Conditional extrema

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Def. $f : X \to \mathbb{R}, M \subset X, x_0 \in M$ — conditional local minimum iff $\exists \varepsilon > 0 : f(x_0) \le f(x) \forall x \in B_\varepsilon(x_0) \cap M$

Usually M is defined by some condition, e.g. level set of a function, i.e.

$$\Phi : X \to \mathbb{R}, M = \{x \in X \mid \Phi(x) = 0\}$$

Also x_0 should be an inner point, otherwise there are no differentials.

$\mathbb{R}^m$ case

$$f : \mathbb{R}^{m+n} \to \mathbb{R}, \Phi : \mathbb{R}^{m+n} \to \mathbb{R}$$

$$M = \{z \in X \mid \Phi(z) = 0\}, z \in M \Leftrightarrow \begin{cases} \Phi_1(z) = 0 \\ \dots \\ \Phi_m(z) = 0 \end{cases}$$

$$\mathrm{d}\Phi(z) = \left(\frac{\partial \Phi_i}{\partial z_j} \right)$$

Def. M is nondegenerate in z if $\operatorname{rank} \mathrm{d}\Phi(z) = m$.

In that case let's choose m independent columns and put them in the end. First n coordinates now are called x and last m — y.

$$\mathrm{d}\Phi(z) = \left(\partial_x \Phi(x, y) \partial_y \Phi(x, y) \right)$$

By implicit function theorem [OG6Z] in a neighborhood $y = \varphi(x)$. Now $z = (x_0, y_0)$ is a conditional minimum iff x is a minimum of $\bar{f}(x) = f(x, \varphi(x)) \Rightarrow \mathrm{d}\bar{f}(x_0) = 0$.

$$\partial_x f(x_0, \varphi(x_0)) + \partial_y f(x_0, \varphi(x_0)) \,\mathrm{d}\varphi(x_0) = 0 \in \mathbb{R}^n$$

So it has a chance of giving a solution (because it removes n degrees of freedom).

$$\forall x \in B(x_0, \varepsilon) : \Phi(x, \varphi(x)) = 0 \Rightarrow \partial_x \Phi(x_0, \varphi(x_0)) + \partial_y \Phi(x_0, \varphi(x_0))\, \mathrm{d}\varphi(x_0) = 0$$

$$\mathrm{d}\varphi(x_0) = (\partial_y \Phi(x_0, \varphi(x_0)))^{-1} \partial_x \Phi(x_0, \varphi(x_0))$$
$$\partial_x f(x_0, y_0) + \partial_y f(x_0, y_0)(\partial_y \Phi(x_0, \varphi(x_0)))^{-1} \partial_x \Phi(x_0, \varphi(x_0)) = 0$$

$$\lambda = \partial_y f(x_0, y_0)(\partial_y \Phi(x_0, \varphi(x_0)))^{-1} \in (\mathbb{R}^m \to \mathbb{R})$$

$$\begin{cases} \partial_x f(x_0, y_0) + \lambda \partial_x \Phi(x_0, \varphi(x_0)) = 0 \\ \partial_y f(x_0, y_0) + \lambda \partial_y \Phi(x_0, \varphi(x_0)) := 0 \end{cases}$$

Lagrange test:

$$\begin{cases} \mathrm{d}f(z_0) - \lambda\, \mathrm{d}\Phi(z_0) = 0 \\ \Phi(z_0) = 0 \end{cases}$$

There are $m + n$ indeterminates in z and m in λ. But also there are $2m + n$ equations, so they might be enough to find z, λ.

Th. 1. *If z_0 is a local conditional extremum of f when $\Phi = 0$ and Φ is nondegenerate in z then $\exists \lambda \in \mathbb{R}^m : \mathrm{d}(f - \lambda\Phi)(z_0) = 0$.*

Geometric meaning

If z_0 is a local conditional minimum then value should increase along every direction in the surface. $\mathrm{d}f(z_0)\, h = 0$ when h is tangent to the surface.

$$\forall h \perp L = \langle \nabla\Phi_1, \ldots \nabla\Phi_m \rangle : \nabla f \perp h$$

Because $(L^\perp)^\perp = L$ (in finite dimensional case, to prove do something with bases):

$$\nabla f \in L \Rightarrow \exists \nabla_1 \ldots \nabla_m : \nabla f(z_0) = \lambda_1 \nabla\Phi_1(z_0) + \cdots + \lambda_m \nabla\Phi_m(z_0)$$

Example: quadratic form on a sphere

$$\mathbb{R}^m, f(x) = \langle Ax, x\rangle, A = A^T, Phi(x) = \|x\|^2 - 1 = \langle x, x\rangle - 1$$
$$\mathrm{d}f(x) = 2\langle A\,\mathrm{d}x\,, x\rangle, \mathrm{d}\Phi(x) = 2\langle x, \mathrm{d}x\rangle$$
$$\mathrm{d}f - \lambda\Phi = 0 \Rightarrow Ax = \lambda x$$

Suppose we found λ_1, v_1:

$$x \perp v_1 \Rightarrow \lambda_1\langle x, v_1\rangle \Rightarrow \langle x, Av_1\rangle \Rightarrow \langle Ax, v_1\rangle$$

So consider $A_1 = A|_{\langle v_1\rangle^\perp}, f_1 = f|_{\langle v_1\rangle^\perp}$ and solve it with more restraints.

Didona's task

$$f(0) = f(1) = 0, J[f] = \int_0^1 f(x)dx \to \max, \Phi[f] = \int_0^1 \sqrt{1 + (f')^2}dx - l = 0$$

Let's use Lagrange multipliers with partial derivatives:

$$\exists\lambda\,\forall h, h(0) = h(1) = 0 : \partial_h(J - \lambda\Phi)(f) = 0$$

Proof. Suppose $\exists h_0 : \partial_{h_0}\Phi(f) \neq 0$. Choose h:

$$g(t,s) = \Phi[f + th + sh_0]$$

Why $\exists s : g(t,s) = 0$? If $\partial_s g(0,0) \neq 0$ we have it from implicit function theorem.

$$\partial_s\Phi[f + th + sh_0] = \mathrm{d}\Phi[f + th + sh_0]\,h_0 = \partial_{h_0}(f + th + sh_0)$$
$$\partial_{h_0}\Phi(f) \neq 0$$

so there is some $s(t)$ in a neighborhood.

$$\varphi(t) = J(f + th = s(t)h_0) \Rightarrow 0 \text{ — maximum point.}$$
$$\varphi'(0) = 0$$
$$\partial_{h+s'(0)+h_0}J(f) = 0$$
$$\partial_h J(f) + s'(0)\partial_{h_0}J(f) = 0$$

And from $\Phi[f + th + s(t)h_0] = 0$:

$$\partial_h\Phi(f) + s'(0)\partial_{h_0}\Phi(f) = 0$$

As in Lagrange method:

$$\partial_h J(f) - \lambda\partial_h\Phi(f) = 0, \lambda\frac{\partial_{h_0}J(f)}{\partial_{h_0}\Phi(f)}$$

□

$$\partial_h(J - \lambda\Phi)(f) = 0$$
$$\partial_h(\int_0^1 (f - \lambda\sqrt{1+(f')^2})dx) = 0$$
$$\partial_2 F(\ldots) - \frac{\mathrm{d}}{\mathrm{d}x}\partial_3 F(\ldots) = 0$$
$$1 - \frac{\mathrm{d}}{\mathrm{d}x}(-\lambda\frac{f'}{\sqrt{1+(f')^2}}) = 0$$
$$-\lambda\frac{f'}{\sqrt{1+(f')^2}} = x + C$$
$$\frac{(f')^2}{1+(f')^2} = \frac{x+C}{\lambda}^2$$
$$\frac{1}{(f')^2} = \frac{\lambda^2}{(x+C)^2} - 1$$
$$f'(x) = \sqrt{\frac{(x+C)^2}{\lambda^2 - (x+C)^2}}$$
$$f(x) = \int \frac{x+c}{\sqrt{\lambda^2 - (x+c)^2}}$$

With $t = x + c$ we get:

$$f(x) = \sqrt{\lambda^2 - (x+C)^2} + C_1$$

So it's a part of a circle with center in $(-\frac{1}{2}, \ldots)$ and radius λ.