Chain line and squeeze maps

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Chain line

Suppose we have a chain from $(0, f_0)$ to $(1, f_1)$ with length l.

$$\begin{cases} \Pi(f) = \int\limits_0^1 f(x)\sqrt{1+(f'(x))^2}\,\mathrm{d}x \to \min \\ \Phi(f) = \int\limits_0^1 \sqrt{1+(f'(x))^2}\,\mathrm{d}x = l \\ f(0) = f_0, f(1) = f_1 \end{cases}$$

Let's apply Lagrange multipliers (here h is possible variation and f is a local minimum):

$$\exists \lambda \forall h : \partial_h (\Pi - \lambda\Phi)(f) = 0$$

$$(\Pi - \lambda\Phi)(f) = \int\limits_0^1 F(x, f(x), f'(x))\,\mathrm{d}x\,, F(x,u,v) = (u-\lambda)\sqrt{1+v^2}$$

Euler-Lagrange equation:

$$\partial_2 F - \frac{\partial}{\partial x}\partial_3 F = 0$$

Because there is no x:

$$F - f'\partial_3 F = C$$

Let's check:

$$\partial_2 F f' + \partial_3 F f'' - f''\partial_3 F - f'\frac{\partial}{\partial dx}\partial_3 F = 0$$

Substitute F:

$$(f(x)-\lambda)\sqrt{1+(f'(x))^2} - f'(x)(f(x)-\lambda)\frac{f'(x)}{\sqrt{1+(f'(x))^2}} = C$$

$$\frac{1}{1+(f')^2} = \left(\frac{C}{f(x)-\lambda}\right)^2$$

$$1+(f')^2 = \left(\frac{f-\lambda}{C}\right)^2$$

$$f' = \sqrt{\frac{(f-\lambda)^2 - C^2}{C^2}}$$

$$\frac{f'}{\sqrt{(f-\lambda)^2 - C^2}} = \frac{1}{C}$$

$$\int \frac{df}{\sqrt{(f-\lambda)^2 - C^2}} = \frac{x}{C}$$

$$\ln\left(f-\lambda+\sqrt{(f-\lambda)^2 - C^2}\right) = \frac{x}{C} + C_1$$

$$f-\lambda = \cosh\left(\frac{x}{C} + C_1\right)$$

Soap film

We have two rings: $x=0, y^2+z^2=f_0^2$ and $x=1, y^2+z^2=f_1^2$. Then

$$S(f) = \int_0^1 2\pi f(x)\sqrt{1+(f'(x))^2}\,\mathrm{d}x$$

It's just previous problem with $\lambda = 0$.

Squeeze maps and differential equations

Th. 1. (X,ρ) — *complete metric space,* $U: X \to X$ — *squeeze map, i.e.* $\exists \gamma < 1 \forall x_1, x_2 \in X$: $\rho(U(x_1), U(x_2)) \leqslant \gamma\rho(x_1,x_2)$ *then* $\exists! x_* : U(x_*) = x_*$.

Proof.

$$x_0 \in X, x_{k+1} = U(x_k)$$

$$\rho(x_{k+1},x_k) \leqslant \gamma\rho(x_k,x_{k-1}) \leqslant \gamma^2\rho(x_{k-1},x_{k-2}) \leqslant gamma^k\rho(x_1,x_0)$$

$$\rho(x_m,x_n) \leqslant \rho(x_1,x_0)\sum_{m-1}^{n}\gamma^k \leqslant \frac{\gamma^n}{1-\gamma}\rho(x_1,x_0)$$

So it's a Cauchy sequence and by completeness of X it has a limit. And $(x_{n+1} = U(x_n)) \to (x_* = U(x_*))$, here we used continuity of U, because it's a squeeze map. $\square$

Th. 2. (X,ρ) — *complete metric space,* $U : X \to X$*, and* $\exists n : U^n$ — *squeeze map then* $\exists! x_* : U(x_*) = x_*$.

Proof. We have x_* that is stationary for U^n. Now let's apply U to it n times. Then all those points are stationary for U^n, and there at least two distinct if $U(x_*) \neq x_*$. □

Now consider a special case of first order ODE:

$$f'(x) = A(x)f(x) + B(x), f \in C^1[a,b], f(a) = 0$$

$$U : f(x) \to \int\limits_a^x A(t)f(t) + B(t)\,\mathrm{d}t$$

We want to find a stationary point for this map. Let's check U^n in $X = \{f \in C[a,b] \mid f(a) = 0\}$:

$$\left\|Uf_1 - Uf_2\right\| = \max_{x\in[a,b]}\left|\int\limits_a^x A(t)f_1(t) - A(t)f_2(t)\,\mathrm{d}t\right|$$

$$\left|\int\limits_a^x A(t)f_1(t) - A(t)f_2(t)\,\mathrm{d}t\right| \leqslant (b-a)\max|A|\left\|f_1 - f_2\right\|$$

it works when $b - a \to 0$, but not in our case.

For some time we'll forget about shift by $B(t)$:

$$U_0 f(x) = \int\limits_a^x A(t)f(t)\,\mathrm{d}t$$

$$U_0^n f(x) = \int\limits_a^x A(t)\left(\int\limits_a^{t_{n-1}} A(t)\left(\ldots\int\limits_a^{t_1} A(t)f(t)\,\mathrm{d}t\right)\right)$$

$$\left\|U_0^n(f_1 - f_2)\right\| \leqslant \left\|f_1 - f_2\right\|\left(\max|A|\right)^n \max\left|\int\int\int\ldots\right| \leqslant \left\|f_1 - f_2\right\|\left(\max|A|\right)^n \frac{(b-a)^n}{n!}$$

And now $U(f) - U(g) = U_0(f) - U_0(g)$, because $B(t)$ cancells at each iteration, so U^n is also squeeze map. Therefore we have $f_*(x) = \int\limits_a^x A(t)f_*(t) + B(t)\,\mathrm{d}t$ and be default it's in $C[a,b]$, but if $A, B \in C^k[a,b]$ then $f_*(x) \in C^k[a,b]$.