Euclidean space and Gram-Schmidt process

10 May 2022

Euclidean space

Def. Euclidean space is $V, \langle . \rangle$, where V is a vector space over $\mathbb{R}$ and $\langle . \rangle$ is a bilinear form called dot product.

What we already know about it:

$$\|x\| = \sqrt{\langle x, x \rangle},\ \rho(x,y) = \|x - y\|,\ |\langle x, y \rangle| \leqslant \|x\|\|y\|$$

Also we can define cosine of angle between vectors:

$$-1 \leqslant \frac{\langle x, y \rangle}{\|x\| y} \leqslant 1$$

And projection:

$$\mathrm{proj}_v : V \to \langle v \rangle, \mathrm{proj}_v\, x = \frac{v}{\|v\|} \|x\| \cos\alpha = \frac{\langle x, v \rangle}{\|v\|^2} v$$

Or projection on $\langle v \rangle^\perp$ by subtracting: $x - \mathrm{proj}_v\, x$.

Gram-Schmidt process

Def. Set of vectors e_i in a Euclidean space is called orthnormal if $e_i \perp e_j$ and $\left\|e_i\right\| = 1$

Def. Orthogonalization is a process which for a set of vectors e_i gives another set f_i, that are orthnormal and $\forall k : \langle e_1 \dots e_k \rangle = \langle f_1 \dots f_k \rangle$

Necessarily $f_1 = \frac{e_1}{\|e_1\|}$, because $\langle f_1 \rangle = \langle e_1 \rangle$, $\left\|f_1\right\| = 1$.

Now, $f_i \perp \langle e_1 \dots e_{i-1} \rangle \Leftrightarrow f_i \perp \langle f_1 \dots f_{i-1} \rangle$. Also it should contain something about e_i in it, so we can just subtract all projections:

$$f_i = e_i - \sum \mathrm{proj}_{f_j} e_i$$

Then just normalize it.