Notes on Financial Modeling

1 Black-Scholes Model

BASIC SETTING

We consider the following processes under the objective measure $\mathbb{P}$:

$$\begin{aligned} \mathrm{d}B_t &= rB_t \mathrm{d}t \\ \mathrm{d}S_t &= \alpha S_t \mathrm{d}t + \sigma S_t \mathrm{d}\bar{W}_t \end{aligned} \tag{S}$$

where r is a constant; α and σ are $\alpha(t, S_t)$ and $\sigma(t, S_t)$ respectively.

DERIVE THE ARBITRAGE-FREE PRICE PROCESS

Assume $\chi = \Phi(S_T)$ is a contingent claim with the date of maturity T. We want to find the arbitrage-free price of $\chi = \Phi(S_T)$.

1. Let $\Pi(t) = F(t, S_t)$ denote the price of χ at time t. Apply Itô's formula:

$$\begin{aligned} \mathrm{d}\Pi &= F_t \mathrm{d}t + F_s \mathrm{d}S + \frac{1}{2} F_{ss} \mathrm{d}[S] \\ &= F_t \mathrm{d}t + F_s \underbrace{(\alpha S \mathrm{d}t + \sigma S \mathrm{d}\bar{W})}_{\text{(S)}} + \frac{1}{2} F_{ss} \underbrace{\sigma^2 S^2 \mathrm{d}t}_{\text{(S)}} \\ &= \alpha_\pi \Pi \ \mathrm{d}t + \sigma_\pi \Pi \ \mathrm{d}\bar{W} \end{aligned}$$

   where $\alpha_\pi = (F_t + F_s \alpha S + \frac{1}{2} F_{ss} \sigma^2 S^2)/\Pi$ and $\sigma_\pi = (F_s \sigma S)/\Pi$.

2. Form a new relative portfolio for the given two assets S and Π, denoted by (u_s, u_π). It means, if we have V dollars, then we use $V \cdot u_s$ dollars to buy stocks (**Note**: totally we can buy $\frac{V_t \cdot u_s(t)}{S_t}$ stocks at time t) and use rest of our money to buy Π. By self-financing,

$$\begin{aligned} \mathrm{d}V &= V(u_s \frac{\mathrm{d}S}{S} + u_\pi \frac{\mathrm{d}\Pi}{\Pi}) \\ &= V\left[u_s \alpha + u_\pi \alpha_\pi\right] \mathrm{d}t + V\left[u_s \sigma + u_\pi \sigma_\pi\right] \mathrm{d}\bar{W} \end{aligned}$$

3. Notice that we can always construct a locally risk-less portfolios by choosing (u_s, u_π) such that the equation $u_s \sigma + u_\pi \sigma_\pi = 0$ holds; so we have

$$\begin{cases} u_s \sigma + u_\pi \sigma_\pi = 0 \\ u_s + u_\pi = 1 \end{cases}$$

   Solve it.

$$\begin{cases} u_s = \frac{\sigma_\pi}{\sigma_\pi - \sigma} \\ u_\pi = \frac{-\sigma}{\sigma_\pi - \sigma} \end{cases} \tag{1}$$

4. Moreover, by the arbitrage-free condition, we must have

$$u_s \alpha + u_\pi \alpha_\pi = r.$$

   In (1), we have solved u_s and u_π. And we also have solved α_π and σ_π. Then we obtain the **Black-Sholes equation**:

$$\begin{cases} F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF = 0 \\ F(T, s) = \Phi(s) \end{cases} \tag{Black-Sholes}$$

5. LAST STEP! By Feynman-Kac formula, the solution to the PDE can be represented as

$$F(t,s) = e^{-r(T-t)}\mathbb{E}^{\mathbb{P}}_{t,s}\left[\Phi(X_T)\right]$$

where X is an Itô diffusion defined by

$$\begin{aligned} \mathrm{d}X_u &= rX_u\mathrm{d}u + \sigma X_u \mathrm{d}W_u, \\ X_t &= s. \end{aligned} \tag{2}$$

Note that W is a Brownian motion under the objective measure $\mathbb{P}$. The only difference between (2) and (S) is that the drift for S is α rather than r. Applying the Girsanov's theorem, X can be explained as the stock price S on a new measure $\mathbb{Q}$. Therefore, we re-write our basic setting under $\mathbb{Q}$ as

$$\mathrm{d}S = rS\mathrm{d}t + \sigma S\mathrm{d}S.$$

Then we have the following useful formula:

$$F(t,s) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}_{t,s}\left[\Phi(S_T)\right] \tag{Price}$$

Example 1.1 (Pricing European Call option)**.** Now consider the simplest case:

$$\begin{aligned} \mathrm{d}B_t &= rB_t\mathrm{d}t \\ \mathrm{d}S_t &= \alpha S_t\mathrm{d}t + \sigma S_t\mathrm{d}\bar{W}_t \end{aligned}$$

where r, α, and σ are constant.

We want to price an European call option with the underlying stock S, the maturity time T, and the strike price K. Let its arbitrage-free price process bet $C(t,s)$. Let $\Phi(x) = (x-K)^+$. We directly use the formula (Price):

$$\begin{aligned} C(t,s) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}_{t,s}\left[(S_T-K)^+\right] \\ &= e^{-r(T-t)}\left\{\underbrace{\mathbb{E}^{\mathbb{Q}}_{t,s}\left[S_T\mathbf{1}\{S_T\ge K\}\right]}_{\text{Part I}} - K\underbrace{\mathbb{E}^{\mathbb{Q}}_{t,s}\left[\mathbf{1}\{S_T\ge K\}\right]}_{\text{Part II}}\right\} \end{aligned}$$

Recall that S is a geometric Brownian motion

$$S_T = S_t\exp\left\{(r-\frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)\right\}.$$

Then we can represent Part I and Part II using the CDF of the standard normal random variable $N(\cdot)$. Finally, we find

$$C(t,s) = e^{-r(T-t)}\left[se^{r(T-t)}N(d_1) - KN(d_2)\right]$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\frac{s}{K} + (r+\frac{1}{2}\sigma^2)(T-t)\right], \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

This result is also known as **Black-Scholes formula**.

Theorem 1.2 (Put-Call parity)**.** *Let $P(T,K)$ and $C(T,K)$ be the price of an European put option and call option at time 0 respectively with underlying asset S, maturity T, and strike K. Then*

$$P(T,K) = Ke^{-r(T-t)} + C(T,K) - S_0.$$

Proof. By (Price), we directly have

$$P(T,K) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(K-S_T)^+ \,|\, S_0\right],$$
$$C(T,K) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(S_T-K)^+ \,|\, S_0\right].$$

Then let $C(T,K) - P(T,K)$:

$$\begin{aligned} P(T,K) - C(T,K) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(K-S_T)^+ - (S_T-K)^+ \,|\, S_0\right] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[K - S_T \,|\, S_0\right] \\ &= Ke^{-rT} - S_0 \end{aligned}$$

□

Example 1.3 (Pricing $\ln S_T$). • We want to find the arbitrage free price of $\Phi(S(T)) = \ln S(T)$. We directly use Theorem 7.8 in the textbook:

$$\Pi(t) = e^{-\mu(T-t)}\mathbb{E}^{Q}_{t,s}[\ln S(T)]. \tag{thm7.8}$$

- Now we consider the distribution of $\ln S(T)$ under the risk-neutral measure Q. Under the standard Black-Scholes model,

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{7.43-7.44}$$

where W is a Brownian motion under the risk-neutral measure Q; and we assume $S_t = s$. We know that S is a geometric Brownian motion; so

$$\begin{aligned} S_T &= s\exp\left((\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)\right) \\ \ln S_T &= \ln s + (\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t) \\ \mathbb{E}^{Q}_{t,s}(\ln S_T) &= \ln s + (\mu - \frac{\sigma^2}{2})(T-t) \end{aligned}$$

- We plug it into (thm7.8):

$$\Pi(t) = e^{-\mu(T-t)} \cdot \left(\ln s + (\mu - \frac{\sigma^2}{2})(T-t)\right).$$

Example 1.4 (Pricing S_T^β). • We want to find the arbitrage free price of $\Phi(S(T)) = S^\beta(T)$. We directly use Theorem 7.8 in the textbook:

$$\Pi(t) = e^{-\mu(T-t)}\mathbb{E}^{Q}_{t,s}[S^\beta(T)]. \tag{thm7.8}$$

- Now we consider the distribution of $S^\beta(T)$ under the risk-neutral measure Q. Under the standard Black-Scholes model,

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{7.43-7.44}$$

where W is a Brownian motion under the risk-neutral measure Q; and we assume $S_t = s$.. Using Itô's formula to $S^\beta(t)$:

$$dS_t^\beta = \left(\mu\beta + \frac{\sigma^2}{2}\beta(\beta-1)\right) S_t^\beta dt + \sigma\beta S_t^\beta dW_t.$$

Obviously, S^β is a geometric Brownian motion. We know the expectation of $S^\beta(T)$ is

$$\mathbb{E}^{Q}_{t,s}(S^\beta(T)) = s^\beta \exp\left((\mu\beta + \frac{\sigma^2}{2}\beta(\beta-1))(T-t)\right).$$

- Therefore, we just plug it into the formula (thm7.8):

$$\Pi_t = s^\beta \exp\left((\mu(\beta-1) + \frac{\sigma^2}{2}\beta(\beta-1))(T-t)\right).$$

Example 1.5 (Pricing $K \cdot \mathbf{1}_{[\alpha,\beta]} \circ S_T$). • First, the binary option is given by

$$\Phi(S_T) = \begin{cases} K & S_T \in [\alpha,\beta] \\ 0 & \text{o.w.} \end{cases}$$

Then we want to compute

$$\Pi(t) = e^{-\mu(T-t)}\mathbb{E}^Q_{t,s}[\Phi(S_T)]. \tag{thm7.8}$$

- Now we directly compute the expectation of $\Phi S(T)$ under the risk-neutral measure Q. Under the standard Black-Scholes model, we have

$$S_T = s\exp\left((\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)\right)$$

where W is a Brownian motion under Q. Obviously, under Q, $W_T - W_t \sim N(0, T-t)$.

Therefore, we compute its expectation

$$\begin{aligned}\frac{1}{K}\mathbb{E}^Q_{t,s}[\Phi(S_T)] &= Q\left[\alpha \le S_T \le \beta\right] \\ &= Q\left[\ln\alpha \le \ln s + (\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T-W_t) \le \ln\beta\right] \\ &= Q\left[\frac{\ln(\alpha/s)}{\sqrt{T-t}\sigma} \le (\mu-\frac{\sigma^2}{2})\sqrt{T-t}/\sigma + (W_T-W_t)/\sqrt{T-t} \le \frac{\ln(\beta/s)}{\sqrt{T-t}\sigma}\right] \\ &= N(B) - N(A)\end{aligned}$$

where N is the cdf of $N(0,1)$, $A = \frac{\ln(\alpha/s)}{\sqrt{T-t}\sigma} - (\mu - \frac{\sigma^2}{2})\sqrt{T-t}/\sigma$, $B = \frac{\ln(\beta/s)}{\sqrt{T-t}\sigma} - (\mu - \frac{\sigma^2}{2})\sqrt{T-t}/\sigma$.

- We plug it into (thm7.8):

$$\Pi(t) = Ke^{-\mu(T-t)}\left(N(B) - N(A)\right)$$

where A and B are given above.

Proposition 1.6. *Linearity*

Note The main idea for Exercise 9.1-9.3 is rewriting the claim as a linear combination of Φ_B, Φ_S and $\Phi_{C,K}$ (given by formula (9.2)-(9.4)). Then apply Proposition 9.1 and formula (9.5)-(9.7) on page 126.

Example 1.7. • Notice that

$$\begin{aligned}\Phi(S_T) &= K\mathbf{1}\{S_T \le A\} + (K + A - S_T)\mathbf{1}\{A < S_T < K + A\} \\ &= K\mathbf{1}\{S_T \le A\} + (K + A - S_T)\left(\mathbf{1}\{S_T \ge A\} - \mathbf{1}\{S_T \ge K + A\}\right) \\ &= K\mathbf{1}\{S_T \le A\} + K\cdot\mathbf{1}\{S_T \ge A\} + (A - S_T)\cdot\mathbf{1}\{S_T \ge A\} - (K + A - S_T)\cdot\mathbf{1}\{S_T \ge K + A\} \\ &= K\cdot\Phi_B(S_T) - \Phi_{C,A}(S_T) + \Phi_{C,K+A}(S_T)\end{aligned}$$

where $\Phi_B(x) := 1$ and $\Phi_{C,K}(x) := (x - K)\cdot\mathbf{1}\{x \ge K\} = \max[x - K, 0]$.

- Then by Proposition 9.1:

$$\begin{aligned}\Pi(t;\Phi) &= K\cdot\Pi(t;\Phi_B) - \Pi(t;\Phi_{C,K}) + \Pi(t;\Phi_{C,K+A}) \\ \text{(Using formula (9.5)-(9.7))} &= K\cdot e^{-r(T-t)} - c\left(t, S(t); K, T\right) + c\left(c, S(t); A+K, T\right)\end{aligned}$$

where the standard Black-Scholes model is given by

$$dS_u = rS_u du + \sigma S_u dW_u.$$

Example 1.8. • Notice that

$$\begin{aligned}\Phi(S_T) &= (K - S_T)\mathbf{1}\{S_T \le K\} + (S_T - K)\mathbf{1}\{K < S_T\} \\ &= (K - S_T)(1 - \mathbf{1}\{S_T > K\}) + \Phi_{C,K} \\ &= (K \cdot \Phi_B - \Phi_S + 2\Phi_{C,K})\,(S_T)\end{aligned}$$

- Then by Proposition 9.1:

$$\begin{aligned}\Pi(t;\Phi) &= K \cdot \Pi(t;\Phi_B) - \Pi(t;\Phi_S) + 2\Pi(t;\Phi_{C,K}) \\ &= K \cdot e^{-r(T-r)} - S(t) + 2c(t, S(t); K, T).\end{aligned}$$

Example 1.9. • Notice that

$$\begin{aligned}\Phi(x) &= B\mathbf{1}\{x > B\} + x\mathbf{1}\{A \le x \le B\} + A\mathbf{1}\{x < A\} \\ &= B\mathbf{1}\{x > B\} + ((x - A + A)\mathbf{1}\{x \ge A\} - (x - B + B)\mathbf{1}\{x \ge B\}) + A\mathbf{1}\{x < A\} \\ &= \Phi_{C,A}(x) - \Phi_{C,B}(x) + A \cdot \Phi_B(x)\end{aligned}$$

- Then by Proposition 9.1:

$$\begin{aligned}\Pi(t;\Phi) &= A \cdot \Pi(t;\Phi_B) + \Pi(t;\Phi_{C,A}) - \Pi(t;\Phi_{C,B}) \\ &= A \cdot e^{-r(T-t)} + c(t, S(t); A, T) - c(t, S(t); B, T).\end{aligned}$$

GREEKS

Let $P(t, s)$ be the pricing function for a portfolio based on a single underlying asset. For example, it could be the price process of an European call option; it means we put all of money in this option. And we are interested in its sensitivity with respect to the price change of the underlying asset or changes in the model parameters.

Definition 1.10 (Greeks)**.**

$$\begin{aligned}\Delta &= \frac{\partial P}{\partial s} \\ \Gamma &= \frac{\partial^2 P}{\partial s^2} \\ \rho &= \frac{\partial P}{\partial r} \\ \Theta &= \frac{\partial P}{\partial t} \\ \mathcal{V} &= \frac{\partial P}{\partial \sigma}\end{aligned}$$

Example 1.11 (Greeks for European call option)**.** Let $N(\cdot)$ be the CDF and φ be the PDF of standard normal distribution. Then the corresponding greeks are

$$\begin{aligned}\Delta &= N(d_1) \\ \Gamma &= \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}} \\ \rho &= K(T-t)e^{-r(T-t)}N(d_2) \\ \Theta &= -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2) \\ \mathcal{V} &= s\varphi(d_1)\sqrt{T-t}\end{aligned}$$

where

$$\begin{aligned}d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\frac{s}{K} + (r + \frac{1}{2}\sigma^2)(T-t)\right], \\ d_2 &= d_1 - \sigma\sqrt{T-t}.\end{aligned}$$

Example 1.12 (Greeks for European put option)**.** • By Put-Call parity:

$$p(t,s) = Ke^{-r(T-t)} + c(t,s) - s \tag{9.11}$$

where $c(t,s) := c(t,s;K,T)$.

- Recall that $\Delta_P := \frac{\partial P}{\partial s}$ where P is the pricing function for some options. We take derivative w.r.t. s on both sides of the equation (9.11):

$$\begin{aligned}\frac{\partial}{\partial s}p &= 0 + \frac{\partial}{\partial s}c - 1\\ \Delta_p &= \Delta_c - 1\end{aligned}$$

  By Proposition 9.5, we know $\Delta_c = N(d_1)$. So

$$\Delta_p = N(d_1) - 1.$$

- For Γ, all steps are same. We take $\frac{\partial^2}{\partial s^2}$ on both sides of the equation (9.11):

$$\begin{aligned}\frac{\partial^2}{\partial s^2}p &= \frac{\partial^2}{\partial s^2}c\\ \Gamma_p &= \Gamma_c\end{aligned}$$

  By Proposition 9.5, we know $\Delta_c = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}$. So

$$\Gamma_p = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}.$$

Theorem 1.13 (Vega-Gamma relation)**.** *content...*

BARRIER OPTIONS

In this subsection, we will consider some more complicated options; for example, the price process may be path-dependent.

DIVIDENDS

2 More Complicated Model

LOCAL VOLATILITY MODEL

Basic Setting We consider a little bit more complicated case:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(t,S_t)S_t\mathrm{d}\bar{W}_t$$

where r, α are constant; and this dynamics are given on the measure $\mathbb{Q}$.

Dupire Formula Now we take derivative with respect to T rather than K.

$$\begin{aligned}\frac{\partial C}{\partial T} &= e^{-rT}\int_0^\infty (x-K)^+ \cdot \frac{\partial p}{\partial T}\mathrm{d}x\\ &=\end{aligned}$$

MODEL-FREE PROPERTIES

Let $C(T,K)$ be the price of European option at time 0 with maturity time T and strike K. And assume $p(x)$ is the density of S_T under the risk-natural measure.

Breeden-Litzenberger Formula The price of European option is

$$\begin{aligned} C(T,K) &= \mathbb{E}^\star\left\{e^{-rT}(S_T-K)^+\right\} \\ &= \int_K^\infty e^{-rT}(x-K)p(x)\mathrm{d}x \end{aligned}$$

Take derivative with respect to K on both sides for twice

$$\begin{aligned} \frac{\partial C}{\partial K} &= -\int_K^\infty e^{-rT}p(x)\mathrm{d}x \\ \frac{\partial^2 C}{\partial K^2} &= e^{-rT}p(K) \end{aligned}$$

Finally, we get a model-free formula,

$$p(K) = e^{rT}\frac{\partial^2 C}{\partial K^2}$$

Carr-Madan Formula We are interested in the arbitrage-free price at time $t=0$ of a simple contingent claim $h(S_T)$. Carr-Madan formula says that this price could be obtained once we know the price of European call option.

$$\begin{aligned} \mathbb{E}^\star\left(e^{-rT}h(S_T)\right) &= \int_0^\infty e^{-rT}h(x)p(x)\mathrm{d}x \\ \text{(B-L formula)} \quad &= \int_0^\infty h(x)\frac{\partial^2 C}{\partial K^2}\mathrm{d}x \\ \text{(Int. by part)} \quad &= (h\cdot\frac{\partial C}{\partial K})\Big|_0^\infty - \int_0^\infty h'\cdot\frac{\partial^2 C}{\partial K^2}\mathrm{d}x \\ \text{(Int. by part)} \quad &= (h\cdot\frac{\partial C}{\partial K})\Big|_0^\infty - \left[(h'\cdot C)\Big|_0^\infty - \int_0^\infty h''\cdot C\mathrm{d}x\right] \end{aligned}$$

Then we get the Carr-Madan formula

$$\mathbb{E}^\star\left(e^{-rT}\Phi(S_T)\right) = h(0)e^{-rT} - h'(0)S_0 + \int_0^\infty h''\cdot C\mathrm{d}x$$

Example 2.1 (European Put Option)**.** Let $h : x \mapsto (K-x)^+$. We find

$$\begin{aligned} h(0) &= K, \\ h'(0) &= -1, \\ h''(x) &= \delta_K. \end{aligned}$$

By the Carr-Madan formula, the price of the European put option $P(T,K)$ is

$$\begin{aligned} P(T,K) &= Ke^{-rT} + S_0 + \int_0^\infty \delta_K\cdot C\mathrm{d}x \\ &= Ke^{-rT} + S_0 + C(T,K) \end{aligned}$$

3 Bonds and Interest Rates

In this section, we will introduce the fixed income market. The simplest example is the zero-coupon bond; if it pays 1$ at maturity time T, we want to decide its value at time 0. Assume the interest rate r is fixed. Then the price of the zero-coupon bond at time t is given by

$$B(t,T) = e^{-r(T-t)}.$$

Or it can be written as

$$
\begin{aligned}
\mathrm{d}B_t &= rB_t\mathrm{d}t,\\
B(T,T) &= 1.
\end{aligned}
$$

Moreover, we are also interested in the relation between the interest rate and the maturity time T. Based on the model above, we can solve r for

$$r(T) = -\frac{1}{T-t}\log B(t,T) \equiv r;$$

it is called the **yield curve**.

EXAMPLES

Example 3.1 (Vasicek model)**.** Assume the rate r for a bond with maturity time T is given by

$$\mathrm{d}r_t = a(m - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

where a, m, and σ are constant. Its price at time t is

$$B(t,T) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s \mathrm{d}s} \,|\, \mathcal{F}_t\right)$$

- In this example, we want to find the precise representation of $B(0,T)$. First, we notice that $\int_0^T r_s\mathrm{d}s$ is a normal distribution. Then we can compute its mean and variance:

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T r_s\mathrm{d}s\right] = \int_0^T \mathbb{E}^{\mathbb{Q}}(r_s)\mathrm{d}s = 0$$

by Fubini theorem; and

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[(\int_0^T r_s\mathrm{d}s)^2\right] &= \int_0^T\int_0^T \mathrm{Cov}(r_s, r_u)\mathrm{d}s\mathrm{d}u\\
&= \int_0^T\int_0^T \frac{\sigma^2}{2a}\left(e^{-a(u-s)} - e^{-a(u+s)}\right)\mathrm{d}s\mathrm{d}u\\
&=
\end{aligned}
$$

Second, by the moment generating function of Gaussian random variable,

$$B(0,T) =$$

- Then we can compute its yield curve.

$$
\begin{aligned}
Y(0,T) &= -\frac{1}{T-t}\log B(0,T)\\
&=
\end{aligned}
$$

- Comment on this curve

Example 3.2 (CIR model)**.** Now assume the rate r is defined as

$$\mathrm{d}r_t = a(m - r_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t.$$

Then its price at time 0 is also

$$P(0,T) = \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T r_s\mathrm{d}s\Big].$$

However, because in this case r_t is chi-square distributed, we cannot write a nice formula for its price.

FORWARD MEASURE

"In finance, a T-forward measure is a pricing measure absolutely continuous with respect to a risk-neutral measure but rather than using the money market as numeraire, it uses a bond with maturity T. The use of the forward measure was pioneered by Farshid Jamshidian (1987), and later used as a means of calculating the price of options on bonds."

(Forward measure - Wikipedia)

Lemma 3.3. *Let (X, Y) be a bi-normal distribution under $\mathbb{P}^\star$; that is,*

$$(X,Y) \sim_{\mathbb{P}^\star} \text{Normal}(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}).$$

And assume $\mathbb{P} \ll \mathbb{P}^\star$ with density

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^\star} = \frac{e^{-\lambda X}}{\mathbb{E}^{\mathbb{P}^\star} e^{-\lambda X}}.$$

Then the distribution of Y under $\mathbb{P}$ is

$$Y \sim_{\mathbb{P}} \text{Normal}\left(\mu_Y - \lambda\rho\sigma_X\sigma_Y, \sigma_Y^2\right).$$

Proof. For convenience, we use $\mathbb{E}^\star$ to represent $\mathbb{E}^{\mathbb{P}^\star}$. Directly compute the distribution function for Y under $\mathbb{P}$ as follow:

$$\begin{aligned} \mathbb{P}(Y \le t) &= \int_\Omega \mathbf{1}_{\{Y \le t\}} \ \mathrm{d}\mathbb{P} = \int_\Omega \mathbf{1}_{\{Y \le t\}} \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^*} \ \mathrm{d}\mathbb{P}^\star \\ &= \mathbb{E}^\star\left(\mathbf{1}_{\{Y \le t\}} \cdot e^{-\lambda X}\right) / \mathbb{E}^\star(e^{-\lambda X}) \end{aligned}$$

Obviously, $\mathbb{E}^\star(e^{-\lambda X})$ doesn't rely on t. So it remains to compute $\mathbb{E}^\star\left(\mathbf{1}_{\{Y \le t\}} \cdot e^{-\lambda X}\right)$.

It is well-known that

$$X|Y \sim_\star N\left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(Y - \mu_Y), \sigma_X^2 - \rho^2\sigma_X^2\right);$$

and it implies that

$$\mathbb{E}^\star(e^{-\lambda X} \,|\, Y) = \exp\left\{ -\left[\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(Y - \mu_Y)\right]\lambda + \sigma_X^2\left[1 - \rho^2\right]\frac{\lambda^2}{2} \right\}.$$

Now we have

$$\begin{aligned} \mathbb{E}^\star\left(\mathbf{1}_{\{Y \le t\}} \cdot e^{-\lambda X}\right) &= \mathbb{E}^\star\left[\mathbf{1}_{\{Y \le t\}} \mathbb{E}^\star\left(e^{-\lambda X}|Y\right)\right] \\ &= \mathbb{E}^\star\left[\mathbf{1}_{\{Y \le t\}} \exp\left\{ -\left[\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(Y - \mu_Y)\right]\lambda + \sigma_X^2\left[1 - \rho^2\right]\frac{\lambda^2}{2} \right\}\right] \\ &= \int_{-\infty}^t \left\{ \exp\left\{ -\left[\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y)\right]\lambda + \sigma_X^2\left[1 - \rho^2\right]\frac{\lambda^2}{2} \right\} \cdot \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}} \right\} \mathrm{d}y \end{aligned}$$

Take derivative on both sides. The PDF of Y under $\mathbb{P}$ is

$$\begin{aligned} f_Y(t) &\propto \exp\left\{ -\rho\lambda\frac{\sigma_X}{\sigma_Y}t \right\} \cdot \exp\{-\frac{(t - \mu_Y)^2}{2\sigma_Y^2}\} \\ &\propto \exp\{\frac{(t + \lambda\rho\sigma_X\sigma_Y - \mu_Y)^2}{2\sigma_Y^2}\} \end{aligned}$$

Therefore, the distribution of Y under $\mathbb{P}$ is

$$Y \sim_{\mathbb{P}} \text{Normal}(\mu_Y - \lambda\rho\sigma_X\sigma_Y, \sigma_Y^2).$$

□

Example 3.4 (Call option on bonds)**.** Assume a Vasicek model for the short-rate process r under the pricing measure $\mathbb{P}^\star$. We would like to price, at time 0, a call option with maturity T and strike K on a zero-coupon bond with maturity $T_1 > T$.

- Let its price at time t with maturity T is $C(t,T)$. Then we directly price the call option at time 0:

$$\begin{aligned} C(0,T) &= \mathbb{E}^\star\left[e^{-\int_0^T r_s \mathrm{d}s}(P(T,T_1)-K)^+\right] \\ &= \mathbb{E}^\star\left[e^{-\int_0^T r_s \mathrm{d}s}(P(T,T_1)-K)\cdot \mathbf{1}\{P(T,T_1)\geq K\}\right] \\ &= \mathbb{E}^\star\left[e^{-\int_0^T r_s \mathrm{d}s}\cdot P(T,T_1)\cdot \mathbf{1}\{P(T,T_1)\geq K\}\right] - K\cdot \mathbb{E}^\star\left[e^{-\int_0^T r_s \mathrm{d}s}\cdot \mathbf{1}\{P(T,T_1)\geq K\}\right] \end{aligned}$$

  For convenience we denote

$$\begin{aligned} (F1) &= \mathbb{E}^\star\left[e^{-\int_0^T r_s \mathrm{d}s}\cdot P(T,T_1)\cdot \mathbf{1}\{P(T,T_1)\geq K\}\right], \\ (F2) &= \mathbb{E}^\star\left[e^{-\int_0^T r_s \mathrm{d}s}\cdot \mathbf{1}\{P(T,T_1)\geq K\}\right]; \end{aligned}$$

  then

$$C(0,T) = (F1) - K\cdot(F2).$$

- **Compute (F1).**

$$\begin{aligned} (F1) &= \mathbb{E}^\star\left[e^{-\int_0^T r_s \mathrm{d}s}\cdot P(T,T_1)\cdot \mathbf{1}\{P(T,T_1)\geq K\}\right] \\ &= P(0,T_1)\cdot \mathbb{E}^\star\left[\frac{e^{-\int_0^T r_s \mathrm{d}s}}{P(0,T_1)}\cdot P(T,T_1)\cdot \mathbf{1}\{P(T,T_1)\geq K\}\right] \end{aligned}$$

  Let

$$\frac{\mathrm{d}P^{T_1}}{\mathrm{d}P^\star} = \frac{e^{-\int_0^T r_s \mathrm{d}s}}{P(0,T_1)}\cdot P(T,T_1).$$

  So (F1) can be written as

$$(F1) = P(0,T_1)\cdot \mathbb{P}^{T_1}\{P(T,T_1)\geq K\}.$$

  It remains to compute $\mathbb{P}^{T_1}\{P(T,T_1)\geq K\}$.

- **Compute (F2).** Similarly, let

$$\frac{\mathrm{d}P^{T}}{\mathrm{d}P^\star} = \frac{e^{-\int_0^T r_s \mathrm{d}s}}{P(0,T)};$$

  then we find

$$\begin{aligned} (F2) &= \mathbb{E}^\star\left[e^{-\int_0^T r_s \mathrm{d}s}\cdot \mathbf{1}\{P(T,T_1)\geq K\}\right] \\ &= P(0,T)\cdot \mathbb{E}^\star\left[\frac{e^{-\int_0^T r_s \mathrm{d}s}}{P(0,T)}\cdot \mathbf{1}\{P(T,T_1)\geq K\}\right] \\ &= P(0,T)\cdot \mathbb{P}^{T}\{P(T,T_1)\geq K\} \end{aligned}$$

- Now we find

$$\begin{aligned} C(0,T) &= (F1) - K\cdot(F2) \\ &= P(0,T_1)p_1 - KP(0,T)p_2 \end{aligned}$$

  where p_1 and p_2 are given in (3) and (4), respectively.

DEFAULTABLE BONDS

Assume the interest rate is constant. We consider the following approaches to price the zero-coupon bond at time 0. (**Note**: when r is not a constant, we change $(T-t)r$ to $\int_t^T r_s \mathrm{d}s$; and in this case, we cannot take it out from the expectation.)

Merton's Approach In this approach, the basic setting is

$$\text{DEFAULT} = \{S_T \leq D\}.$$

In this setting, the price of bonds is

$$\begin{aligned} P^D(t,T) &= \mathbb{E}^\star_{t,s}\left[e^{-r(T-t)}\mathbf{1}_{\{S_T>D\}}\right] \\ &= e^{-r(T-t)}\mathbb{P}^\star_{t,s}\left[S_T > D\right] \\ &= e^{-r(T-t)}N(d_2) \end{aligned}$$

where

$$d_2 = \frac{\log(\frac{x}{D}) + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Black-Cox Approach Now we consider the first default time τ

$$\text{DEFAULT} = \{\min_{0\leq t\leq T} S_t \leq D\}$$

In this setting, the price of bonds at time 0 is

$$P^D(0,T) = e^{-rT}\left(N\left(d_2^+\right) - \left(\frac{x}{B}\right)^{1-k} N\left(d_2^-\right)\right)$$

where $k = 2r/\sigma^2$ and

$$d_2^\pm = \frac{\pm\log\left(\frac{x}{B}\right) + \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

Intensity Based Approach In this setting, we are given the intensity of $\mathbf{1}_{\{\tau>T\}}$. Then the bond price is

$$P^D(0,T) = e^{-rT}\mathbb{E}^\star\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\right]$$

FORWARD RATE AND YIELD CURVE

In this subsection, we summarize the previous results together.

Forward Rate

$$f(t,T) = \frac{E^\star\left\{r_T e^{-\int_t^T r_s ds}|r_t\right\}}{E^\star\left\{e^{-\int_t^T r_s ds}|r_t\right\}}$$

Yield Curve

$$\begin{aligned} P(0,T) &= e^{-Y(0,T)T} \\ Y(0,T) &= -\frac{1}{T}\log P(0,T) \end{aligned}$$

Yield Spread

$$\begin{aligned} P^D(0,T) &= e^{-(Y(0,T)+Y^S(0,T)T} \\ Y^S(0,T) &= -\frac{1}{T}\log\frac{P^D(0,T)}{P(0,T)} \end{aligned}$$

A Important Processes

Geometric Brownian Motion

Definition Let S be the solution to

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t;$$

then S follows a GBM. And S can be represented as

$$S_t = S_0 \exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma W_t\right].$$

Properties

$$\mathbb{E}S_t = S_0 e^{\mu t}$$
$$\mathrm{Var}S_t = S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$$

Ornstein–Uhlenbeck process

Definition Let r be the solution to

$$\mathrm{d}r_t = \theta(\mu - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t;$$

then r follows an O-U process. And r can be represented as

$$r_t = r_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} \mathrm{d}W_s$$

Properties r_t is normal distributed with

$$r_t \sim \mathrm{Normal}\Big(\mu + (y - \mu)e^{-\theta t}, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\Big).$$

And the covariance is given by

$$\mathrm{Cov}(r_s, r_t) = \frac{\sigma^2}{2\theta}\left(e^{-\theta|t-s|} - e^{-\theta(t+s)}\right).$$

B Binomial Tree Approximation

Basic Setting

We model the stock price as

$$S_n = S_0 R_1 \dots R_n$$

where

$$R_i \overset{iid}{\sim} \begin{cases} = u & p^\star = \frac{1-d}{u-d} \\ = d & 1 - p^\star \end{cases}$$

Let $n = \lfloor \frac{t}{\epsilon} \rfloor$. Define the stock price at time t as

$$\tilde{S}_t := \lim_{\epsilon \downarrow 0} S_{\lfloor \frac{t}{\epsilon} \rfloor}.$$

Rescaling

Rescale u and d:

$$\begin{cases} u &= e^{a\sqrt{\epsilon}} \\ d &= e^{-a\sqrt{\epsilon}} \end{cases}$$

Then we consider the Taylor expansion near $\epsilon = 0$,

$$\begin{aligned} p^\star &= \frac{1 - e^{-a\sqrt{\epsilon}}}{e^{a\sqrt{\epsilon}} + e^{-a\sqrt{\epsilon}}} \\ &\approx \frac{1}{2} + \frac{a}{4}\sqrt{\epsilon} \end{aligned}$$

Limiting

Now we compute the limit distribution of S_n:

$$\log \frac{S_n}{S_0} = \sum_{i=1}^{\lfloor \frac{t}{\epsilon} \rfloor} \log R_i;$$

by the central limit theorem, it is approximated normal distribution. It remains to determine its mean and variance. Because

$$\begin{aligned} &\mathbb{E} \log R_i = \frac{a^2\epsilon}{2}, \\ &\operatorname{Var} \log R_i = (\log \frac{u}{d})^2 p^\star (1 - p^\star), \end{aligned}$$

we find

$$\log \frac{S_n}{S_0} \sim \text{Normal}(-\frac{a^2}{2}t, a^2 t).$$