Implicit and inverse function theorem

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Th. 1 (Implicit function theorem). *X, Y, Z — normed.*

Basic example

$$\begin{cases} g_1(x_1 \dots x_m, y_1 \dots y_n) = 0 \\ \dots \\ g_n(x_1 \dots x_m, y_1 \dots y_n) = 0 \end{cases}$$

$$\leftrightarrow$$

$$\begin{cases} y_1 = f_1(x_1 \dots x_m) \\ \dots \\ y_n = f_1(x_1 \dots x_m) \end{cases}$$

It's a surface with dimension m (because every point is determined by it's x coordinates) in $\mathbb{R}^{n+m}$.

Th. 2. *Y — complete normed space, $U \in L(Y, Y)$, $\|U\| < 1$, I — identity $\rightarrow \exists (I - U)^{-1} \in L(Y, Y)$*

Proof. First proof.

Existence.

Want to prove that $\forall u \in Y \exists y \in Y : (I - U)y = u$.

$$y_{n+1} = u + Uy_n$$

$$y_{n+1} - y_n = Uy_n - Uy_{n-1}$$

$$\left\|y_{n+1}-y_n\right\| = \left\|U(y_n-y_{n-1})\right\| \leqslant \|U\|\left\|y_n-y_{n-1}\right\|$$

If we iterate this squeeze mapping, we will get the unique solution y_0.

Continuity.

$$y_n \to y_0$$
$$(I-U)y_n = u_n \to u_0$$

???

□

Proof. Second proof.

$$(I-U)^{-1} = I+U+U^2+\dots$$

This series converges absolutely, i.e. $\sum\left\|U^k\right\| \leqslant \frac{1}{1-\|U\|}$ converges.

$L(Y,Y)$ is complete (link?), so every Cauchy sequence converges. And absolutely converging series are Cauchy sequences.

Now, consider S:

$$S_n = I+U+\dots+U^n \to S \in L(Y,Y)$$
$$S_n(I-U) = (I-U)S_n = I-U^{n+1} \to I$$
$$S(I-U) = (I-U)S = I \to S = (I-U)^{-1}$$

□

Th. 3.

$$U \in L(Y,Z), \exists U^{-1} \in L(Z,Y) \forall V \in L(Y,Z)$$

Y — complete

$$\|V\| < \frac{1}{\|U^{-1}\|} \to \exists (U \pm V)^{-1} \in L(Z,Y)$$

Proof.

$$U+V = U(I_y)$$

□

Th. 4. *As Th. 1, but require continuity of G and $\mathrm{d}G$ not only in the point, but in a nenighborhood. Then f will be continuous in a nenighborhood of x_0.*

Proof. If (x_1, y_1) is $\square$

Th. 5. *As Th. 1, but $\exists \mathrm{d}G$ then $\exists \mathrm{d}f$ and:*

$$\mathrm{d}f(x_0) = -(\frac{\partial}{\partial y}G(x_0, y_0))^{-1}\frac{\partial}{\partial x}G(x_0, y_0)$$

and all three parts are in $L(X,Y), L(Z,Y), L(X,Z)$ correspondingly.

Proof.

$$G(x,y) = G(x_0,y_0)+\frac{\partial}{\partial x}G(x_0,y_0)(x-x_0)+\frac{\partial}{\partial y}G(y-y_0)+o(\left\|x-x_0\right\|+\left\|y-y_0\right\|)$$

$$y = f(x) : 0 = G(x, f(x)) = \frac{\partial}{\partial x}G(x_0,y_0)(x-x_0)+\frac{\partial}{\partial y}G(f(x)-f(x_0))+o(\left\|x-x_0\right\|+\left\|f(x)-f(x_0)\right|$$

Divide by $\frac{\partial}{\partial y}G(x_0, y_0)$:

$$f(x)-f(x_0) = -(\frac{\partial}{\partial y}G(x_0,y_0))^{-1}\frac{\partial}{\partial x}G(x_0,y_0)(x-x_0)+(\frac{\partial}{\partial y}G(x_0,y_0))^{-1}(o(\ldots))$$

Let $C_1 = \left\|(\frac{\partial}{\partial y}G(x_0,y_0))^{-1}\right\|\left\|\frac{\partial}{\partial x}G(x_0,y_0)\right\|, C_2 = \left\|\frac{\partial}{\partial y}\right\|\left\|(\frac{\partial}{\partial y}G(x_0,y_0))^{-1}\right\|, \varepsilon \to 0$ from $o(\ldots)$:

$$\left\|y-y_0\right\| = C_1\left\|x-x_0\right\| + C_2\varepsilon(\left\|x-x_0\right\| + \left\|y-y_0\right\|)$$

$\square$

Th. 6 (Corollary)**.** *If G is k times differentiable in a nenighborhood, then f is too.*

Th. 7 (Inverse function theorem)**.** *Y — complete, $F : Y \to X, F(y_0) = x_0, \exists \mathrm{d}F$ in a nenighborhood, $\exists(\mathrm{d}F(y_0))^{-1} \in L(X,Y)$*
then exist nenighborhoods $U, V : x_0 \in U, y_0 \in V$ such that $F : V \to U$ — bijection and

$$(\mathrm{d}F^{-1})(x_0) = (\mathrm{d}F(y_0))^{-1}$$

Proof.

$$G : X \times Y \to X : G(x, y) = x - F(y)$$

It's suitable for Th. 1:

$$\frac{\partial}{\partial y} G(x_0, y_0) = -ddF, G(x_0, y_0) = 0$$

Then by using it we get $f(x)$:

$$\exists U, V \forall x \in U, y \in V : x = F(y) \leftrightarrow G(x, y) = 0 \leftrightarrow y = f(x)$$

Now it's already a map and even an injection, but we want to make $V = F(U)$ to make it a bijection. And we need y_0 to be internal in it.

$$\mathrm{d}f(x_0) = -(\frac{\partial}{\partial y} G(x_0, y_0))^{-1} \frac{\partial}{\partial x} G(x_0, y_0) = (\mathrm{d}F(y_0))^{-1}$$

$$\exists (\mathrm{d}f(x_0))^{-1} = \mathrm{d}F(y_0) \in L(Y, X)$$

So for any point close to y_0 its preimage is close to x_0, and y_0 is internal in $F(U)$. Now f and F are mutually inverse and $V = F(U), U = f(V)$, so they are bijections. $\square$