Orthogonal matrices and least squares

16 May 2022

Def. $Q \in M_n(\mathbb{R})$ — orthogonal iff the columns form an orthonormal basis.
It's equivalent to $Q^T Q = E_n$.

Properties

1. $\|Qx\| = \left\| x^T Q^T Q x \right\| = x^T x = \|x\|$
2. Also it preserves scalar products.
3. All orthogonal matrices preserve distances and zero, and in other direction it's true too.
4. All transition matrices between orthonormal bases are orthogonal.

Least squares method

$$x = \arg\min_{x \in \mathbb{R}^n} \|Ax - b\|$$

$$\langle Ax - b, Ax - b \rangle = x^T A^T A x - x^T A^T b - b^T A x - b^T b$$

We can transpose 1×1 matrices, so $x^T A^T b = b^T A x$:

$$x^T A^T A x - 2x^T A^T b \to \min$$

After extracting the full square:

$$x = (A^T A)^{-1} A^T b$$

Or if $A = QR$:

$$Rx = Q^T b$$

And it can be solved in $O(n^2)$.