Final Project: Animated Heat Equation and Traveling Wave Equation(Modified Midterm)

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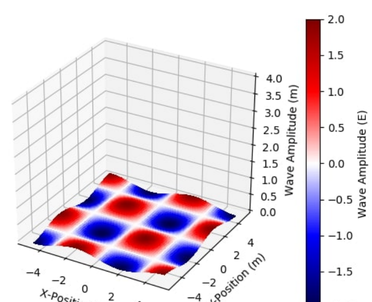
1 Section One: The Heat Equation

1.1 Introduction

In this section we will explore various solutions to the heat equation in 3-Dimensions. This will be conducted by exploring the evolution of the heat equation within a black body system contained within a box. Our goal here is to create a general purpose code that can be modified to animate more complex systems in the future. We want to empower the user to input specified initial conditions and boundary conditions such that the time-resolved solution can be animated with ease. Through the animation we gain insight into the solutions of the differential equation and allow us to visualize how physical systems would thermally diffuse in everyday systems. Ideally the code can be modified to incorporate other geometries in order to broaden it's usefulness.

In the past, we have successfully created an animations for the wave equation. This differential equation is used to measure the motion of waves propagating through mediums and is very useful in understanding excitation energies of photons and other electromagnetic waves. Below we present the 3rd excited state of the wave equation using python.



Solution to the 3rd excited state of the Wave Equation.

Using the knowledge gained from the previous project, as stated before, we want to create a python code that will take in a few parameters, pass them through a function, and output a time-evolved figure. Thus, a general understanding of the parameters, by the user, is essential for effective use of the code. These parameters are the boundary conditions for each wall, the thermal diffusivity, and the general position and time arrays. Of these, the only values of general interest are the thermal diffusivity and the boundary conditions.

1.2 Theory

In the fields of physics and mathematics, the heat equations is a fundamental equation that has been thoroughly explored since it's discovery in the early 1800s. It is a partial differential equation that describes how heat is diffused and distributed over time within a specified medium. This diffusion is spontaneous and occurs due to interactions between atoms within a physical system. This concept of heat diffusion, or energy diffusion, is known as entropy and is a centerpiece within the heat equation itself (although not explicitly).

When working with the heat equation we expect that our models will have this entropic tendency as discovered by Joseph Fourier in 1822. His discovery was brought about from his studies into modeling heat flow. Now the heat equation is used in many areas of physics and has lead to the further discovery of Brownian motion (via the Fokker-Planck equation) and 'Random Walks' equations. Due to the fundamental nature of the equation it is used in many sub-categories of physics such as condensed matter physics, astrophysics, applied physics, etc. Likewise, it is even used outside of physics and mathematics. Once such example is within finance.

As stated previously, the heat equation is defined by the positional dimensions and by the thermal values. This is shown in the equation below.

$$u(x,t) = X(x)T(t)$$

Above is the heat equation with respect to time. The left hand side defines the general solution to the heat equation as a function u(x,t). The right hand side denotes the two primary variables associated with the equation. X(x) denotes the spatial position (can be expanded to Y(y) and Z(z). The second term on the RHS is associated with the thermal value. This value is described at each point in the spatial domain and is a function of time.

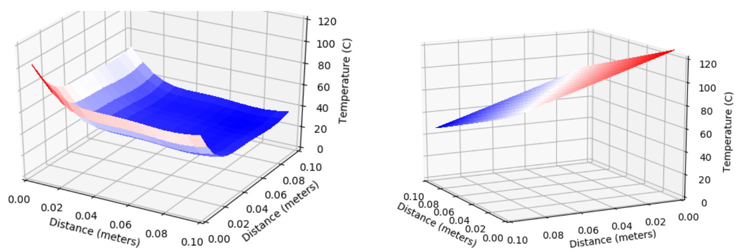
1.3 Experiment

Building onto a previously constructed framework (Wave Equation), I have created and attached the Python code that performs the tasks that were previously stated. The code was constructed using the specified initial conditions and positional and time arrays and results were animated and printed. The animation transposing the new solution onto the already existing plot to give the appearance of an animation. In a sense, it is much like a flip book where the image slowly changes from one solution to the next so we can follow with it's evolution. Spatial and time steps were created in order to cycle the array of values through the equation and output the results, but no data sets were required to obtain the animation as the code creates its own solution based on the input array. This code was created with the explicit goal of being modified to perform other calculations in the future thus it is a continuous project.

1.4 Conclusion

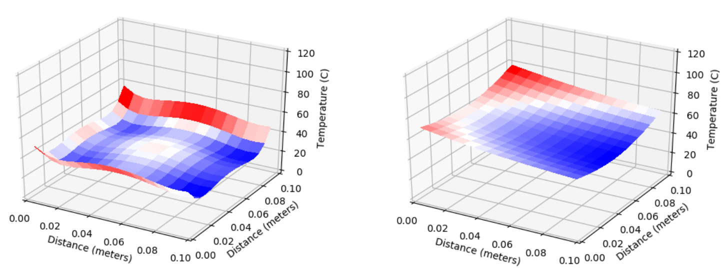
From the observations of the results, a conclusion can be made that the code does in fact perform the tasks required. Animations of the solutions to the heat equation are displayed and the user has the ability to manipulate the thermal diffusivity of the medium, temperature of the boundary walls, and the temperature in specific regions of the space. This fact is illustrated below.

System One:



Solution to the heat equation for system one. The figure to the left is the initial thermal condition of the system before it diffuses and the figure to the right is the final thermal solution. Thermal diffusivity = 0.000228. Thermal boundary conditions: [Wall One: 62 deg. C, Wall Two: 20 deg. C, Wall Three: 71 deg. C, Wall 4: 21 deg. C].

System Two:



Solution to the heat equation for system two. The figure to the left is the initial thermal condition of the system before it diffuses and the figure to the right is the final thermal solution. Thermal diffusivity = 0.000068. Thermal boundary conditions: [Wall One: 31 deg. C, Wall Two: 13 deg. C, Wall Three: 43 deg. C, Wall 4: 53 deg. C].

Fundamentally, the job of a physicist or a mathematician is to break complicated problems down into their component parts and access how to construct a model that describes it. Differential equations is simply one tool used in order to construct a more accurate model, but now that we have computers to assist us in our computations codes like this one can become fundamental tools as well. We have showed that, for differing systems, we can accurately model the thermal heat equation. This visualization of the function also gives us insight into how the geometrical dimensions, thermal diffusivity of the medium, and the boundary conditions all play a role in the final solution, thus we can say that this code can be used as a fundamental tool for future systems.

2 Section Two: The Traveling Wave Equation and Propagated Wave Aspirations

2.1 Theory

The wave equation is a fundamental function used commonly in many fields of physics. It is a valued second-order linear partial differential equation that gives a description classical wave mechanics. This is akin to understanding and measuring the dynamics of water waves, sound wave, seismic waves, light waves, and other classical waves. The function also arises commonly in electromagnetism, fluid mechanics, thermal dynamics, and optics. Because of it's broad use the differential equation has been well studied and many solutions and derivations of it have been explored.

The wave equation is defined by the following partial differential equation:

$$\ddot{u} = c^2 \nabla^2 u$$

In this equation the left hand side is the general function evaluated as the double derivative with respect to time. On the right hand side we have the derivative of the general function partially derived with respect to space.

This is shown more clearly with the following equation:

$$\frac{\partial^2 u}{\partial t^2} \;=\; c^2 \big(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \cdots + \frac{\partial^2 u_n}{\partial x_n^2}\big)$$

Here we can see that the left hand side is the partial derivative of the general function (u) with respect to time, but on the right hand side we can clearly see that the general function (u) is being derived by each spatial parameter. This is due to the multiplication of the general function with the Laplacian. In this case the function is also multiplied by a constant squared (c**2).

Traveling Wave Equation:

$$y = A \sin \frac{2\pi(x - vt)}{\lambda}.$$

First attempts at modifying the midterm code in order to create the propagated wave animations guided me to the traveling wave function shown above. On the RHS of the equation we are given a few parameters and variables. A denotes the amplitude of the wave at each spatial point along x and y. The x value is input as the spatial array in the 1-dimensions but this can be expanded into 2 dimensions. v dictates the velocity of the traveling wave and t dictates the time array of which the general solution will evolve in. Parameters of the equation is given by the lambda function which is the speed that the wave can travel through the medium. The LHS of the equation is the general solution to the equation.

Inhomogenous Wave Equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u = v,$$

In our efforts to modify the code to computationally animate wave propagation, we attempt to solve a system of equations for the inhomogenous wave equation. Regrettably however, we were not able to computationally mimic the inhomogenous wave equation, thus we had to display the solutions to the traveling wave equation above. This general solution (v) is given by the second derivative with respect to time of the speed of the wave propagation subtracted by the second derivative of the spatial variables. Each of these terms is multiplied by the function u, thus distributions and separation of variables is required to find a general system of equations to model this. However, I ran into the issue of complex numbers and could not reconcile that into the code I developed so this will have to be modified in the future.

The parameters present within the equations are listed as follows:

1. Time - In the equation t denotes time. In the code, time is accounted for by an initial time ("ti") and an array which designates the amount of time

that passed since "ti". Likewise, a time step parameter is added to allow us to evolve the equation through the length of the array. This is given by "dt".

2. Space - From the equation we see that the spacial parameters are given by the number of spatial dimensions we are looking to observe. This is given by the Laplacian which attributes the number of dimensions (x,y,z) to the partial derivatives. Within the code this is given by x and y as we are only measuring two spatial domains and one time domain. this parameter also has a step size in the code which is given by "fx" and "fy"

3. Wave Speed - The final parameter is the wave speed. In the equation this is dictated by the constant value "c". Typically in ideal systems this value is equal to the speed of light but in this case we can set it to what ever value we want. This parameter is listed in the code as well.

Historically, the main equation was explored via the solution to the one dimensional wave equation. This is very similar to the dynamics of a wave travelling along a string. In fact, this is exactly how the first wave equation was discovered. In 1746, a French mathematician named Jean le Rond d'Alembert was studying the dynamics of how a string vibrated when plucked on an musical instrument. He later discovered the first 1-Dimensional wave equation. He was later joined by Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange in the studies of the function. Within 10 years of the initial discovery Euler discovered the 3-Dimensional wave equation. Years after the initial discoveries we are still studying the dynamics of these fundamental equations within more complex systems, all thanks to the work of a few interested mathematicians and physicists within the 1700s.

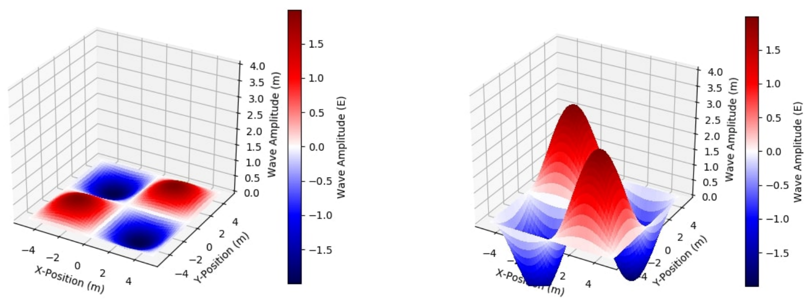
2.2 Experiment

As stated previously, for the final project I would like to modify my midterm code in order to produce an inhomogenous wave guide that will simulate sound waves or any other type of traveling wave. In order to do this I must reconstruct the model for the wave equation using python as my medium. The final goal is to explore these systems by changing the initial conditions and to animate them in order to gain an understanding of the fundamental dynamics of the systems in question and how waves propagate through them. Accompanied with this document is an attached code. The goal is to have to the user input different initial parameters and boundary conditions in order to have a wave propagate through the medium as stated previously. Ultimately, the insertion of objects within the medium is desired in order to see how waves interact with barriers.

2.3 Conclusions from the Midterm

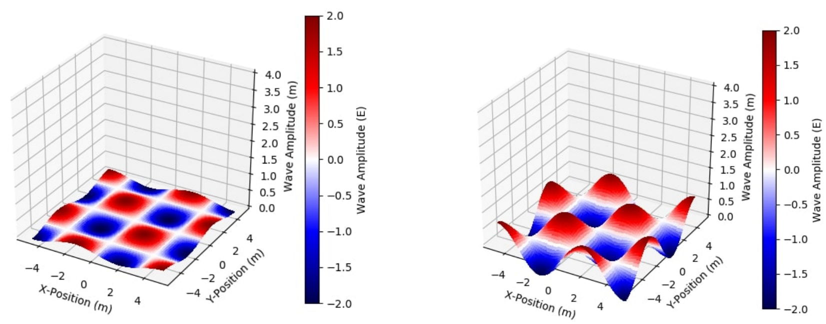
Here we have the solution for the n = 2 excited state of the wave equation. This function is distinctly characterized by the two wave formations in both the x and y domains. Animations of the the function are shown below.

System Three:



To the left we have the n=2 wave functions at their inflection point (where they flip). On the right, is the second image of the animation just before the peaks descend. The energy is characterized by the height of each peak.

System Four:



To the left is the inflection point for the n = 3 excited state wave form. Noticeably the number of wave forms added up along the x and y domains is 3. Just like the previous excited state, we have the n=3 excited state just after it's inflection on the right side. In this figure we do not see it's maximum but we can clearly see the waves rising.

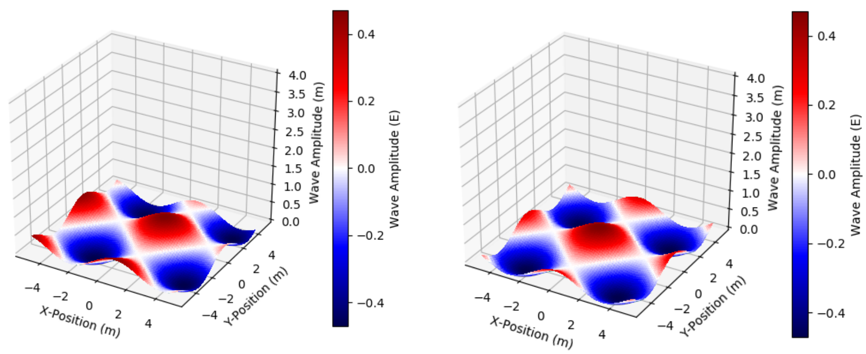
We have successfully created a code that displays the n number of excited states of the wave equation in 3 dimensions. After experimentation and testing of the code I have selected two cases of an n = 2 and n = 3 excitation case for the wave equation. The position space was measured in meters from the origin and the wave amplitude was attributed to the wave energy (the higher the amplitude the higher the energy present in the wave within that region in space.)

2.4 Conclusions from Modifying the Midterm

We begin this section by discussing our failure to produce an animation for the propagated waveform as stated earlier. Ultimately, the implementation of the inhomogenous wave equation is required in order to continue, but the limitations at the moment come in the form of understanding how to use complex integers whilst looping over them with an array of real numbers. As we know, real and imaginary numbers not like to play together so getting python to understand the logic behind our equation will require a bit of finesse.

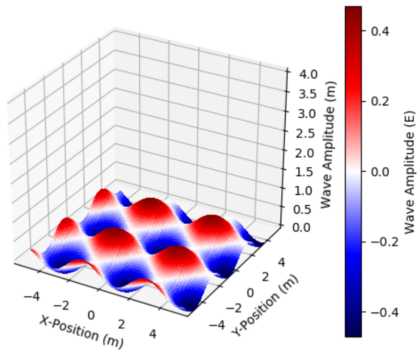
Regardless of our setbacks, we were manage to manipulate the code to allow for a wave to travel through a medium as opposed to standing in a single spatial domain. This is a step forward toward our final goal so this is something worth discussing. Below are examples of two systems where this was displayed. We will attempt to mimic the case of the n=2 and n=3 solutions to the traveling wave equation as we have explored this previously.

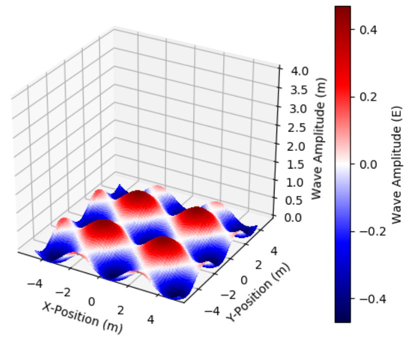
System Five:



For system five, we have the traveling wave guide for an n=2 excitation state. The figure on the left is the wave guide at the initial time and the figure to the right is the time-evolved wave.

System Six:





For system six, we have the traveling wave guide for an n=3 excitation state. The figure on the left is the wave guide at the initial time and the figure to the right is the time-evolved wave.

From systems five and six, we can clearly see that we have obtained the modified form of the midterm code and created a tool that allows for wave propagation through a medium. However, what we didn't obtain was a code that could propagate from a single source and move around barriers within a medium. Future work is needed in order to modify the code to perform these tasks.

2.5 References: Midterm

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2. Dall, P. "String Wave Solutions." Wave Equation, Wave Packet Solution, 12 May 2017, hyperphysics.phy-astr.gsu.edu/hbase/Waves/wavsol.html.

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5. Hannon. "The Wave Equation." The Physics Classroom, 23 May 2018, www.physicsclassroom.com/class/waves/Lesson-2/The-Wave-Equation.

2.6 References: Heat Equation and Traveling Wave/Inhomogenous Wave Equation

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2. Stanford University. Web.Stanford.edu/Heat Equation. 24 Mar. 2013, web.stanford.edu/class/math220b/handouts/heateqn.pdf.

3. Fitzpatrick, Richard. "Solution of Inhomogeneous Wave Equation." Solution of Inhomogeneous Wave Equation, 27 June 2014.

4. School of Physics, Sydney Australia, UNSW. "Travelling Sine Wave." Travelling Sine Wave: from Physclips, 2 Apr. 2017.

2.7 PART 2 : Finding your data

For this section of the project I didn't need any data sets as the animation actually creates its own data. The only things that were required as inputs into the function were simply initial conditions for the code to run. These were informed values based on specific excited states of interest. Wave equation data sets could be available for specific systems such as waves traveling through a pool or light traveling from a distant galaxy but these are simply measured by changing the bounds for the wave function to develop over. Thus working with data doesn't seem to make sense here as I am simply making an animation of such excited states. The project itself was very complicated to produce and additional datasets would further complicate it but it can be pursued in the future.