Implicit and inverse function theorem

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Th. 1 (Implicit function theorem)**.** *X, Y, Z — normed, Y — full,*
$W \subset X \times Y$ — open, $(x_0, y_0) \in W$,
$G : W \to Z$, G continuous at (x_0, y_0), $G(x_0, y_0) = 0$,
$\exists \partial_y G$ in W and it's continuous at (x_0, y_0),
$\exists (\partial_y G(x_0, y_0))^{-1} \in L(Z, Y)$.
Then $\exists U, V$: neighborhoods of $x_0 y_0$ and $f : U \to V$ continuous at x_0 such that $G(x, f(x)) = 0$.

Proof. Let's use Newton's method to find y:

$$g_x : y \mapsto y - (\partial_y G(x_0, y_0))^{-1} G(x, y)$$

$$g_x : Y \to Y$$

Consider it's differential:

$$\mathrm{d}g_x(y) = I_y - (\partial_y G(x_0, y_0))^{-1} \,\mathrm{d}G(x, y)$$

$$(G(x, y) \to G(x_0, y_0)) \to (\mathrm{d}g_x \to 0)$$

Then in some neighborhood of (x_0, y_0) $\left\|\mathrm{d}g_x(y)\right\| < \frac{1}{2}$. Also in some neighborhood of x_0: $\left\|g_x(y_0) - g_{x_0}(y_0)\right\| < \varepsilon$ by continuity of G. For the chosen $\varepsilon < \frac{\Delta}{2}$ and x in the chosen δ-neighborhood, with $L_{+\infty}$ norm:

$$g_x(B(y_0, 2\varepsilon)) \subset B(y_0, 2\varepsilon)$$

Let's prove it:

$$\left\|g_x(y) - y_0\right\| = \left\|g_x(y) g_{x_0}(y_0)\right\| \leqslant$$

$$\left\|g_x(y) - g_x(y_0)\right\| + \left\|g_x(y_0) - g_{x_0}(y_0)\right\| \leqslant$$

$$\sup \left\|\mathrm{d}g_x\right\| \left\|y - y_0\right\| + \varepsilon \leqslant \frac{1}{2}\varepsilon + \varepsilon$$

So now we take x in $B(x_0, \min(\delta, \Delta))$, y start fromy_0 and stay in $B(y_0, \Delta)$, therefore g_x is squeezing and we finally arrive at unique y where $G(x, y) = 0$.

And f is continuous at x_0 because we can make ε smaller by taking δ smaller. □

Basic example

$$\begin{cases} g_1(x_1 \dots x_m, y_1 \dots y_n) = 0 \\ \dots \\ g_n(x_1 \dots x_m, y_1 \dots y_n) = 0 \end{cases}$$

$$\leftrightarrow$$

$$\begin{cases} y_1 = f_1(x_1 \dots x_m) \\ \dots \\ y_n = f_1(x_1 \dots x_m) \end{cases}$$

It's a surface with dimension m (because every point is determined by it's x coordinates) in $\mathbb{R}^{n+m}$.

Th. 2. *Y — complete normed space, $U \in L(Y,Y)$, $\|U\| < 1$, I — identity* $\to \exists (I-U)^{-1} \in L(Y,Y)$

Proof. First proof.

Existence.

Want to prove that $\forall u \in Y \exists y \in Y : (I-U)y = u$.

$$y_{n+1} = u + Uy_n$$

$$y_{n+1} - y_n = Uy_n - Uy_{n-1}$$

$$\left\|y_{n+1} - y_n\right\| = \left\|U(y_n - y_{n-1})\right\| \leqslant \|U\| \left\|y_n - y_{n-1}\right\|$$

If we iterate this squeeze mapping, we will get the unique solution y_0 (here we used completeness of Y).

Continuity.

Now we consider $u_n \to u_0$, for each we find y_n and want to show $y_n \to y_0$.

$$(I-U)y_n = u_n, (I-U)y_0 = u_0 \Rightarrow (y_n - y_0) = U(y_n - y_0) + (u_n - u_0)$$

$$\left\|y_n - y_0\right\| \leqslant \left\|u_n - u_0\right\| + \|U\| \left\|y_n - y_0\right\|$$

$$0 \leqslant (1 - \|U\|) \left\|y_n - y_0\right\| \leqslant \left\|u_n - u_0\right\| \to 0$$

$$1 - \|U\| > 0 \Rightarrow \left\|y_n - y_0\right\| \to 0$$

$\square$

Proof. Second proof.

$$(I-U)^{-1} = I + U + U^2 + \dots$$

This series converges absolutely, i.e. $\sum \left\|U^k\right\| \leqslant \frac{1}{1-\|U\|}$ converges.

$L(Y,Y)$ is complete (link?), so every Cauchy sequence converges. And absolutely converging series are Cauchy sequences.

Now, consider S:

$$S_n = I + U + \dots + U^n \to S \in L(Y,Y)$$
$$S_n(I-U) = (I-U)S_n = I - U^{n+1} \to I$$
$$S(I-U) = (I-U)S = I \to S = (I-U)^{-1}$$

□

Interesting fact $A \in L(X,Y), B \in L(Y,X), AB = I_x$, then:

$$\frac{1}{\|B\|} = \inf_{\|x\|=1} \frac{1}{\|Bx\|} = \inf_{\|Bx\|=1} \frac{\|x\|}{\|Bx\|} = \inf_{\|y\|=1} \frac{\|Ay\|}{\|y\|} = \inf_{\|y\|=1} \|Ay\|$$

Th. 3.

$$Y \text{ — } \textit{complete}\ , U \in L(Y,Z), \exists U^{-1} \in L(Z,Y) \forall V \in L(Y,Z)$$
$$\|V\| < \frac{1}{\|U^{-1}\|} \to \exists (U \pm V)^{-1} \in L(Z,Y)$$

Proof.

$$U + V = U(I + U^{-1}V)$$
$$(U+V)^{-1} = (I + U^{-1}V)^{-1}U^{-1}$$

Now use completeness of Y and the previous theorem:

$$\left\|U^{-1}V\right\| < 1 \Rightarrow \exists (I + UV^{-1})^{-1}$$

□

Th. 4. *As Th. 1, but require continuity of G and* dG *not only in the point, but in a neighborhood. Then f will be continuous in a neighborhood of x_0.*

Proof. Consider Δ-neighborhood from Th. 1 where f exists. If we take (x_1, y_1) from there then $\exists(\partial_y G(x_1,y_1))^{-1} \in L(Z,Y)$ by previous lemma, so we can apply Th. 1 to that point and f will be the same, but now with continuity at that point too.

Note that neighborhood where it holds might be smaller than Δ because we didn't have any requirements on where $\exists(\partial_y G(x_1,y_1))^{-1} \in L(Z,Y)$. □

Th. 5. *As Th. 1, but* $\exists\, \mathrm{d}G$ *then* $\exists\, \mathrm{d}f$ *and:*

$$\mathrm{d}f(x_0) = -(\frac{\partial}{\partial y}G(x_0,y_0))^{-1}\frac{\partial}{\partial x}G(x_0,y_0)$$

and all three parts are in $L(X,Y), L(Z,Y), L(X,Z)$ *correspondingly.*

Proof.

$$G(x,y) = G(x_0,y_0) + \frac{\partial}{\partial x}G(x_0,y_0)(x-x_0) + \frac{\partial}{\partial y}G(x_0,y_0)(y-y_0) + o\Big(\|x-x_0\| + \|y-y_0\|\Big)$$

Consider $y = f(x)$ (and $G(x,y) = 0$ now):

$$0 = \frac{\partial}{\partial x}G(x_0,y_0)(x-x_0) + \frac{\partial}{\partial y}G(x_0,y_0)\big(f(x)-f(x_0)\big) + o\Big(\|x-x_0\| + \|f(x)-f(x_0)\|\Big)$$

Divide by $\frac{\partial}{\partial y}G(x_0,y_0)$ and find $f(x)-f(x_0)$:

$$f(x)-f(x_0) = -\left(\frac{\partial}{\partial y}G(x_0,y_0)\right)^{-1}\left(\frac{\partial}{\partial x}G(x_0,y_0)(x-x_0) + o\Big(\|x-x_0\| + \|f(x)-f(x_0)\|\Big)\right)$$

Now we need to show that $o(\dots)$ is small enough and f is continuous:

Let $C_1 = \left\|\left(\frac{\partial}{\partial y}G(x_0,y_0)\right)^{-1}\right\|\left\|\frac{\partial}{\partial x}G(x_0,y_0)\right\|, C_2 = \left\|\left(\frac{\partial}{\partial y}G(x_0,y_0)\right)^{-1}\right\|, \varepsilon \to 0$ from $o(\dots)$:

$$\|y-y_0\| \leqslant C_1\|x-x_0\| + C_2\varepsilon\Big(\|x-x_0\| + \|y-y_0\|\Big)$$

Then for $\varepsilon < C_2^{-1}$:

$$\|y-y_0\| \leqslant \frac{C_1 + C_2\varepsilon}{1 - C_2\varepsilon}\|x-x_0\|$$

□

Th. 6 (Corollary)**.** *If* G *is* k *times differentiable in a neighborhood, then* f *is too.*

Th. 7 (Inverse function theorem)**.** Y *— complete,* $F: Y \to X, F(y_0) = x_0, \exists\, \mathrm{d}F$ *in a neighborhood,* $\exists(\mathrm{d}F(y_0))^{-1} \in L(X,Y)$
then exist neighborhoods $U, V : x_0 \in U, y_0 \in V$ *such that* $F: V \to U$ *— bijection and*

$$(\mathrm{d}F^{-1})(x_0) = (\mathrm{d}F(y_0))^{-1}$$

Proof.

$$G : X \times Y \to X : G(x,y) = x - F(y)$$

It's suitable for Th. 1:

$$\frac{\partial}{\partial y} G(x_0, y_0) = -\mathrm{d}F\,, G(x_0, y_0) = 0$$

Then by using it we get $f(x)$:

$$\exists U, V \forall x \in U, y \in V : x = F(y) \leftrightarrow G(x,y) = 0 \leftrightarrow y = f(x)$$

Now it's already a map and even an injection, but we want to make $V = F(U)$ to make it a bijection. And we need y_0 to be internal in it.

$$\mathrm{d}f\,(x_0) = -(\frac{\partial}{\partial y} G(x_0, y_0))^{-1} \frac{\partial}{\partial x} G(x_0, y_0) = (\mathrm{d}F\,(y_0))^{-1}$$

$$\exists (\mathrm{d}f\,(x_0))^{-1} = \mathrm{d}F\,(y_0) \in L(Y, X)$$

So for any point close to y_0 its preimage is close to x_0, and y_0 is internal in $F(U)$. Now f and F are mutually inverse and $V = F(U), U = f(V)$, so they are bijections. □