Higher derivatives

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Simplest case: $\mathbb{R}^m \to \mathbb{R}$

Consider $f : \mathbb{R}^m \to \mathbb{R}$, differentiable on $\mathbb{R}^m$. Then $\exists \frac{\partial}{\partial x_i} : \mathbb{R}^m \to \mathbb{R}$. If it's again differentiable, $\exists \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} : \mathbb{R}^m \to \mathbb{R}$. And so on.
$\frac{\partial^p f}{\partial x_{i_1} \ldots \partial x_{i_p}}$ denotes the derivative of p-th order.

Th. 1 (Order of differentiation). *If $\frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial^2 f}{\partial x_i \partial x_j}$ exist in a neighborhood and are continuous, they are equal.*

Proof. For simplicity, $i = 1, j = 2, m = 2$. Also consider the discrete derivative:

$$F(h_1, h_2) = (f(x_1+h_1, x_2+h_2) - f(x_1+h_1, x_2)) - (f(x_1, x_2+h_2) - f(x_1, x_2))$$

And another helper function:

$$\varphi(t) = f(x_1 + h_1, x_2 + th_2) - f(x_1, x_2 + th_2)$$

Let's use mean value theorem along x_2:

$$F(h_1, h_2) = \varphi(1) - \varphi(0) = \varphi'(\theta_2)$$

$$\varphi'(t) = h_2(\frac{\partial f}{\partial x_2}(x_1 + h_1, x_2 + th2) - \frac{\partial f}{\partial x_2}(x_1, x_2 th2))$$

Now use it again along x_1:

$$\varphi'(t) = h_1 h_2 \frac{\partial^2 f}{\partial x_1 \partial x_2}$$

$$F(h_1,h_2)=h_1h_2\frac{\partial^2 f}{\partial x_1\partial x_2}, \exists\theta_1,\theta_2\in(0,1)$$

$$\lim_{h\to 0}\frac{F(h_1,h_2)}{h_1h_2}=\frac{\partial^2 f}{\partial x_1\partial x_2}(x) \text{ (by continuity of } \frac{\partial^2 f}{\partial x_1\partial x_2}\text{)}$$

Similarly, we can do it in other order and get:

$$\lim_{h\to 0}\frac{F(h_1,h_2)}{h_1h_2}=\frac{\partial^2 f}{\partial x_2\partial x_1}(x)$$

□

Th. 2 (Order of differentiation, corollary)**.** *If a function has continuous derivatives of n-th order, they don't depend on the order of differentiation.*

Proof. Suppose there are two orders for which the results differ, and the differnce consists of swapping two neighbors. All other permutations can be derived from it.
We can remove their common prefix, because after differentiating it we got the same functions. Here we apply the previous theorem, so we get the same functions again. And that remains is their suffixes, which are equal too. □

More complicated case: $X\to Y$

Consider $f: X\to Y$, differentiable on X. Then $\exists\, \mathrm{d}f: X\to L(X,Y)$. If it's again differentiable, $\exists\, \mathrm{d}(\mathrm{d}f): X\to L(X,X,Y)$ (or $L(X,L(X,Y))$). It is the second differential and can be denoted $\mathrm{d}^2 f$. If it's differentiable too we can go on further.

Th. 3 (Relation of the first differential to directed derivatives)**.**

$$\mathrm{d}f\,(x)h=\frac{\partial f}{\partial h}(x)=\lim_{t\to 0}\frac{f(x+th)-f(x)}{t}$$

Th. 4 (Relation of the second differential to directed derivatives)**.**

$$\mathrm{d}^2 f\,(x)(h_1,h_2)=\mathrm{d}(\mathrm{d}f)\,(x)(h_1,h_2)=(\frac{\partial}{\partial h_1}(\mathrm{d}f))(x)(h_2)$$

$$=\left(\lim_{t\to 0}\frac{\mathrm{d}f\,(x+th_1)-\mathrm{d}f\,(x)}{t}\right)(h_2)=\lim_{t\to 0}\frac{\mathrm{d}f\,(x+th_1)h_2-\mathrm{d}f\,(x)h_2}{t}$$

$$= \lim_{t \to 0} \frac{\frac{\partial f}{\partial h_2} - \frac{\partial f}{\partial h_2}}{t} = \frac{\partial}{\partial h_1}\left(\frac{\partial f}{\partial h_2}\right)(x)$$

And we can continue applying it by induction.

Th. 5 (Symmetricity of differential form). *If* $\exists\, \mathrm{d}^n f\,(x)$ *and is continuous, then the order of its arguments doesn't matter.*

Proof. Let's use the form of directed derivative and consider $n = 2$. So we want to prove that:

$$\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2} f(x) = \frac{\partial}{\partial h_2}\frac{\partial}{\partial h_1} f(x)$$

Again let's introduce some helper functions:

$$F(h_1, h_2, t) = f(x + th_1 + th_2) - f(x + th_1) - f(x + th_2) + f(x)$$

$$\varphi(v) = f(x + th_1 + tv) - f(x + tv), v \in [0, h_2]$$
$$F(h_1, h_2, t) = \varphi(h_2) - \varphi(0)$$

Now we'll show $\|F(h_1, h_2, t) - t^2\, \mathrm{d}^2 f\,(x)(h_1, h_2)\| = o(1)$:

$$\left\|\varphi(h_2) - \varphi(0) - t^2\, \mathrm{d}^2 f\,(x)(h_1)(h_2)\right\| \leqslant \sup_{\theta_2 \in (0,1)} \left\|\mathrm{d}\varphi(\theta_2 h_2) - t^2\, \mathrm{d}^2 f\,(x)(h_1)\right\| \|h_2\| = \ldots$$

$$(\mathrm{d}\varphi\,(\theta_2 h_2) = t(\mathrm{d}f\,(x + th_1 + t\theta_2 h_2) - \mathrm{d}f\,(x + t\theta_2 h_2)))$$
$$\cdots = |t|\|h_2\| \sup \left\|\mathrm{d}f\,(x + \theta_2 h_2 t + th_1) - \mathrm{d}f\,(x + \theta_2 h_2 t) - t\, \mathrm{d}^2 f\,(x)(h_1)\right\| \leqslant$$
$$|t|\|h_2\| \sup \sup \left\|\mathrm{d}^2 f\,(x + \theta_2 h_2 + \theta_1 h_1) - \mathrm{d}^2 f\,(x)\right\| \|th_1\| = o(t^2)$$

□

Relation of differential to the partial derivatives

Consider $f : \mathbb{R}^n \to \mathbb{R}$, its $\mathrm{d}^n f$ is a polylinear map, therefore can be expressed as:

$$\mathrm{d}^n f\,(x)(h_1 \ldots h_n) = \sum_{1 \leqslant j_1 \ldots j_n \leqslant n} \mathrm{d}^n f\,(x)(e_{j_1} \ldots e_{j_n}) h_1^{(j_1)} \ldots h_n^{(j_n)}$$

From that we get that n-th differential is a polylinear map and its coefficients are the partial derivatives.

Taylor formula

Again, $f : \mathbb{R}^m \to \mathbb{R}, f \in C^{n+1}[x, x+h]$, then (here $\mathrm{d}^k f\,(x)h^k$ means substitution of k entries of h):

$$f(x+h) = f(x) + \sum_{k=1}^{n} \frac{\mathrm{d}^k f\,(x)h^k}{k!} + \frac{\mathrm{d}^{n+1} f\,(x+\theta h)h^{k+1}}{(k+1)!}$$

Proof.

$$\varphi(t) = f(x+th), \varphi \in C^{n+1}[0,1]$$

$$\varphi(1) = \varphi(0) + \sum_{k=1}^{n} \frac{\varphi^{(k)}(0)}{k!} + \frac{\varphi(k+1)(\theta)}{(k+1)!}$$

□