CS21
Decidability and Tractability

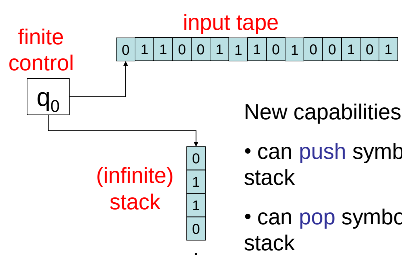
Lecture 5

January 13, 2010

Outline

- Context-Free Grammars and Languages
  - ambiguity
  - normal form
- equivalence of NPDAs and CFGs

Pushdown Automata



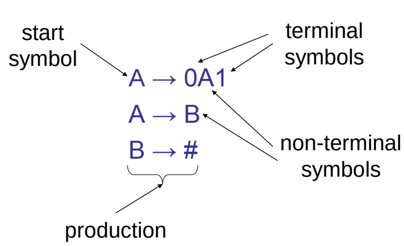
New capabilities:

- can push symbol onto stack
- can pop symbol off of stack

Context-free grammars and languages

- languages recognized by a (N)FA are exactly the languages described by regular expressions, and they are called the regular languages
- languages recognized by a NPDA are exactly the languages described by context-free grammars, and they are called the context-free languages

Context-Free Grammars



Context-Free Grammars

- generate strings by repeated replacement of non-terminals with string of terminals and non-terminals
  - write down start symbol (non-terminal)
  - replace a non-terminal with the right-hand-side of a rule that has that non-terminal as its left-hand-side.
  - repeat above until no more non-terminals

Context-Free Grammars

Example:

A ⇒ 0A1 ⇒ 00A11 ⇒ 000A111 ⇒ 000B111 ⇒ 000#111

A → 0A1
A → B
B → #

- a derivation of the string 000#111
- set of all strings generated in this way is the language of the grammar L(G)
- called a Context-Free Language

Context-Free Grammars

- Natural languages (e.g. English) structure:

shorthand for multiple rules with same lhs

<sentence> → <noun-phrase><verb-phrase>
<noun-phrase> → <cpx-noun> | <cpx-noun><prep-phrase>
<verb-phrase> → <cpx-verb> | <cpx-verb><prep-phrase>
<prep-phrase> → <prep><cpx-noun>
<cpx-noun> → <article><noun>
<cpx-verb> → <verb>|<verb><noun-phrase>
<article> → a | the
<noun> → dog | cat | flower
<verb> → eats | sees
<prep> → with

Generate a string in the language of this grammar.

Context-Free Grammars

- CFGs don't capture natural languages completely
- computer languages often defined by CFG
  - hierarchical structure
  - slightly different notation often used "Backus-Naur form"
  - see next slide for example

Example CFG

<stmt> → <if-stmt> | <while-stmt> | <begin-stmt> | <asgn-stmt>
<if-stmt> → IF <bool-expr> THEN <stmt> ELSE <stmt>
<while-stmt> → WHILE <bool-expr> DO <stmt>
<begin-stmt> → BEGIN <stmt-list> END
<stmt-list> → <stmt> | <stmt>; <stmt-list>
<asgn-stmt> → <var> := <arith-expr>
<bool-expr> → <arith-expr><compare-op><arith-expr>
<compare-op> → < | > | ≤ | ≥ | =
<arith-expr> → <var> | <const> | (<arith-expr><arith-op><arith-expr>)
<arith-op> → + | - | * | /
<const> → 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9
<var> → a | b | c | … | x | y | z

CFG formal definition

- A context-free grammar is a 4-tuple

(V, Σ, R, S)

where
  - V is a finite set called the non-terminals
  - Σ is a finite set (disjoint from V) called the terminals
  - R is a finite set of productions where each production is a non-terminal and a string of terminals and non-terminals.
  - S ∊ V is the start variable (or start non-terminal)

CFG formal definition

- u, v, w are strings of non-terminals and terminals, and A → w is a production:

"uAv yields uwv" notation: $uAv \Rightarrow uwv$

also: "yields in 1 step" notation: $uAv \Rightarrow^1 uwv$

- in general:

"yields in k steps" notation: $u \Rightarrow^k v$

  - meaning: there exists strings $u_1, u_2, \ldots u_{k-1}$ for which $u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \ldots \Rightarrow u_{k-1} \Rightarrow v$

CFG formal definition

- notation: $u \Rightarrow^* v$
  - meaning: $\exists$ k ≥ 0 and strings $u_1,\ldots,u_{k-1}$ for which $u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \ldots \Rightarrow u_{k-1} \Rightarrow v$
- if u = start symbol, this is a derivation of v
- The language of G, denoted L(G) is:

  $\{w \in \Sigma^* : S \Rightarrow^* w\}$

CFG example

- Balanced parentheses:
  - ( )
  - (( ) ((( ) ( ))))
- a string w in Σ* = { (,) }* is balanced iff:
  - # "("s equals # ")"s, and
  - for any prefix of w, # "("s ≥ # ")"s

Exercise: design a CFG for balanced parentheses.

CFG example

- Arithmetic expressions over {+,*,(,),a}
  - (a + a) * a
  - a * a + a + a + a + a
- A CFG generating this language:

  <expr> → <expr> * <expr>
  <expr> → <expr> + <expr>
  <expr> → (<expr>) | a

CFG example

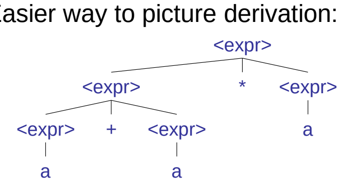
<expr> → <expr> * <expr>
<expr> → <expr> + <expr>
<expr> → (<expr>) | a

- A derivation of the string: a+a*a

  <expr> ⇒ <expr> * <expr>
  ⇒ <expr> + <expr> * <expr>
  ⇒ a + <expr> * <expr>
  ⇒ a + a * <expr>
  ⇒ a + a * a

Parse Trees

- Easier way to picture derivation: parse tree



- grammar encodes grouping information; this is captured in the parse tree.

CFGs and parse trees

<expr> → <expr> * <expr>
<expr> → <expr> + <expr>
<expr> → (<expr>) | a

- Is this a good grammar for arithmetic expressions?
  - can group wrong way (+ precedence over *)
  - can also group correct way (ambiguous)

Solution to first problem

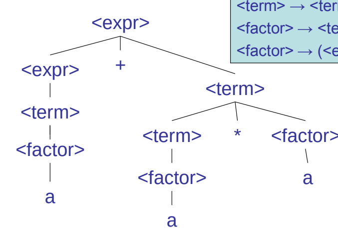
<expr> → <expr> + <term> | <term>
<term> → <term> * <factor> | <factor>
<factor> → <term> * <factor>
<factor> → (<expr>) | a

- forces correct precedence in parse tree grouping
  - within parentheses, * cannot occur as ancestor of + in the parse tree.

Parse Trees

- parse tree for a + a * a in new grammar:

<expr> → <expr> + <term> | <term>
<term> → <term> * <factor> | <factor>
<factor> → <term> * <factor>
<factor> → (<expr>) | a



Ambiguity

- Second problem: ambiguous grammar
- Definitions:
  - a string is derived ambiguously if it has two different parse trees
  - a grammar is ambiguous if its language contains an ambiguously derived string
- ambiguity sometimes undesirable
- some CFLS are inherently ambiguous

Ambiguity

- Definition in terms of derivations (rather than parse trees):
  - order in which we replace terminals in shouldn't matter (often several orders possible)
  - define leftmost derivation to be one in which the leftmost non-terminal is always the one replaced
  - a string is ambiguously derived if it has 2 leftmost derivations

Chomsky Normal Form

- Useful to deal only with CFGs in a simple normal form
- Most common: Chomsky Normal Form (CNF)
- Definition: every production has form

  A → BC or S → ε or
  A → a

  where A, B, C are any non-terminals (and B, C are not S) and a is any terminal.

Chomsky Normal Form

<u>Theorem</u>: Every CFL is generated by a CFG in Chomsky Normal Form.

<u>Proof</u>: Transform any CFG into an equivalent CFG in CNF. Four steps:
- add a new start symbol
- remove "ε-productions" A → ε
- eliminate "unit productions" A → B
- convert remaining rules into proper form

Chomsky Normal Form

- add a new start symbol
  - add production $S_0 \rightarrow S$
- remove "ε-productions" $A \rightarrow \varepsilon$
  - for each production with A on rhs, add production with A's removed: e.g. for each rule $R \rightarrow uAv$, add $R \rightarrow uv$
- eliminate "unit productions" $A \rightarrow B$
  - for each production with B on lhs: $B \rightarrow u$, add rule $A \rightarrow u$

Chomsky Normal Form

- convert remaining rules into proper form
  - replace production of form:

$$A \rightarrow u_1U_2u_3\ldots u_k$$

with:

$A \rightarrow U_1A_1$ $U_1 \rightarrow u_1$

$A_1 \rightarrow U_2A_2$

$A_2 \rightarrow U_3A_3$ $U_3 \rightarrow u_3$

:

$A_{k-2} \rightarrow U_{k-1}U_k$ $U_{k-1} \rightarrow u_{k-1}$ $U_k \rightarrow u_k$

U_2 already a non-terminal

Some facts about CFLs

- CFLs are closed under
  - union (proof?)
  - concatenation (proof?)
  - star (proof?)
- Every regular language is a CFL
  - proof?