Different estimation strategies and nonlinearities

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Outline

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1. A Menu of Estimation Strategies

2. Nonlinearities
    Non-Parametric Techniques

Running Example: monetary policy shocks

- **Variables:** GDP (y_t), Unemployment (u_t), Inflation (π_t), Nominal Rate (i_t).

- **Identification:** use Aruoba and Drechsel (2024) shock (z_t^m).
    - For VAR-based, add it to the VAR first and use recursive ID.
    - For LP-based, use it as perfect measure of the shock.
      [Equivalent to LP-IV if we normalize interest rate response]
    - In all cases, include a constant and 4 lags of all variables.

- We discuss several **estimation** strategies.
    - **VAR-based**: Standard VAR, Bias Corrected VAR, Bayesian VAR.
    - **LP-based**: LP, Bias Corrected LP, Penalized LP.

- **Goal:** provide a (non-extensive) summary of some recent methods
  Disclaimer: won't cover inference, large literature, can provide references

VAR-based: Standard and Bias-Correction

- Let $y_t = [z_t^m, y_t, u_t, \pi_t, i_t]$.
- **Standard VAR**
  - Fit by OLS
$$y_t = c + \sum_{\ell=1}^{p} A_\ell y_{t-\ell} + u_t \tag{1}$$
    get $\hat{A}_\ell$, $\hat{\Sigma} = \widehat{Var}[u_t]$.
  - Use Choslesky Decomposition to get $\hat{C}\hat{C}' = \hat{\Sigma}$
  - Response at time 0 is first column of $\hat{C}$. Get other horizons by iterating on (1).
  - If desired, rescale IRFs to get unit response in some variable.

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  - If desired, rescale IRFs to get unit response in some variable.
- **Bias-Corrected VAR**
  - It turns out that OLS estimates $\hat{A}_\ell$ have T^{-1} order bias (Pope, 1990):
  $$E[\hat{A}_\ell - A] = -\mathbb{B}_\ell / T + O(T^{-3/2})$$
  where $\mathbb{B}_\ell$ can be computed and corrected. Then proceed as above.
  See Kilian and Lütkepohl (2017) ch 2.3.3 for details.

VAR-based: Bayesian VAR

- Treat (1) as a Normal, Inverse Wishart Bayesian regression.
- Assume $u_t \sim N(0, \Sigma)$, collect $\beta = vec([c, A_1, \ldots, A_p]')$.
- **Prior**

$$\begin{aligned} \Sigma &\sim IW(\Psi, d) \\ \beta|\Sigma &\sim N(b, \Sigma \otimes \Omega\lambda^2) \end{aligned}$$

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- **Posterior**:

$$
\begin{aligned}
\beta|\Sigma, y &\sim N(\hat{\beta}(\lambda^2), \hat{V}(\lambda^2)) \\
\hat{\beta}(\lambda) &= (x'x + (\Omega\lambda^2)^{-1})^{-1}(x'y + (\Omega\lambda^2)^{-1}\bar{b}) \\
\hat{V}(\lambda) &= \Sigma \otimes (x'x + (\Omega\lambda^2)^{-1})^{-1}
\end{aligned}
$$

where $y = [y_p, \ldots, y_T]$ and x contains a constant and p lags of y_t.

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  where $y = [y_p, \ldots, y_T]$ and x contains a constant and p lags of y_t.
- High λ: loose prior, little shrinkage. [Note: $\lambda \to \infty$ yields OLS estimates]

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- Low λ: tight prior, high shrinkage.

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\end{aligned}$$

  where $y = [y_p, \ldots, y_T]$ and x contains a constant and p lags of y_t.
- How do we choose optimal λ?
- Giannone et al. (2015): derive the likelihood as function of hyperparameters. Then do Bayesian analysis.

LP-based: Standard and Bias-Correction

- **Standard LP**
  - Fit by OLS
$$y_{j,t+h} = c_{j,h} + \beta_{j,h} z_t^m + \text{ controls } + u_t \tag{2}$$
    for all horizons and variables of interest. Obtain $\{\hat{\beta}_{j,h}\}$
  - If desired, rescale IRFs to get unit response in some variable.

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    for all horizons and variables of interest. Obtain $\{\hat{\beta}_{j,h}\}$
  - If desired, rescale IRFs to get unit response in some variable.
- **Bias-Corrected LP**
  - Herbst and Johannsen (2024) show that $\hat{\beta}_{j,h}$ also has T^{-1} bias
    $$E[\hat{\beta}_{j,h} - \beta_{j,h}] = -\mathbb{B}_{j,h}^{LP}/T + O(T^{-3/2})$$
    where $\mathbb{B}_{j,h}^{LP}$ can be computed and corrected. Then proceed as above.
  - Bias can be substantial if T is small.

LP-based: Penalized Local Projections

- LP is very noisy, since we estimate $\beta_{j,h}$ for each horizon independently.

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- In particular, assume $\beta_{j,h} \approx \sum_{k=1}^{K} b_{j,k} B_k(h)$, where $B_k(x)$ is a set of B-spline basis.
  We'll see why they choose this in a second.

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- Ignore controls. Doing the same for the constant, we can write:

$$y_{j,t+h} \approx \sum_{k=1}^{K} c_{j,k} B_k(h) + \sum_{k=1}^{K} b_{j,k} B_k(h) z_t^m + u_t \tag{3}$$

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- Stack this for all h as:

$$\mathcal{Y}_t = \mathcal{X}_t \theta + \mathcal{U}_t \tag{4}$$

Where $\theta_j = (c_{j,1}, \cdots, c_{j,K}, b_{j,1}, \cdots, b_{j,K})$, $\mathcal{Y}_t = [y_t, \cdots, y_{t+H}]$ and $\mathcal{X}_t$ is a matrix that contains elements $B_k(k)$ and $B_k(h) z_t^m$

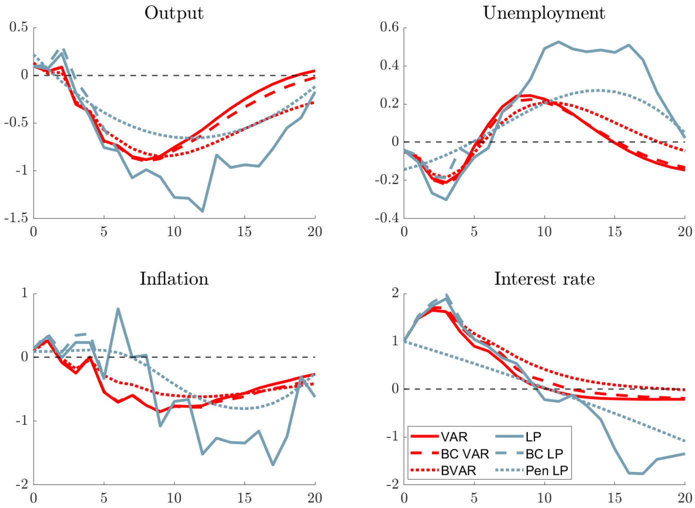
LP-based: Penalized Local Projections, cont.

- Penalized LP solves:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} ||\mathcal{Y} - \mathcal{X}\theta||^2 + \lambda\theta'\mathbf{P}\theta \tag{5}$$

- If $\lambda = 0$, this is OLS: we just rotated the regressors by writing them in other basis.
- With this basis, choosing $\mathbf{P}$ we can shrink the resulting β_h towards a polynomial.
- How do we choose λ? k-fold cross validation
- In our example, shrink towards quadratic.

IRFs to Aruoba-Drechsel shock



Discussion

- VARs and LP are only two out of a larger menu of estimation strategies.
- **Key trade-off**: bias and variance.
    - Imposing more structure (VARs, BVARs or Penalized LPs) reduces variance, but at the cost of bias.

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- **Key trade-off**: bias and variance.
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  - Li et al. (2024): In many cases, estimators other than pure VAR or LP are better.
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    Pen LP retain conceptual simplicity of LPs but eliminate their noisy behavior.
    BVARs improve specially for short horizons, and can handle larger number of variables.
  - Montiel Olea et al. (2024): however, the extra structure of VARs creates vulnerabilities.
    Under mispecification, confidence intervals are off-centered so coverage probability is incorrect.

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Parametric vs Non-parametric approaches

- **Parametric Approach:**
  1. Idea: commit to a parametric, non-linear (econometric) model of the macro-economy
     Often chosen to speak to a particular kind of possible non-linearity: state dependence, sign dependence, zero lower bound ▸ Examples
  2. Although popular, this approach has some issues:
     - **Relation to structural modeling**: linearized models imply SVMAs, but there are no similar guarantees for non-linear models. In fact globally solved DSGEs may not map into the typical reduced-form models in the literature. Model mis-specification is thus likely.
     - **Vulnerability to mis-specification**: identification & estimation rely heavily on functional form assumptions & the entire model being correctly specified. Not plausible in practice.
       Compare with our previous linear analysis—only second moments matter.
- Natural alternative: **non-parametric approaches**

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Recent push: bringing microeconometric insights to macro

- How should we think about potential outcomes in a time series context?
  Reference is Rambachan and Shephard (2021).
  - **Main takeaways**
    1. External instruments robustly estimate weighted average treatment effects. Internal instruments are more fragile.
       This point is quite general in econometrics, see nice summary by Hull (2024).
    2. What do standard (linear) LPs identify?
       Akin to cross-sectional literature on OLS: some weighted average treatment effects.
    3. What do **non-linear** LPs identify? How we can use them to learn about state, sign and size dependence?

Rambachan and Shephard (2021): General DGP

- **Notation**: scalar outcome y_t, (scalar) shock $\varepsilon_{1,t}$ of interest, $\varepsilon_{-1,t}$ vector of other shocks, $\varepsilon_t = [\varepsilon_{1,t}, \varepsilon'_{-1,t}]'$. Generalize linear SVMA to:
$$y_t = g\left(\{\varepsilon_{t-\ell}\}_{\ell \geq 0}\right) \tag{6}$$
Solution of all macro models has this form.

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$$y_t = g\left(\{\varepsilon_{t-\ell}\}_{\ell \geq 0}\right) \tag{6}$$

  Solution of all macro models has this form.
- Fix horizon of interest h, then:

$$y_{t+h} = g(\underbrace{\{\varepsilon_{t+i}\}_{i=1}^{h}}_{\text{shocks after } t}, \varepsilon_{1,t}, \underbrace{\varepsilon_{-1,t},}_{\text{other shocks at } t} \underbrace{\{\varepsilon_{t-j}\}_{i=0}^{\infty}}_{\text{lags}}) = g^h(\varepsilon_{1,t}, u_{h,t}) \tag{7}$$

  where $u_{h,t}$ contains everything other than $\varepsilon_{1,t}$.

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where $u_{h,t}$ contains everything other than $\varepsilon_{1,t}$.

- Define the *average structural function*

$$G_h(\varepsilon) = \mathbb{E}\left[y_{t+h} \mid \varepsilon_{1,t} = \varepsilon\right] = \mathbb{E}\left[g^h(\varepsilon_{1,t}, u_{h,t}) \mid \varepsilon_{1,t} = \varepsilon\right]$$

This function gives the counterfactual average value of y_{t+h} if the shock was equal to ε, averaging out over all other shocks.

Identification schemes in a non-paramtetric world

- Rambachan and Shephard (2021) essentially distinguish two main cases

  1. **Observe outcome & shock** [or IV]: the average structural function is then non-parametrically identified as

  $$G_h(\varepsilon) = \mathbb{E}\left[y_{t+h} \mid \varepsilon_{1,t} = \varepsilon\right]$$

     Could be estimated with enough (i.e., a huge amount of) data.

  2. **Only observe outcomes**: (heavily) restrict the functional form to make progress.
     [See Section 7.1: impose linearity + invertibility + X. Otherwise nonstandard interpretation.]

- We'll now dig deeper into 1.
  - Probably hopeless to fully characterize $G_h(\varepsilon)$, in particular if want to additionally condition on something else, e.g., state of the economy
  - Instead: can we give an interpretation to the estimand of *simple* estimation strategies?

Linear methods in non-linear models

- Suppose we observe $\varepsilon_{1,t}$ and run the linear local projection

$$y_{t+h} = \beta_h \varepsilon_{1,t} + \text{controls} + \text{error}$$

- Then what do we estimate for an arbitrary general non-linear DGP?

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  1. You in general estimate a **best linear approximation** to the underlying $g(\bullet)$ function
     Plagborg-Møller and Wolf (2021)

     - Formally: estimand are entries of a pseudo-SVMA $\sum_{\ell=0}^{\infty} \Theta_\ell^* \varepsilon_{t-\ell}$ where

$$(\Theta_0^*, \Theta_1^*, \dots) \in \operatorname*{argmin}_{\tilde{\Theta}_0, \tilde{\Theta}_1, \dots} \mathbb{E}\left[\left(g(\{\varepsilon_{t-\ell}\}_{\ell \geq 0}) - \sum_{\ell=0}^{\infty} \tilde{\Theta}_\ell \varepsilon_{t-\ell}\right)^2\right]$$

     - Analogous to best linear predictor property of OLS. Can we say more?

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- Then what do we estimate for an arbitrary general non-linear DGP?

  2. You in general estimate a particular **weighted average of causal effects for small shocks**

     - Again let $G_h(\epsilon)$ denote the average counterfactual future outcome for y if the shock—e.g., a policy instrument—was set to some value ϵ: $G_h(\epsilon) \equiv \mathbb{E}[y_{t+h} \mid \varepsilon_{1,t} = \epsilon]$
     - Then, as we will show on the next slide:

$$\beta_h = \frac{\text{Cov}(y_{t+h}, \varepsilon_t)}{\text{Var}(\varepsilon_t)} = \lim_{\delta \to 0} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \omega(\epsilon) \frac{G_h(\epsilon + \delta) - G_h(\epsilon)}{\delta} d\epsilon$$

       where $\omega(\epsilon)$ is a weight function (non-negative and integrates to one), $supp(\varepsilon_t) \subset (-\bar{\varepsilon}, \bar{\varepsilon})$
     - Practical takeaway: plot $\omega(\epsilon)$. Most weight on positive/negative/small/large shocks?

Linear methods in non-linear models

- Let $F(\bullet)$ denote the cdf of $\varepsilon_{1,t}$, assume mean zero. Begin with the numerator:

$$\begin{aligned}
\mathrm{Cov}(y_{t+h}, \varepsilon_t) &= \mathbb{E}\left[\mathbb{E}\left(y_{t+h} \mid \varepsilon_{1,t}\right)\varepsilon_{1,t}\right] = \mathbb{E}\left[G_h(\varepsilon_{1,t})\varepsilon_{1,t}\right] \\
&= \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} G_h(\epsilon)\epsilon dF(\epsilon) = \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \left(\int_{-\bar{\varepsilon}}^{\epsilon} G_h'(a)da\right)\epsilon dF(\epsilon) \\
&= \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \underbrace{\left(\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \mathbf{1}_{a\leq\epsilon}\epsilon dF(\epsilon)\right)}_{=\mathrm{Cov}(1_{a\leq\epsilon},\epsilon)} G_h'(a)da
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&= \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} G_h(\epsilon)\epsilon dF(\epsilon) = \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \left(\int_{-\bar{\varepsilon}}^{\epsilon} G_h'(a)da\right)\epsilon dF(\epsilon) \\
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\end{aligned}$$

- Next consider the denominator:

$$\text{Var}(\varepsilon_{1,t}) = \text{Cov}(\varepsilon_{1,t}, \varepsilon_{1,t}) = \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \text{Cov}(\mathbf{1}_{a\leq\epsilon}, \epsilon)da$$

- The overall weight function is thus

$$\omega(a) \equiv \frac{\text{Cov}(\mathbf{1}_{a\leq\epsilon}, \epsilon)}{\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \text{Cov}(\mathbf{1}_{b\leq\epsilon}, \epsilon)db}$$

Note: if $\varepsilon_{1,t}$ is mean-0 and normal then it can be shown that $\omega(a) \propto f(a)$.

The arguments in this slide come from Yitzhaki (1996) and Plagborg-Møller and Kolesar (2024).

State dependence in non-linear models

- Now suppose we observe $\varepsilon_{1,t}$ and run the simple state-dependent local projection

$$y_{t+h} = \beta_{0,h}(1 - s_{t-1})\varepsilon_{1,t} + \beta_{1,h} s_{t-1}\varepsilon_{1,t} + \text{controls} + \text{error}$$

  - Interpretation: s_{t-1} here would typically be the state of the economy, e.g. recession versus expansion. E.g. see Ramey and Zubairy (2018).

- Then what do we estimate for an arbitrary general non-linear DGP?
  - Assume that $\varepsilon_{1,t}$ is also independent of s_{t-1}. Then by a similar argument to above:

$$\beta_{s,h} = \lim_{\delta \to 0} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \tilde{\omega}(\epsilon, s) \frac{\tilde{G}_h(\epsilon + \delta, s) - \tilde{G}_h(\epsilon, s)}{\delta} d\epsilon$$

    where $\tilde{G}_h(\epsilon, s) \equiv \mathbb{E}[y_{t+h} \mid \varepsilon_{1,t} = \epsilon, s_{t-1} = s]$ and $\tilde{\omega}(\epsilon, s)$ now additionally conditions on s
  - Takeaway: we now estimate a weighted average of state-dependent causal effects

Sign and Size dependence in non-linear models

- Consider the (population) regressions:
$$y_{t+h} = \beta_{1,h}\varepsilon_t + \beta_{2,h} f\left(\varepsilon_t\right) + \text{controls} + \text{error}$$
Where $f(x)$ is some non-linear transformation, such as x^2, x^3 or $|x|$. (Ascari and Haber (2021),Ben Zeev, Ramey, and Zubairy (2023), Forni et al. (2023))

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- Caravello and Martinez-Bruera (2024): if the shock $\varepsilon_{1,t}$ has symmetric distribution, then we can test for sign and size separately by picking f appropriately.

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- Caravello and Martinez-Bruera (2024): if the shock $\varepsilon_{1,t}$ has symmetric distribution, then we can test for sign and size separately by picking f appropriately.
- Following a similar procedure as before:

Lemma 1 (Weighted average representation)

$$\beta_{n,h} = \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \omega_n(a) G_h'(a) da \quad n = 1, 2$$

where $\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \omega_1(a) da = 1$ and $\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \omega_2(a) da = 0$

Even-Odd decomposition

- Decompose $m(a) = G_h'(a)$ as

$$m(a) = m^{Even}(a) + m^{Odd}(a)$$

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- Recall:
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Even-Odd decomposition

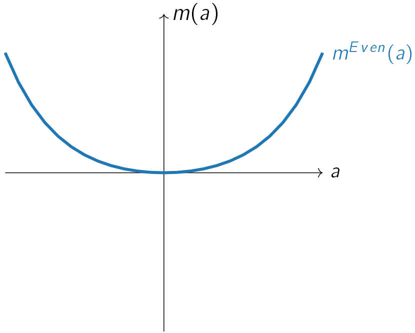
- Decompose $m(a) = G_h'(a)$ as

$$m(a) = m^{Even}(a) + m^{Odd}(a)$$

- Recall:
  - By definition, $m^{Even}(a) = m^{Even}(-a)$ and $m^{Odd}(a) = -m^{Odd}(-a)$
  - Decomposition always exists and is unique, given by $m^{Even}(a) = \frac{m(a)+m(-a)}{2}$ and $m^{Odd}(a) = \frac{m(a)-m(-a)}{2}$
  - Examples: x^2 and $|x|$ are even, x^3 is odd.

Even part captures only pure size differences

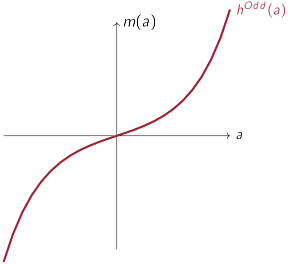
- Decompose $m(a) = G_h'(a)$ as $m(a) = m^{Even}(a) + m^{Odd}(a)$



- Since $m^{Even}(a) = m^{Even}(-a)$, **sign doesn't matter**

On average, odd part captures only sign differences

- Decompose $m(a) = G_h'(a)$ as $m(a) = m^{Even}(a) + m^{Odd}(a)$



- Since $m^{Odd}(a) = -m^{Odd}(-a) \Rightarrow \frac{m^{Odd}(a)+m^{Odd}(-a)}{2} = 0$
- **When averaging, size doesn't matter**
- Our estimands are averages, so this works well for our purposes!

Main Result

- We can also decompose $\omega_2(a) = \omega_2^{Odd}(a) + \omega_2^{Even}(a)$. Thus:

$$\beta_2 = \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \omega_2(a) m(a) da$$

Main Result

- We can also decompose $\omega_2(a) = \omega_2^{Odd}(a) + \omega_2^{Even}(a)$. Thus:

$$\beta_2 = \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \omega_2(a) m(a) da = \underbrace{\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \omega_2^{Odd}(a) m^{Odd}(a) da}_{\text{Sign}} + \underbrace{\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \omega_2^{Even}(a) m^{Even}(a) da}_{\text{Size}}$$

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  - If $\omega_2(a)$ purely even $\rightarrow$ only size non-linearities!

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Proposition 1 Proof

Assume the unconditional distribution of ε_t is symmetric. Then

- If $f(\cdot)$ is even $\implies$ $\omega_2(a)$ is odd (i.e $\omega_2^{Even}(a) = 0 \quad \forall a$)
- If $f(\cdot)$ is odd $\implies$ $\omega_2(a)$ is even (i.e $\omega_2^{Odd}(a) = 0 \quad \forall a$)

Practical Takeaways

1. Sign non-linearities $\rightarrow$ use even transformation.
2. Size non-linearities $\rightarrow$ use odd transformation.
3. Test for $\beta_2 = 0$ using standard inference.
4. For interpreation, compute and plot weights (Plagborg-Møller and Kolesar, (2024)), and its even-odd decomposition.

Empirical Application: Oil Supply News Shocks

- We fit local projections using oil shock series (Känzig (2021)). Details

$$y_{t+h} = \alpha_h + \beta_{h,1}\varepsilon_t^{oil} + \beta_{h,2} f(\varepsilon_t^{oil}) + \text{ controls } + u_{t+h} \tag{8}$$

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$$y_{t+h} = \alpha_h + \beta_{h,1}\varepsilon_t^{oil} \qquad + \text{ controls } + u_{t+h} \tag{8}$$

- Linear: no $f(\cdot)$ term.

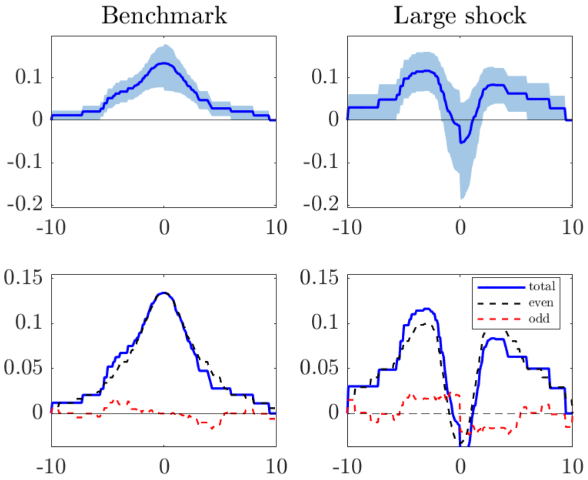
Empirical Application: Oil Supply News Shocks

- We fit local projections using oil shock series (Känzig (2021)).

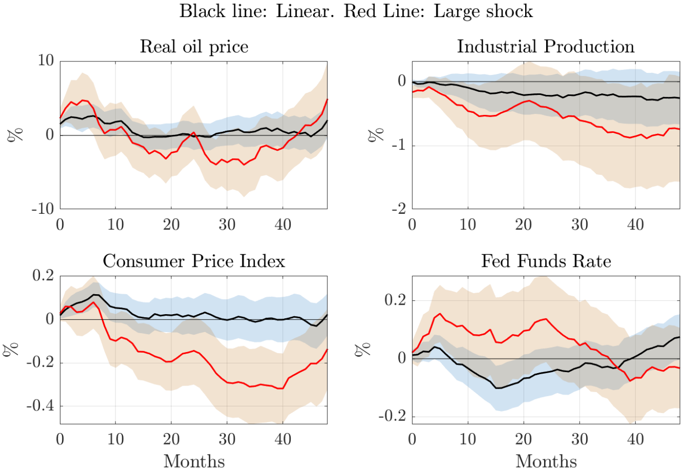
$$y_{t+k} = \alpha_k + \beta_{k,1}\varepsilon_t^{oil} + \beta_{k,2} f(\varepsilon_t^{oil}) + \text{ controls } + u_{t+k} \tag{9}$$

- For non-linear terms, use piecewise linear functions for ease of interpretation.
  - Sign: use $f(x) = |x|$
  - Size: use $f(x) = \mathbf{1}_{\{x \leq -\bar{b}\}}(x + \bar{b}) + \mathbf{1}_{\{x \geq \bar{b}\}}(x - \bar{b})$

Weights: linear only vs large shock ($\beta_{k,1} + \beta_{k,2}$)



Linear IRFs vs Large shock IRF ($\beta_{k,1} + \beta_{k,2}$)



Open issues

Literature on non-linear time series methods has recently seen a bit of a revival, still lots of interesting **open questions**

- **Methodological**
  - What kind of parametric models—if any—capture well the non-linearities implied by non-linearly solved DSGE models? [See Aruoba et al. (2017)]
  - How feasible are non-parametric estimators of general potential outcomes functions? [See Gonçalves et al. (2024)]
- **Applied**
  - Do non-linearities matter in the aggregate? If so, what are the relevant non-linearities? ZLB, downward wage rigidity, state-dependence in price setting or investment/durables?
  - What kind of non-linearities are most important? By size, sign, or state of the economy? E.g. Barnichon et al. (2022) for government spending multipliers.

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Appendix

Parametric Approach: Examples

Back

- We will briefly review a couple of the most notable examples:

1. **Regime-switching models**: model coefficients differ by *regime* of the macro-economy
2. **Time-varying parameter models**: model coefficients evolve continuously over time
3. **Censored autoregression** [mostly for ZLB constraint]

Regime-switching models

- **Framework**: allow parameters of model to differ by **regime**. E.g.:

$$y_t = \{A_0(1 - s_{t-1}) + A_1 s_{t-1}\} y_{t-1} + \eta_t, \quad \eta_t | s_{t-1} \sim N(0, \Sigma_0(1 - s_{t-1}) + \Sigma_1 s_{t-1})$$

- Popular special cases:

1. **Threshold Autoregression:**

$$s_t = \mathbf{1}_{x_t \leq c}$$

where x_t could be something exogenous or itself depend on y_t, and c is a parameter

2. **Smooth Transition Autoregression:**

$$s_t = \frac{1}{1 + \exp(\beta x_t - c)} \in [0, 1]$$

where again x_t may evolve exogenously or be a function of y_t, and β, c are parameters

3. **Markov Switching:** s_t evolves exogenously as a Markov Process.

Models with time-varying parameters

- **Framework**: parameters are **varying continuously**. E.g.:

$$y_t = A_t y_{t-1} + \Sigma_t^{1/2} \varepsilon_t, \quad \varepsilon_t \sim N(0, I_n)$$

- Popular special cases:
  - **Autoregressive component** $\beta_t \equiv \mathrm{vec}(A_t)$: independent random walks
  - **Volatility** Σ_t: multivariate stochastic volatility model
- Many well-known examples of these kind of approaches in the VAR literature
  Cogley and Sargent (2002),Primiceri (2005).

Models with censoring

- **Framework**: censoring of a subset of variables. E.g.:

$$\begin{aligned} y_{1,t} &= \max\{A_{1,\bullet} y_{t-1} + H_{1,\bullet} \varepsilon_t, 0\} \\ y_{2:n,t} &= A_{2:n,\bullet} y_{t-1} + H_{2:n,\bullet} \varepsilon_t \\ \varepsilon_t &\sim N(0, I_n) \end{aligned}$$

- Main application: binding **lower bound on nominal interest rates**
    - Assumption: linear system is adequate up to constraints on monetary policy. Echoes solution methods for structural models with occasionally binding constraints. See Guerrieri and Iacoviello (2015)

Summarizing $g(\bullet)$: dynamic causal effects

What are **dynamic causal effects** in these environments?

- No unique def'n. Note that such models generally imply recursive representations:

$$y_{t+h} = g_h\left(\varepsilon_{t+h}, \varepsilon_{t+h-1}, \ldots, \varepsilon_{t+1}, \varepsilon_t, y_{t-1}\right)$$

- Suggests two possible natural defn's:

1. **Set other shocks to zero**

$$\theta_h(\delta_1, \delta_2; y_{t-1}) \equiv g_h(0, 0, \ldots, 0, \delta_2 \times e_1; y_{t-1}) - g_h(0, 0, \ldots, 0, \delta_1 \times e_1; y_{t-1})$$

2. **Average over other shocks**

$$\begin{aligned}\theta_h(\delta_1, \delta_2; y_{t-1}) \equiv \mathbb{E}_{\varepsilon_{t+h},\ldots,\varepsilon_{t+1},\varepsilon_{2:n,t}} [&g_h(\varepsilon_{t+h}, \varepsilon_{t+h-1}, \ldots, \varepsilon_{t+1}, (\delta_2, \varepsilon_{2:n,t}); y_{t-1}) \\ &- g_h(\varepsilon_{t+h}, \varepsilon_{t+h-1}, \ldots, \varepsilon_{t+1}, (\delta_1, \varepsilon_{2:n,t}); y_{t-1})]\end{aligned}$$

Note: definitions coincide in linear models and there depend only on $\delta_2 - \delta_1$.