Bilinear and quadratic forms

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Bilinear form

$h : V \times V \to K$ where V is a vector space and h is polylinear is called bilinear form.

As usually, bilinear form is characterized by $a_{i,j} = h(e_i, e_j)$ which is called the matrix of h in basis e. Here is how to evaluate h through A:

$$h(u, v) = [u]_e^T A [v]_e$$

Use cases

o Dot product: $x, y \mapsto \sum x_i y_i$

o Convolution?: $f, g \mapsto \int_0^1 f(x) g(x) w(x) \, \mathrm{d}x$

Symmetricity

$\forall v, u : h(u, v) = h(v, u) \Rightarrow h$ — symmetric. It's equivalent to A being symmetric.

Examples of assymetricity:

o Something with $\mathbb{C}$: $x, y \mapsto \bar{x} y$

o Another convolution (it's usually antisymmetric): $f, g \mapsto \int_0^1 f'(x) g(x) \, \mathrm{d}x$

Positive definite form

$$\forall x \neq 0 : h(x, x) > 0$$

Use cases

It allows decomposition into a direct sum by taking orthogonal vectors.

Def. $x \perp y \Leftrightarrow h(x, y) = 0$

Def (Orthogonal complement). $U \leqslant V, U^\perp = \{x \in V \mid \forall y \in U x \perp y\}$

Th. 1. *$h > 0$ — symmetric bilinear, $U \leqslant V$ then $V = U \oplus U^\perp$*

Proof. 1. Suppose $0 \neq x \in U \cap U^\perp$. Then $x \neq 0 \Rightarrow h(x,x) > 0$, but $x \in U \cap U^\perp \Rightarrow h(x,x) = 0$.

2. Now let's show $\dim U + \dim U^\perp = \dim V$. Consider what equatioins define $U^\perp$:

$$\forall y \in U : h(x, e_i) = 0$$

Here e is the basis of U, and it's enough to check only with basis elements. So we have not more (they might be dependent, though they aren't) than $\dim U$ equations. So $\dim U^\perp \geqslant \dim V - \dim U$. □

Quadratic form

Def. $q : V \to K$ which is a homogenous quadratic polynomial in every basis.

For $q(x) = \sum_{i \leqslant j} b_{i,j} x_i x_j$ let's define A this way:

$$a_{i,j} = \begin{cases} \frac{b_{i,j}}{2}, i < j \\ \frac{b_{j,i}}{2}, i > j \\ b_{i,j}, i = j \end{cases}$$

Now, both symmetric bilinear forms and quadratic forms both correspond to symmetric matrices, so they correspond to each other, and mapping is very simple: $h(x,y) = \frac{q(x+y)-q(x)-q(y)}{2}$.

Change of basis

Suppose h is $x^T A X$ in e and $y^T B y$ in f, then:

$$x^T A x = (Cy)^T ACy = y^T (C^T AC) y, C = [id]_e^f$$

Note that if we find an orthogonal basis than A will be a diagonal matrix.

Th. 2. *$q : V \to K, \exists e$ in which A (the matrix of q) will be diagonal.*

Proof. Here $a_{ij} = a_{ji}$:

$$q(x) = \sum a_{ij} x_i x_j = a_{11}(x_1 + \sum \frac{a_{1j}}{a_{11}})^2 + q(x_{2\ldots n}) - p(x_{2\ldots n})$$

Where:

$$p(x_{2\ldots n}) = \sum_{j \geqslant 2} \frac{a_{1j}^2}{a_{11}}$$

Then we continue to extract squares from $q-p$ on $x_{2\dots n}$ and the first basis element is:

$$e_1 = x_1 + \sum \frac{a_{1j}}{a_{11}} x_i$$

Final transition matrix will be upper-triangular with ones on the main diagonal, so it will be nondegenerate and give a basis. If $a_{11} = 0$ we can swap this row with another. If all $a_{ii} = 0$, take some $i, j : a_{ij} \neq 0$ and go to $x_i', x_j' = x_i + x_j, x_i - x_j$, there will appear a difference of squares. $\square$

We want to compute the main diagonal of matrix after this process.

Th. 3. *$q(x) = x^T A x, d_i \neq 0$ — principal minors (i.e.* $\det A[:i, :i]$*).*

The main diagonal will be $\frac{d_i}{d_{i-1}}$, where $d_0 \overset{def}{=} 0$ (i.e. they won't change).

Proof. It's enough to proof that multiplying $A \in M_n(K)$ by $C \in UT_n(K)$ with ones on the main diagonal doesn't change it's pricipal minors.

Actually, we consider only top-left $k \times k$ blocks and find the determinant of their product. But the determinant of UT_n with ones on the diagonal is 1, so the product is equal to the original determinant.

Now we need to show, that on each iteration $a_{ii} \neq 0$ so that only the main branch of the algorithms works, otherwise transition matrices wouldn't have the right form.

Consider the k-th step of the algorithm:

$$A^{(k)} = \begin{pmatrix} c_1 & & & 0 & 0 \\ & \dots & & 0 & 0 \\ & & c_k & 0 & 0 \\ 0 & 0 & 0 & x & \dots \\ 0 & 0 & 0 & \dots & \dots \end{pmatrix}$$

Here $x = a_{k+1,k+1} \neq 0$, otherwise the $(k+1)$-th minor would be zero. More precisely, $x = \frac{d_{k+1}}{d_k}$. $\square$

Th. 4 (LU decomposition from the diagonalized form)**.** *After the process from the previous theorem we get $A = C^T D C$, where $C \in UT_n(K)$ with ones on the diagonal, D — diagonal.*

Real case

By scaling the basis elements we can get $q(x) = \sum_{i=1}^{k} x_i^2 - \sum_{k+1}^{l} x_i^2$.

Def. (k, l) or simply $k - l$ is called signature

Th. 5. $k = \max\{\dim U \mid U \leqslant V, q|_U > 0\}$

Proof. If $q(x) = \sum_{i=1}^{k} x_i^2 - \sum_{k+1}^{l} x_i^2$ is true for some (k,l), then $\max\{\dim U \mid U \leqslant V, q|_U > 0\} \geqslant k$, because we can take $\langle e_1 \dots e_k \rangle$.

Now suppose we found $U \leqslant V : \dim U \geqslant k+1$ and $q|_U > 0$. Also consider $W = \langle e_{k+1} \dots e_n \rangle$. We know $q|_W \leqslant 0$ and $U \cap W \neq \{0\}$ (from Grassman's formula). So we have a nonzero vector lying in the both forms, which is a contradiction. □

Th. 6 (Stronger version of Silvester criterion)**.** $q(x) = x^T A x, A^T = A, d_i \neq 0, c_i = \frac{d_i}{d_{i-1}} \Rightarrow$ *l =number of sign changes in $1, d_1, d_2 \dots d_n$, because each negative element on the diagonal (in canonical form) changes the sign of this and next minors.*

Th. 7. *Following are equivalent:*

1. $\forall i : d_i > 0$

2. $\exists C \in UT_n(\mathbb{R}), \det C \neq 0 : A = C^T C$

3. $\exists C, \det C \neq 0 : A = C^T C$

4. $q > 0$

Proof. $1 \Rightarrow 2$: We can find $A = C^T D C$, then $A = (C^T \sqrt{D}^T)(\sqrt{D} C)$ as they are positive.

$2 \Rightarrow 3$: Trivial.

$3 \Rightarrow 4$: $x^T C^T C x > 0$ because with $y = Cx$, $y^T y \geqslant 0$ and $y \neq 0$ because $x \neq 0, \det C \neq 0$.

$4 \Rightarrow 1$: Positive definite form being restricted on anything is still positive (semi)definite. And k-th principal minor is a restriction on $\langle e_1 \dots e_k \rangle$. And it's determinant is nonzero because $A = C^T C$, where $|C| \neq 0$, so $|A| = |C|^2 > 0$. □