CFD 3 Assignment 1

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March 8, 2021

1 — Defining the physical and spectral domain

The physical domain is characterized by a 3d space (x, y, z) each with a domain of

$$[0, 2\pi\ (1 - \frac{1}{N})] \tag{1.1}$$

The spectral domain is characterized by ξ_x, ξ_w, ξ_z
Ordered as such:

$$0, 1, 2, 3, ..., \frac{N}{2} - 1, -\frac{N}{2}, -\frac{N}{2} + 1, ..., -3, -2, -1 \tag{1.2}$$

Where N is the number of points on an axis of length 192.

2 — Flow Compressibility

The divergence is described as:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{2.1}$$

To solve in the spectral domain we begin with the given:

$$u = \sum_{\xi_x} \hat{u}(\boldsymbol{\xi}) e^{i\xi \cdot \mathbf{x}} \tag{2.2}$$

This is differentiated with respect to x giving us:

$$\frac{\partial u}{\partial x} = \sum_{\xi_x} i\xi_x \hat{u}(\boldsymbol{\xi}) e^{i\xi \cdot \mathbf{x}} \tag{2.3}$$

Same procedure is done for y, z

As python does not have an RMS I computed it as the following:

$$rms_{spectral} = \sqrt{\sum_{\boldsymbol{\xi}} \left[\left(\frac{\partial u}{\partial x} \right)_i + \left(\frac{\partial v}{\partial y} \right)_j + \left(\frac{\partial w}{\partial z} \right)_k \right]^2} \tag{2.4}$$

The physical domain was easier to calculate as numpy has a second order accurate gradient function.

$$rms_{physical} = \sqrt{\sum_{\boldsymbol{x}} \left[\left(\frac{\partial u}{\partial x} \right)_i + \left(\frac{\partial v}{\partial y} \right)_j + \left(\frac{\partial w}{\partial z} \right)_k \right]^2} \tag{2.5}$$

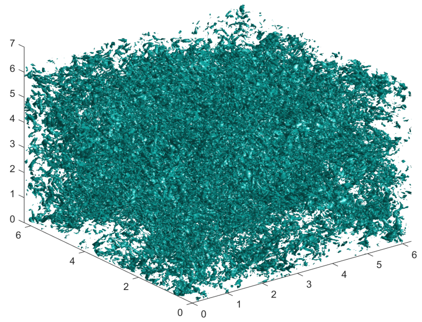
This gave me a spectral rms of $-3.4305539425757516e-08$ and a physical one of 0.655 hence the flow is incompressible. The large difference is due to the nature of the spectral rms having at most machine error whereas the numerical method (using 2 adjacent points) has inherent scheme error dominating. Hence the spectral value is more accurate.

3 — Q-criterion

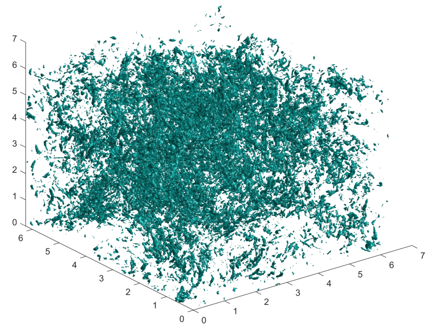
Using the following:

$$Q = -\frac{1}{2}\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \tag{3.1}$$

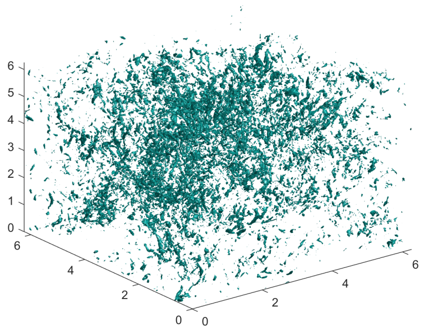
I obtained the following figures :



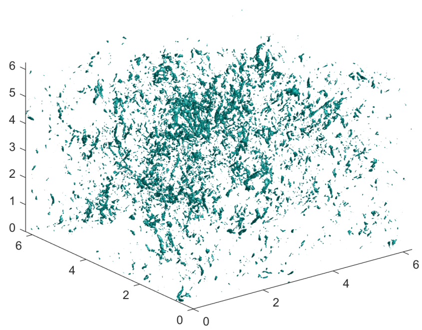
(a) Q = 1000



(b) Q = 2000



(c) Q = 3000



(d) Q = 4000

4 — Box-averaged Turbulence Kinetic Energy

$$
\begin{gathered}
k = \left\langle \frac{1}{2}\mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \right\rangle_{\mathcal{L}} \\
\sum_{\boldsymbol{\xi}} \frac{1}{2}\hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot \hat{\mathbf{u}}^{*}(\boldsymbol{\xi})
\end{gathered}
\tag{4.1}
$$

The top equations gives the box averaged turbulence kinetic energy in physical space whereas the bottom prescribes it in the spectral domain, we get the same value for the physical and spectral kinetic energy to be 2.506, them being the same shows that kinetic energy is indeed conserved.

5 — Dissipation Rate

The Dissipation rate is calculated using the following equation

$$\varepsilon = \nu \sum_{\boldsymbol{\xi}} \boldsymbol{\xi}^2 \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot \hat{\mathbf{u}}^*(\boldsymbol{\xi}) \tag{5.1}$$

Where $\hat{\mathbf{u}}^*$ is the complex conjugate of $\hat{\mathbf{u}}$. The obtained value is $\epsilon = 0.84482$

6 — Wave Number Magnitudes & 3-D Kinetic Energy Density Spectrum

Considering:

$$E\left(\xi_n = \xi_0 n\right) = \sum_{\xi_0\left(n-\frac{1}{2}\right) \leq |\xi| < \xi_0\left(n+\frac{1}{2}\right)} \frac{1}{2} \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot \hat{\mathbf{u}}^*(\boldsymbol{\xi}) \tag{6.1}$$

Where the modulus of the wave number is prescribed as:

$$|\boldsymbol{\xi}|_{ijk} = \left(\xi_{x,i}^2 + \xi_{y,j}^2 + \xi_{z,k}^2\right)^{1/2} \tag{6.2}$$

One observes that the equation describes the domain being divided up to concentric sphere. Where the center is radius zero and boundary is at 192. As such the energy spectrum is calculated summed for each consecutive shell n in the domain.

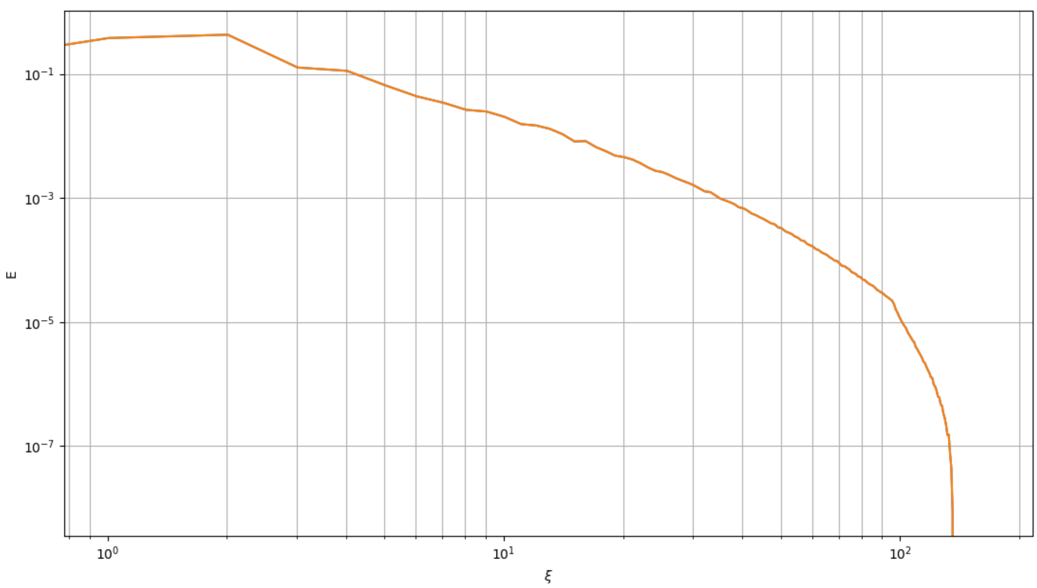


Figure 6.1: Energy Spectrum

The decrease in energy at larger wave numbers actually correctly prescribes the cascade effects of larger turbulent scales.

7 — Dissipation Spectrum

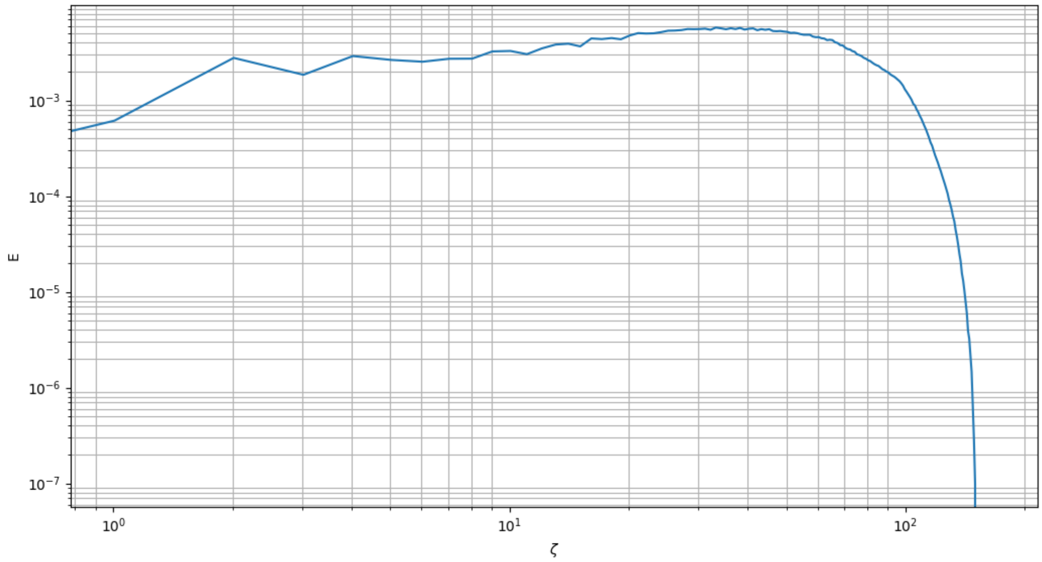


Figure 7.1: Dissipation Spectrum

The following equations describe the kinetic

$$k = \sum_n E(\xi_n) \quad = 2.498 and \quad \varepsilon = \sum_n D(\xi_n) = 0.84482 \tag{7.1}$$

8 Wave Number Range & Kolmogorov Constant

The inertial range was from $\xi = [4, 45]$

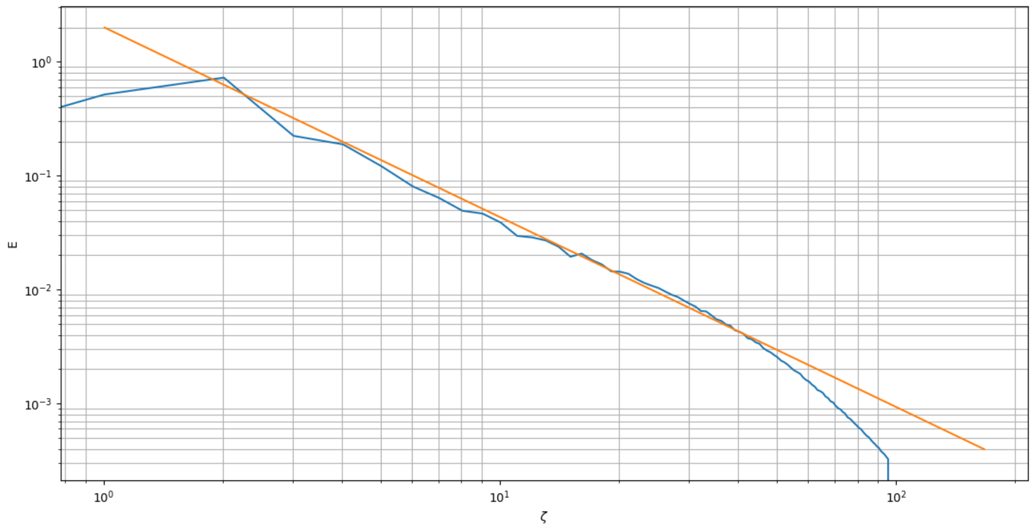


Figure 8.1: Energy Spectrum

$$\tilde{C}(\xi) = E(\xi)\varepsilon^{-2/3}\xi^{5/3} \tag{8.1}$$

The Kolmogorov constant of 1.1627. This is in the range of the experimental values.

9 — Reynolds Number and Integral Length Scale

The reynolds number of large scales is given by:

$$Re_L = \frac{k^2}{\varepsilon \nu} = 9232.69 \tag{9.1}$$

$$Re_\lambda = \frac{\lambda_g \sqrt{\frac{2}{3}k}}{\nu} = 248.09 \tag{9.2}$$

Where $\lambda_g = 0.2722$

The integral length scale is as following:

$$L = \frac{k^{3/2}}{\varepsilon} = 4.7 \tag{9.3}$$

10 — Application of Filter

The filter width is given by:

$$\Delta_{LES} = \frac{2\pi\left(1 - \frac{1}{N_{LES}}\right)}{N_{LES} - 1} \tag{10.1}$$

Where the cutoff frequency ξ_c is as such:

$$\xi_c = \frac{\pi}{\Delta_{LES}} \tag{10.2}$$

The filter is defined as following, where the input is $|\boldsymbol{\xi}|$

$$G(|\boldsymbol{\xi}|) = \frac{2\xi_N}{\pi|\xi|} \sin \frac{\pi|\boldsymbol{\xi}|}{2\xi_N} \tag{10.3}$$

$$|\boldsymbol{\xi}|_{ijk} = \left(\xi_{x,i}^2 + \xi_{y,j}^2 + \xi_{z,k}^2\right)^{1/2} \tag{10.4}$$

This gives a filter of shape:

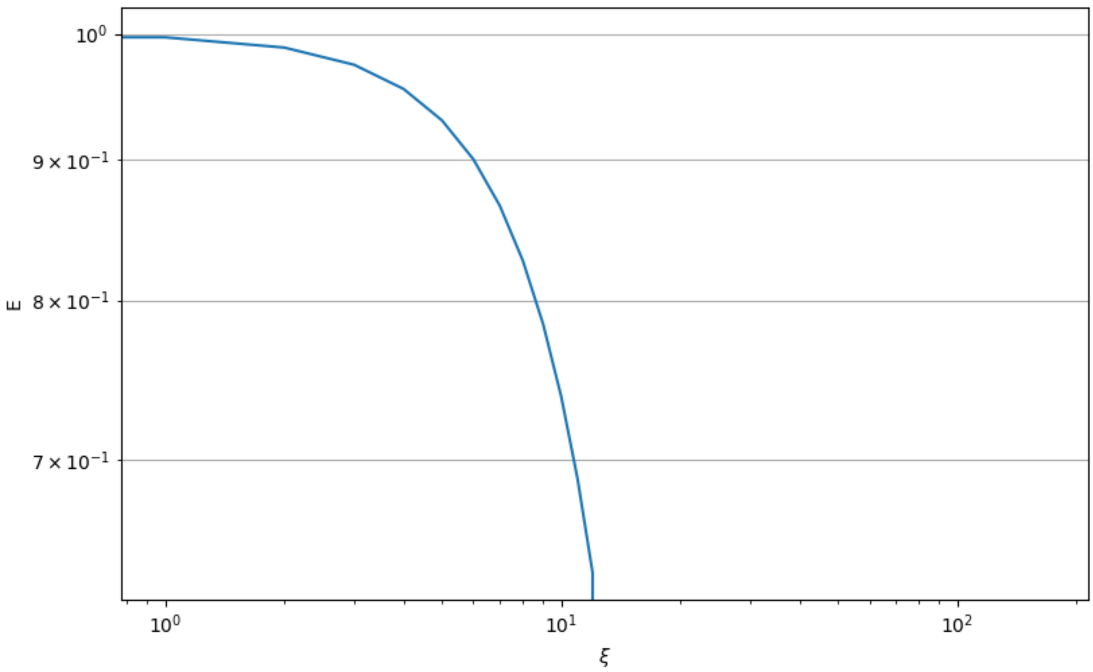


Figure 10.1: Filter Strength with respect to Wavenumber

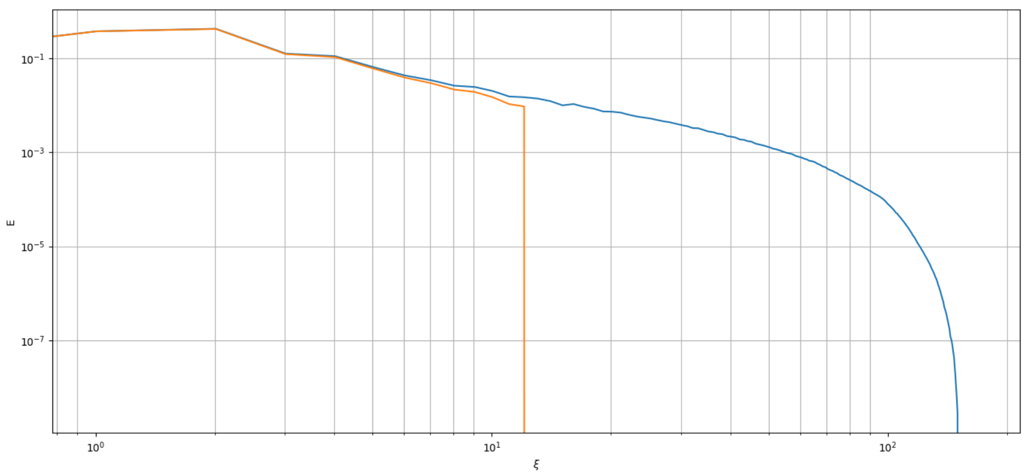


Figure 10.2: Filtered Energy Spectrum

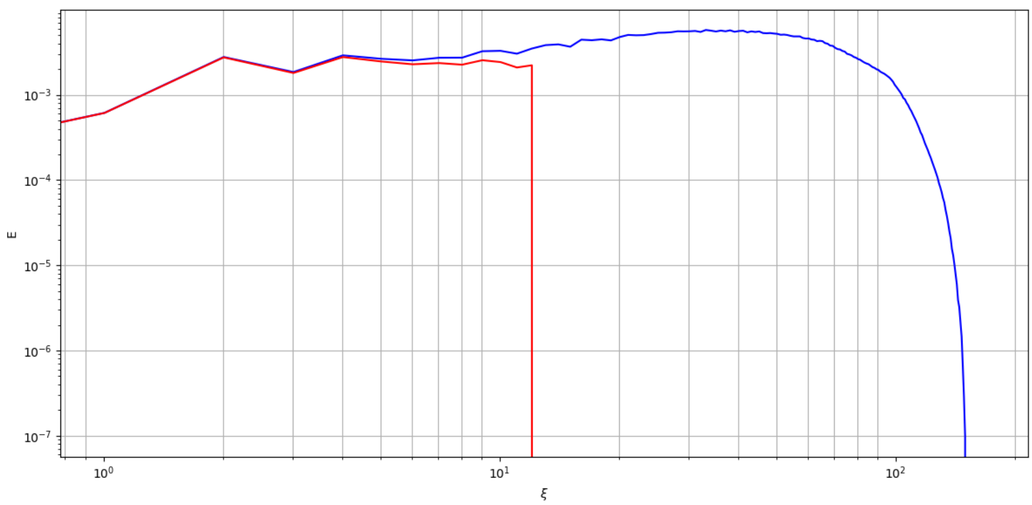


Figure 10.3: Filtered Dissipation Spectrum

One can observe in both graphs the filter correctly cutting off values after the cutoff frequency of 12.

11 — DNS Vorticity and Velocity Magnitude

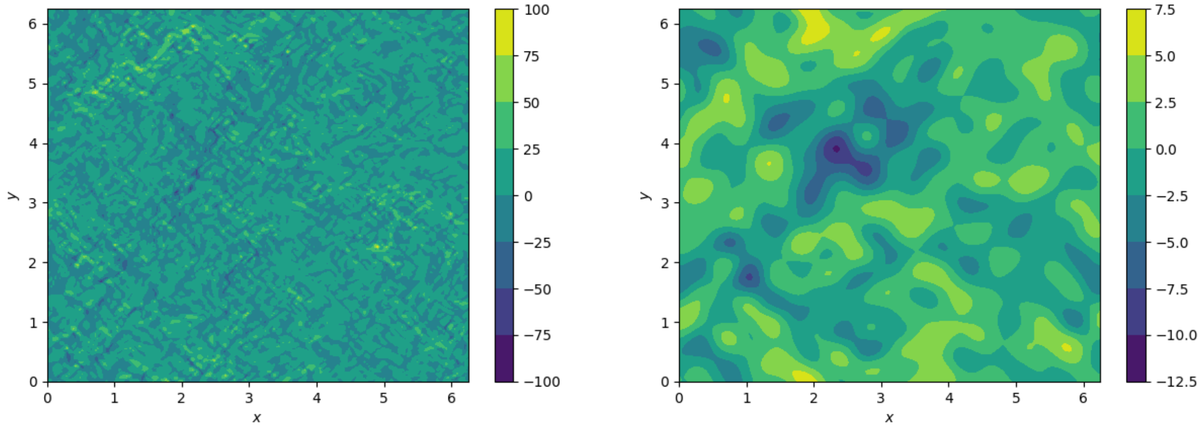


Figure 11.1: Curl Field

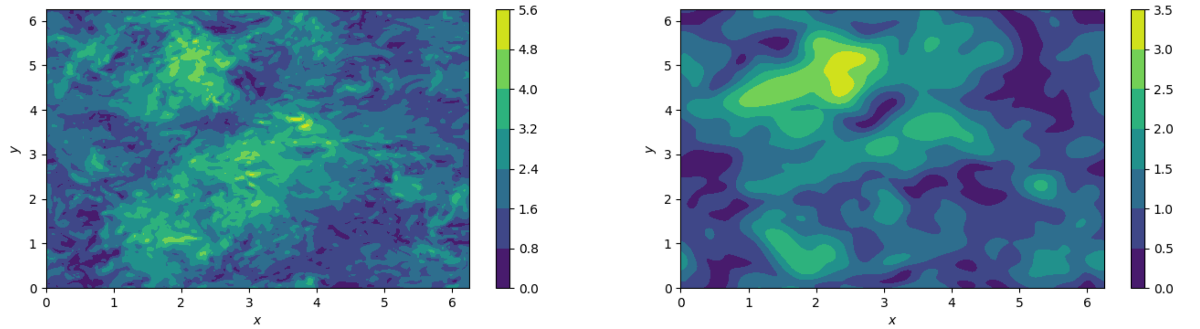


Figure 11.2: Velocity magnitude

12 — SGS Stress Tensor

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j \tag{12.1}$$

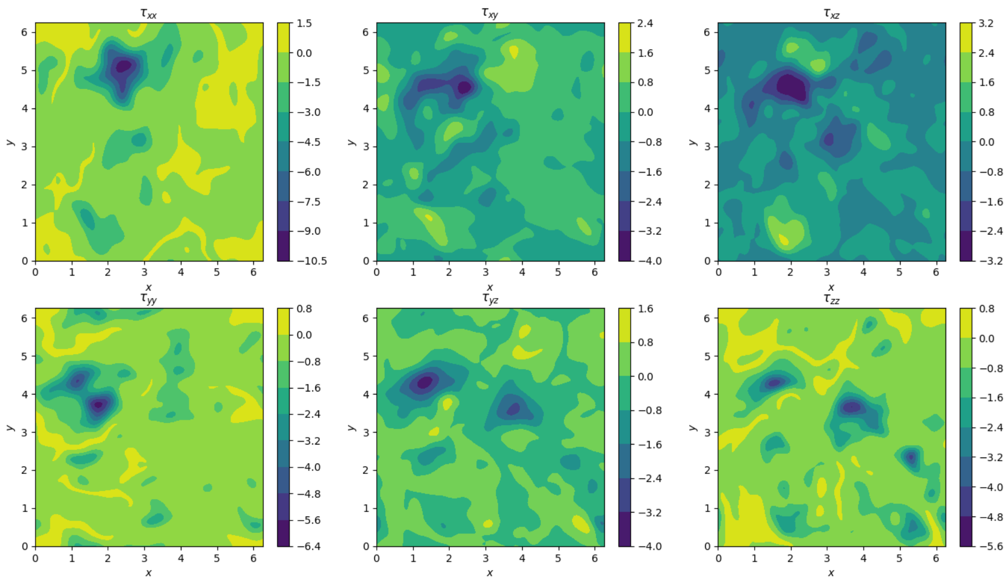


Figure 12.1: SGS Tensor at the 31st plane

13 — Smagorinsky Model

The Smagorinsky model eddy viscosity is given as

$$v = (C_s \Delta)^2 \sqrt{2\bar{S}_{ij}\bar{S}_{ij}} \tag{13.1}$$

Where Cs = 0.173 (from lecture notes).

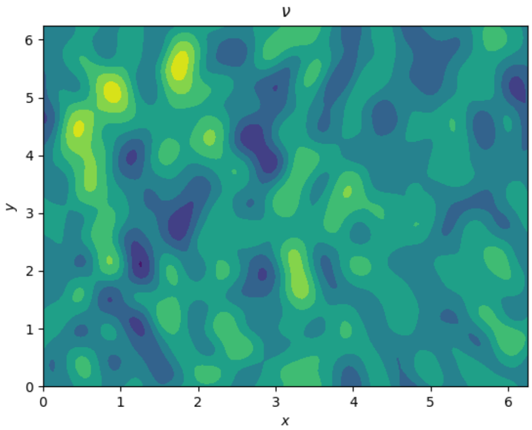


Figure 13.1: Viscosity

The stress tensor is given as

$$\tau_{ij} = -2v_{SGS}\bar{S}_{ij} \tag{13.2}$$

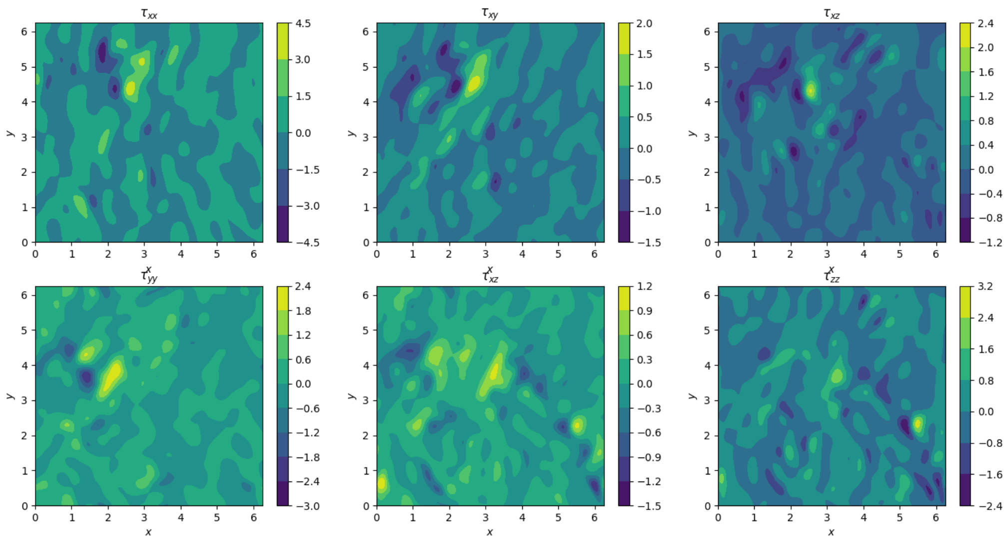


Figure 13.2: Smagorinsky tensor at the 31st plane

14 — Bardina Model

The Galilei Invariant Version of the Bardina stress tensor is given as:

$$\tau_{ij} = \overline{\bar{u}u} - \bar{\bar{u}}\bar{\bar{u}} \tag{14.1}$$

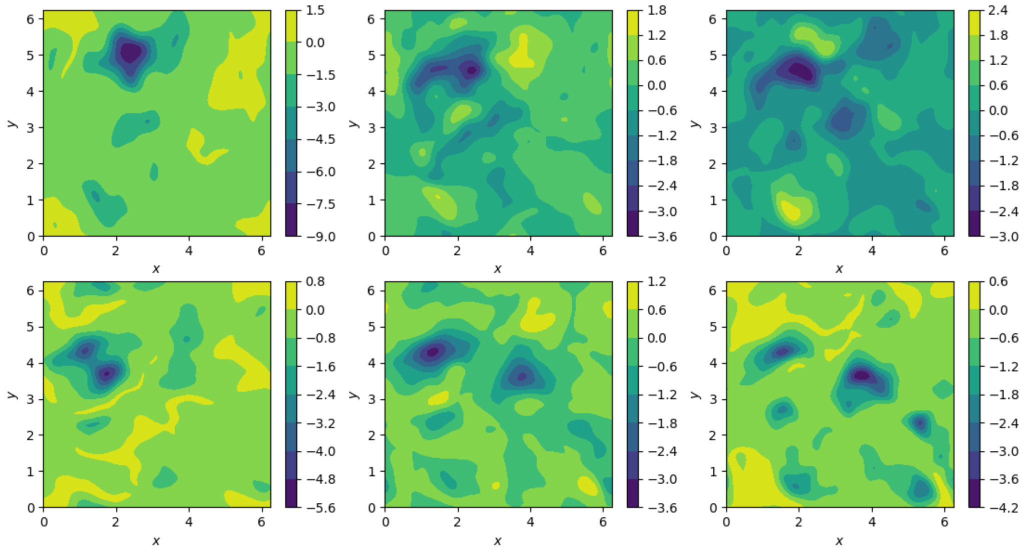


Figure 14.1: Bardina Tensor at the 31st plane

One sees that the Bardina model much more closely resembes the exact stress tensor compared to the Smagorinsky model. That being said though the Bardina model has the same maximas as the SGS tensor, the minimas are smaller in magnitude. Some smoothing by the Bardina model is present as well. The Smagorinsky model is almost incomparable compared to the Bardini model. The Smagorinsky model has a much more aggressive evening the domain. The maximas and minimas exhibit smaller magnitudes compared to both the SGS and the Bardini model