeter.

$$f(y) = \int_0^y e^{-x^2}\,dx$$

$$e^{-x^2} = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{k!} + R_{n+1}$$

$$R_{n+1} \le \frac{x^{2n+2}}{(n+1)!}$$

$$\Rightarrow \left| \int_0^{50} e^{-x^2}\,dx - \sum_{k=0}^{n} \frac{(-1)^k (50)^{2k+1}}{k!(2k+1)} \right| \le \frac{(50)^{2n+3}}{(2n+3)(n+1)!}$$

kaç terim lazım?

7

$$\frac{(50)^{2n+3}}{(2n+3)\left(\frac{n}{e}\right)^n} \le 10^{-100000}$$

$$\sim \left(\frac{7500}{n}\right)^n \le 10^{-100000}$$

$n = 20{,}000$ yeter.

Algoritması

1. $$\alpha = \sum_{k=0}^{20{,}000} \frac{(-1)^k (50)^{2k+1}}{k!(2k+1)}$$ hesaplayınız.

2. $4\alpha^2 \longrightarrow \pi$ için iyi yaklaşım

$$|\pi - 4\alpha^2| < 10^{-100000}$$