Seifert fibered 3-manifolds and Turaev-Viro invariants volume conjecture

Shashini Marasinghe

Michigan State University

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MICHIGAN STATE UNIVERSITY

Outline

1. Motivation
2. Preliminary concepts
   - Oriented Seifert fibered 3-manifolds??
   - $SO(3)$-Turaev-Viro invariants
   - $SU(2)$-Witten-Reshetikhin-Turaev invariants
   - Turaev-Viro invariants volume conjecture
3. Results
   - What I used, and what I did
   - Results

Motivation

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$$T(a,b) = \big(D \times [0,1]\big)/\sim,$$

where

- $(x,1) \sim \big(\pi(x), 0\big)$ for $x \in D$ and
- $\pi : D \to D$ denote the counter-clockwise rotation of the unit disk $D \subset \mathbb{C}$ by $\frac{2\pi b}{a}$, with $\gcd(a,b) = 1$.

Oriented Seifert fibered 3-manifold??

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- If $a > 1$, the middle fiber is called an *exceptional fiber*.

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- If $a > 1$, the middle fiber is called an *exceptional fiber*.

- The *2-orbifold base* of M is the quotient of $\Sigma \times S^1$ by identifying each exceptional fiber to a point, where Σ is genus-g surface.
  The base is a closed connected surface, orientable or non-orientable.

Standard notation

Oriented Seifert Fibered 3-Manifold

The symbol of an oriented Seifert fibered 3-manifold M that fibers over a genus g 2-orbifold base is

$$(\epsilon, g; (a_1, b_1), \ldots, (a_n, b_n)),$$

where:

- $n \geq 0$ and $g > 0$ are integers, and
- each (a_j, b_j) is a pair of coprime integers with $a_j > 0$, representing an exceptional fiber for $j = 1, \ldots, n$.
- For an orientable 2-orbifold base, we write $\epsilon = o$; for a non-orientable base, $\epsilon = n$.

Outline

$SO(3)$-Turaev-Viro invariants

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$SO(3)$-Turaev-Viro invariants

- The Turaev-Viro invariants were introduced as a state sum for a triangulation of 3-manifolds.
- These are real valued invariants indexed by integers r, which depend on the choice of a root of unity.
- For any 3-manifold M, odd integers $r \geq 3$, and $2r$-th root of unity q, these invariants are denoted by $TV_r(M, q = e^{\frac{2\pi i}{r}})$.

Constructing $SO(3)$-Turaev-Viro invariants

- Let M be a compact orientable 3-manifold and consider $\mathcal{T}$ as a triangulation.
- If $\partial(M) \neq 0$, $\mathcal{T}$ is a (partially) ideal triangulation.

- $I_r = \{0, 2, 4, \ldots, r-3\}$

- V - set of interior vertices that are not on ∂M,
- E - set of interior edges ,
- F - set of interior faces, and
- T - set of interior tetrahedra in $\mathcal{T}$.

Admissible coloring of $(M, \mathcal{T})$ at level r - assign elements of I_r in a way 6-tuple assigned to the edges of tetrahedrons of $(M, \mathcal{T})$ satisfies the admissibility conditions.

Constructing $SO(3)$-Turaev-Viro invariants - Notations and Definitions

Now, we need the following notations and definitions to explain *Admissible coloring of $(M,\mathcal{T})$ at level r*.

Quantum integer $\{n\}$

$$\{n\} = q^n - q^{-n} = 2\sin(\frac{2\pi}{r})[n]$$

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{2\sin(\frac{2n\pi}{r})}{2\sin(\frac{2\pi}{r})}$.

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Quantum factorial $\{n\}!$

$$\{n\}! = \prod_{i=n}^{n} \{i\}.$$

Constructing $SO(3)$-Turaev-Viro invariants - Notations and Definitions

- A triple (i, j, k) is an *admissible triple* of elements in I_r if it satisfies the inequalities:

$$i \leq j + k, \quad j \leq i + k, \quad k \leq i + j, \quad i + j + k \leq 2(r - 2)$$

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- For an admissible triple (i, j, k), define

$$\Delta(i,j,k) := \zeta_r^{\frac{1}{2}} \left(\frac{\left\{\frac{i+j-k}{2}\right\}! \left\{\frac{i+k-j}{2}\right\}! \left\{\frac{j+k-i}{2}\right\}!}{\left\{\frac{i+j+k}{2}+1\right\}!} \right)^{\frac{1}{2}}$$

where $\zeta_r = 2\sin(\frac{2n\pi}{r})$.

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where $\zeta_r = 2\sin(\frac{2n\pi}{r})$.

- A 6-tuple (i,j,k,l,m,n,o) is an *admissible 6-tuple* of elements in I_r if the triples $F_1 = (i,j,k)$, $F_2 = (j,l,n)$, $F_3 = (i,m,n)$, and $F_4 = (k,l,m)$ are admissible.

Constructing $SO(3)$-Turaev-Viro invariants - Notations and Definitions

- For an admissible 6-tuple (i, j, k, l, m, n, o), define *quantum 6j-symbol* at the root q as follows.

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = (\zeta)^{-1}(\sqrt{-1})^{\lambda} \prod_{a=1}^{4} \Delta(F_k) \sum_{z=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}} \frac{(-1)^z\{z+1\}!}{\prod_{b=1}^{4}\{z-T_b\}! \prod_{c=1}^{3}\{Q_c-z\}!}$$

where $\lambda = i+j+k+l+m+n$, and

$$T_1 = \frac{i+j+k}{2}, \quad T_2 = \frac{i+m+n}{2}, \quad T_3 = \frac{j+l+n}{2}, \quad T_4 = \frac{k+l+m}{2},$$
$$Q_1 = \frac{i+j+l+m}{2}, \quad Q_2 = \frac{i+k+l+n}{2}, \quad Q_3 = \frac{j+k+m+n}{2}.$$

Constructing $SO(3)$-Turaev-Viro invariants - Admissible coloring explained!

- Let c be an admissible coloring of $(M, \mathcal{T})$ at level r.

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$$\rightarrow |e|_c = (-1)^{c(e)}[c(e)+1]$$

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- For each face f of $(M, \mathcal{T})$ with edges e_1, e_2, and e_3

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- For each tetrahedron Δ in $(M, \mathcal{T})$ with coloring c, $|\Delta|_c$ is the quantum 6-j symbol associated with the admissible 6-tuple assigned to Δ by c.

$SO(3)$-Turaev-Viro invariants - Definition

Definition

Let $A_r(\mathcal{T})$ be the set of $SO(3)$-admissible coloring of $(M, \mathcal{T})$ at level r. Define $SO(3)$-version of r-th Turaev-Viro invariant as

$$TV_r(M,q) = (\eta')_r^{2|V|} \sum_{c \in A_r(\mathcal{T})} \frac{\prod_{e \in E} |e|_c \prod_{\Delta \in T} |\Delta|_c}{\prod_{f \in F} |f|_c}$$

where $\eta' = \frac{2\sin(\frac{2\pi}{r})}{\sqrt{r}}$.

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$SU(2)$-Witten-Reshetikhin-Turaev invariants - Definition

- For any 3-manifold M, any integer $r \geq 2$, and a root of unity $e^{\frac{\pi i}{r}}$, this theory produces a complex-valued invariant $RT_r(M, e^{\frac{\pi i}{r}})$.

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- The Witten-Reshetikhin-Turaev TQFT is a functor

Category of (2+1)-dimensional cobordisms

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- The double $D(M)$ is formed by gluing M and its orientation-reverse $\overline{M}$ along their common boundary.
  eg: double of $M{=}(\epsilon, 2g; (a_1, b_1), \ldots, (a_n, b_n), (a_1, -b_1), \ldots, (a_n, -b_n))$

Relationship between $SU(2)$-$RT_r(M, e^{\frac{\pi i}{r}})$ and $SO(3)$-$TV_r(M, e^{\frac{2\pi i}{r}})$

Theorem (Benedetti-Petronio 1996)

Let M be a 3-manifold with boundary, and r be an odd integer. Then,

$$TV_r(M, e^{\frac{2\pi i}{r}}) = \eta^{-\chi(M)} RT_r(D(M), e^{\frac{\pi i}{r}}),$$

where $\chi(M)$ is the Euler characteristic of M and $\eta_r = RT_r(S^3) = \frac{2sin(\frac{2\pi}{r})}{\sqrt{r}}$.

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Theorem (Benedetti-Petronio 1996)

For M an oriented compact 3-manifold with empty or toroidal boundary and $r \geq 3$ an odd integer, we have

$$TV_r(M, e^{\frac{2\pi i}{r}}) = ||RT_r(M, e^{\frac{\pi i}{r}})||^2.$$

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Turaev-Viro invariants volume conjecture

Conjecture (Turaev-Viro invariants volume conjecture Detcherry-Kalfagianni 2020)

For every compact orientable 3-manifold M with an empty or toroidal boundary, we have

$$LTV(M) = \limsup_{r \longrightarrow \infty} \frac{2\pi}{r} log|TV_r(M, q)| = v_3||M||,$$

where r runs over all odd integers.

Turaev-Viro invariants volume conjecture

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where r runs over all odd integers.

- I studied the large r asymptotic behaviour of the Turaev-Viro invariants of oriented Seifert fibered 3-manifolds at the root $q = e^{\frac{2\pi i}{r}}$.

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What I did, and what I used

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Theorem (Detcherry-Kalfagianni 2020)

Let M be a compact, orientable 3-manifold that is Seifert fibered. Then, there exist constants $A > 0$ and $N > 0$, depending on M, such that

$$TV_r(M, q) \leq Ar^N.$$

Thus, $LTV(M) \leq 0$.

What I did, and what I used

- Used some previous results.
- $SU(2)$-Witten-Reshetikhin-Turaev invariants of oriented Seifert fibered 3-manifolds.

$RT_r(M, e^{\frac{i\pi}{r}})$ of oriented Seifert fibered manifold

Lemma (Hansen 2001)

Let M be closed, oriented Seifert fibered manifold with symbol $(\epsilon, g; (a_1, b_1), \ldots, (a_n, b_n))$, where $n \geq 0$ and $g > 0$ are integers, and where $a_j \geq 0$ and b_j are coprime pairs of integers for $j = 1, \ldots, n$. The rational Euler number of the Seifert fibration is $e(M) = -\sum_j b_j/a_j$. Then,

$$RT_r(M, e^{\frac{i\pi}{r}}) = e^{\frac{i\pi}{2r}[3(a_\epsilon - 1)sgn(e(M)) - e(M) - 12\sum_{j=1}^{n} s(b_j, a_j)]}$$

$$\times (-1)^{a_\epsilon g} \frac{i^n r^{a_\epsilon g/2 - 1}}{2^{n + a_\epsilon g/2 - 1}\sqrt{\prod_j a_j}} e^{i\frac{3\pi}{4}(1 - a_\epsilon)sgn(e(M))} Z_{(\epsilon, r)}(M, e^{\frac{\pi i}{r}})$$

Continued

Lemma (Hansen 2001)

$$Z_{(\epsilon,r)}(M, e^{\frac{\pi i}{r}}) =$$

$$\sum_{(\gamma,\boldsymbol{\mu})} \left\{ \frac{(-1)^{\gamma a_\epsilon g} e^{\frac{i\pi e(M)\gamma^2}{2r}} \prod_{j=1}^{n} \left(\mu_j e^{\frac{-i\pi\gamma\mu_j}{a_j r}} \right)}{\sin^{n+a_\epsilon g-2}(\pi\gamma/r)} \sum_{\boldsymbol{m}} \prod_j e^{-i\cdot\left(\frac{2\pi m_j(\gamma+\mu_j b_j^*)}{a_j} + \frac{2\pi r m_j^2 b_j^*}{a_j} \right)} \right\}$$

where

- j *ranges over* $\{1, \ldots, n\}$
- $(\gamma, \boldsymbol{\mu}, \boldsymbol{m}) = (\gamma, (\mu_1, \ldots, \mu_n), (m_1, \ldots, m_n))$ *ranges over* $\{1, 2, \ldots, r-1\} \times \{\pm 1\}^n \times \mathbb{Z}/a_1\mathbb{Z} \times \ldots \times \mathbb{Z}/a_n\mathbb{Z}$.

What I did, and what I used

- Used some previous results.
- $SU(2)$-Witten-Reshetikhin-Turaev invariants of oriented Seifert fibered 3-manifolds.
    - Easy formula was obtained by computing it for $D(M) = (\epsilon, 2g; (a_1, b_1), \ldots, (a_n, b_n), (a_1, -b_1), \ldots, (a_n, -b_n))$ and assuming r divisible by $A := \mathrm{lcm}(a_1, \ldots, a_n)$ for mod A.

Observation

$$Z_{(\epsilon,r)}(D(M),e^{\frac{\pi i}{r}}) = \sum_{(\gamma,\boldsymbol{\mu})}\left\{\frac{\prod_{j=1}^{n}(\mu_j\mu_{n+j})e^{-i\pi\gamma\cdot\sum_{j=1}^{n}\frac{(\mu_j+\mu_{n+j})}{a_j r}}}{\sin^{2n+2a_\epsilon g-2}(\pi\gamma/r)}\cdot\right.$$

$$\left.\prod_{j=1}^{n}\sum_{m_j}e^{-i\cdot\left(\frac{2\pi m_j(\gamma+\mu_j b_j^*)}{a_j}\right)}\cdot\prod_{j=1}^{n}\sum_{m_j}e^{-i\cdot\left(\frac{2\pi m_j(\gamma-\mu_{n+j}b_j^*)}{a_j}\right)}\right\}$$

- For $Z_{(\epsilon,r)}(D(M),e^{\frac{\pi i}{r}}) \neq 0$, there must exist some γ that satisfy

$$\gamma+\mu_j b_j^* \equiv 0 (\mathrm{mod}\ a_j) \quad \text{and} \quad \gamma-\mu_{n+j}b_j^* \equiv 0 (\mathrm{mod}\ a_j).$$

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$$\left. \prod_{j=1}^{n}\sum_{m_j} e^{-i\cdot\left(\frac{2\pi m_j(\gamma+\mu_j b_j^*)}{a_j}\right)} \cdot \prod_{j=1}^{n}\sum_{m_j} e^{-i\cdot\left(\frac{2\pi m_j(\gamma-\mu_{n+j} b_j^*)}{a_j}\right)} \right\}$$

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$$\gamma + \mu_j b_j^* \equiv 0 (\mathrm{mod}\ a_j) \quad \text{and} \quad \gamma - \mu_{n+j} b_j^* \equiv 0 (\mathrm{mod}\ a_j).$$

- i.e., $\boldsymbol{\mu}$ which contribute to the existence of γ is of the form $(\mu_1, \ldots, \mu_n, -\mu_1, \ldots, -\mu_n)$.

Observation

$$Z_{(\epsilon,r)}(D(M),e^{\frac{\pi i}{r}}) = \sum_{(\gamma,\boldsymbol{\mu})}\left\{\frac{\prod_{j=1}^{n}(\mu_j\mu_{n+j})e^{-i\pi\gamma\cdot\sum_{j=1}^{n}\frac{(\mu_j+\mu_{n+j})}{a_j r}}}{\sin^{2n+2a_\epsilon g-2}(\pi\gamma/r)}\cdot\right.$$

$$\left.\prod_{j=1}^{n}\sum_{m_j} e^{-i\cdot\left(\frac{2\pi m_j(\gamma+\mu_j b_j^*)}{a_j}\right)}\cdot\prod_{j=1}^{n}\sum_{m_j} e^{-i\cdot\left(\frac{2\pi m_j(\gamma-\mu_{n+j} b_j^*)}{a_j}\right)}\right\}$$

- For $Z_{(\epsilon,r)}(D(M),e^{\frac{\pi i}{r}}) \neq 0$, there must exist some γ that satisfy

$$\gamma+\mu_j b_j^* \equiv 0(\text{mod } a_j) \quad \text{and} \quad \gamma-\mu_{n+j} b_j^* \equiv 0(\text{mod } a_j).$$

- i.e., $\boldsymbol{\mu}$ which contribute to the existence of γ is of the form $(\mu_1,\ldots,\mu_n,-\mu_1,\ldots,-\mu_n)$.
- That is, the expression $Z_{(\epsilon,r)}(D(M),e^{\frac{\pi i}{r}}) \neq 0$ if and only if there exist a solution $\gamma \in \{1,\ldots,A-1\}$ for $\gamma+\mu_j b_j^* \equiv 0(\textbf{mod } a_j)$ for $j=1,\ldots,n$ where $\boldsymbol{\mu}$ are of the form $(\mu_1,\ldots,\mu_n,-\mu_1,\ldots,-\mu_n)$.

What I did, and what I used

- Used some previous results.
- Goal is $LTV(M) \geq 0$.
- $SU(2)$-Witten-Reshetikhin-Turaev invariants of oriented Seifert fibered 3-manifolds.
  - Easy formula was obtained by computing it for $D(M) = (\epsilon, 2g; (a_1, b_1), \ldots, (a_n, b_n), (a_1, -b_1), \ldots, (a_n, -b_n))$ and assuming r divisible by $A := \mathrm{lcm}(a_1, \ldots, a_n)$ for mod A.
- Relationship between $SO(3)$-Turaev-Viro invariants and $SU(2)$-Witten-Reshetikhin-Turaev invariants.

Lemma

Lemma (Marasinghe 2025)

Let M be an oriented Seifert fibered 3-manifold described by the symbol

$$(\epsilon, g; (a_1, b_1), \ldots, (a_n, b_n)),$$

and let $D(M)$ be the double of M. Suppose that there is an integer $\gamma > 0$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ with $\mu_j = \pm 1$ such that $\gamma + \mu_j b_j^* \equiv 0 (mod\ a_j)$ for $j = 1, \ldots, n$. Then, for r divisible by $A := lcm(a_1, \ldots, a_n)$,

$$|TV_r(\boldsymbol{D(M)}, e^{\frac{2\pi i}{r}})| > 1,$$

and

$$|TV_r(\boldsymbol{M}, e^{\frac{2\pi i}{r}})| > 1.$$

What I did, and what I used

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- Goal is $LTV(M) \geq 0$.
- $SU(2)$-Witten-Reshetikhin-Turaev invariants of oriented Seifert fibered 3-manifolds.
  - Easy formula was obtained by computing it for $D(M)$ double of M and assuming r divisible by $A := \mathrm{lcm}(a_1, \ldots, a_n)$ for mod A.
- Relationship between $SO(3)$-Turaev-Viro invariants and $SU(2)$-Witten-Reshetikhin-Turaev

Outline

What I proved: Condition

Theorem (Marasinghe 2025)

Let M be an oriented Seifert fibered 3-manifold with boundary, described by the symbol

$$(\epsilon, g; (a_1, b_1), \ldots, (a_n, b_n)).$$

Suppose that there is an integer $\gamma > 0$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ with $\mu_j = \pm 1$ such that

$$\gamma + \mu_j b_j^* \equiv 0 (mod\ a_j)$$

for $j = 1, \ldots, n$. Then, M and $D(M)$ satisfy Turaev-Viro invariants volume conjecture. That is, $LTV(M) = LTV(D(M)) = 0$.

What I proved: Applications

Corollary (Marasinghe 2025)

Let M be an oriented Seifert fibered 3-manifold with boundary, described by the symbol

$$(\epsilon, g; (a_1, b_1), \ldots, (a_n, b_n)).$$

If either

(a) $a_1, \ldots, a_n$ *are relatively coprime; or*

(b) $a_1 = \ldots = a_n = a$ *and there are* $\mu_1, \ldots, \mu_n$ *such that* $\mu_1 b_1^* \equiv \ldots \equiv \mu_n b_n^* \pmod{a}$,

then M and $D(M)$ satisfy Turaev-Viro invariants volume Conjecture.

What I proved: Applications

Corollary (Marasinghe 2025)

Suppose M and $D(M)$ are 3-manifolds in the statement of the main Theorem. Let L be a link in M or $D(M)$ that has simplicial volume (a.k.a. Gromov norm) zero. Then, Turaev-Viro invariants volume conjecture is true for L.

Thank You!
Any Questions?

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