Computational Linear Algebra, Module 9

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Due: April 11th, 2018

1. Yes, this does draw the function $g(x) = x - \frac{5}{3}x^3$

2. When more terms are added to the approximation function, we see that it looks like the sine function

3. Errors for interval $[0, 2\pi]$:

n=3	44.03945
n=5	38.71829
n=7	18.99292
n=13	0.22859

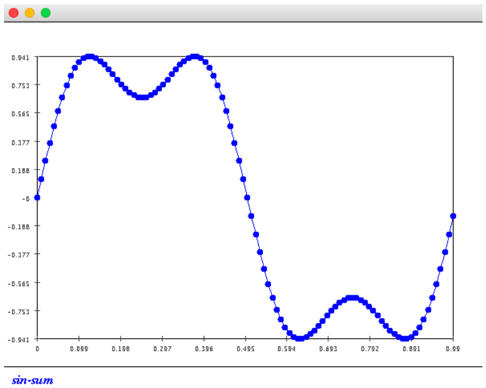
   The difference between $n = 3$ and $n = 13$ is $44.03945 - 0.22859 = 43.81086$. The errors for $n = 3$ and $n = 13$ for the interval $[0, \pi]$ are: $n = 3 \to 1.13981$ and $n = 13 \to 4.30594 \times 10^{-6}$

   For the interval $[0, \pi/2]$, the errors are $n = 3 \to 0.01836$ and $n = 13 \to 4.30594 \times 10^{-11}$

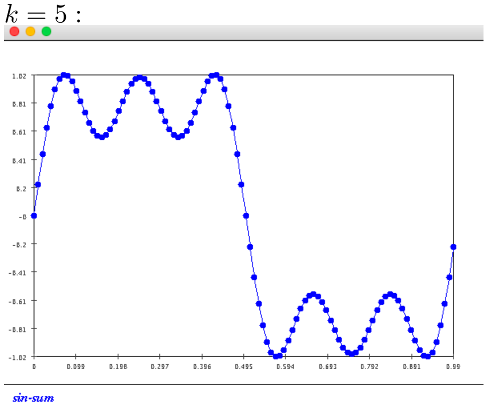
   The error at each point is multiplied by deltaX because we are calculating the area between the two curves to find the error.

4. Taylor series expansion: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$. The coefficients printed in the alpha array are the same as those we saw in the earlier exercise.

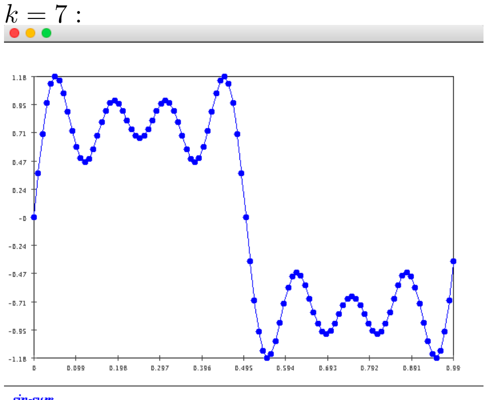
5. $k = 3$:


0.941
0.753
0.565
0.377
0.188
-0
-0.188
-0.377
-0.565
-0.753
-0.941
0
0.099
0.198
0.297
0.396
0.495
0.594
0.693
0.792
0.891
0.99
sin-sum

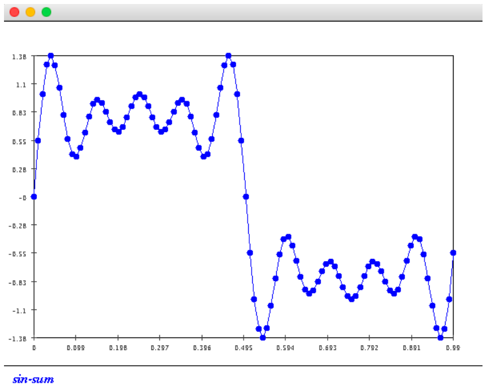
$k = 5:$


1.02
0.81
0.61
0.41
0.2
-0
-0.2
-0.41
-0.61
-0.81
-1.02
0
0.099
0.198
0.297
0.396
0.495
0.594
0.693
0.792
0.891
0.99
sin-sum

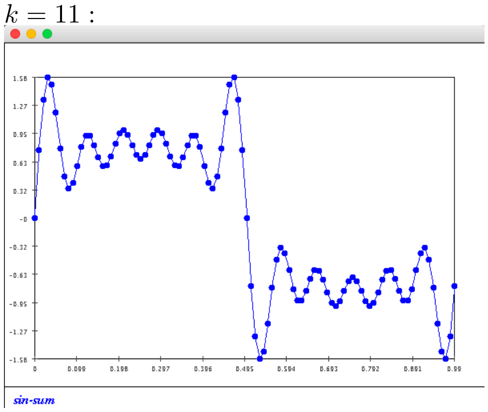
$k = 7:$


1.18
0.95
0.71
0.47
0.24
-0
-0.24
-0.47
-0.71
-0.95
-1.18
0
0.099
0.198
0.297
0.396
0.495
0.594
0.693
0.792
0.891
0.99
sin-sum

$k = 9:$



$k = 11$:



6. No linear combination of $\sin(2\pi kx)$ can approximate any step function. For example, $h(x) =$

$$\begin{cases} 1 & x < 0 \\ 3 & 0 \leq x \leq 20 \\ 5 & 20 < x \end{cases}$$

7. The second graph that is output by this program is a subsection of the first graph, and this is a fractal so the subsection looks like the full graph.

8. $k = 0 \rightarrow$ evaluates to 1
   $k = 1 \rightarrow$ evaluates to n
   $k = n - 1 \rightarrow$ evaluates to n, which is the same as when $k = 1$
   $k = n \rightarrow$ evaluates to 1, which is the same as when $k = 0$

9. $\binom{n}{k} = \frac{n!}{k!(n-k)!} \rightarrow$ this is derived by taking the total number of combinations possible, and dividing out the repeated combinations.

k	$\binom{n}{k}$
0	1
1	5
2	10
3	10
4	5
5	1

10. We know that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, $\binom{n-1}{k} = \frac{(n-1)!}{k!(n-1-k)!}$ and $\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!}$.
So now we have:

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!(n-k) + k(n-1)!}{k!(n-k)!} \\
&= \frac{n! - k(n-1)! + k(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}$$

Therefore, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ ■

$\binom{n}{k}$ is a symmetric function because of how choosing combinations works.
Pascal's Triangle:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Each row of Pascal's Triangle gives the values of $\binom{n}{k}$, where k is the column of the row and n is the row number.

11.

n	numCalls	numCallsRecursive
5	62	58
10	222	2037
20	842	2097131

For large enough n, the recursive definition is no longer efficient.

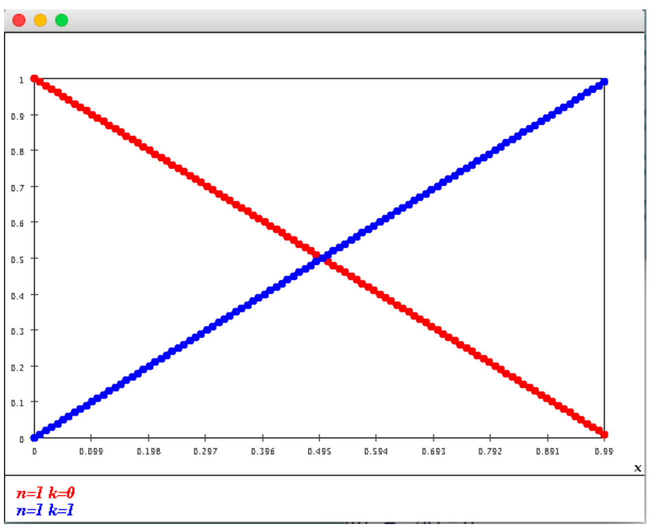
12.

$$
\begin{aligned}
RHS &= \frac{n}{k}\binom{n-1}{k-1} \\
&= \frac{n}{k}\left(\frac{(n-1)!}{(k-1)!(n-1-(k-1))!}\right) \\
&= \frac{n}{k}\left(\frac{(n-1)!}{(k-1)!(n-k)!}\right) \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}
$$

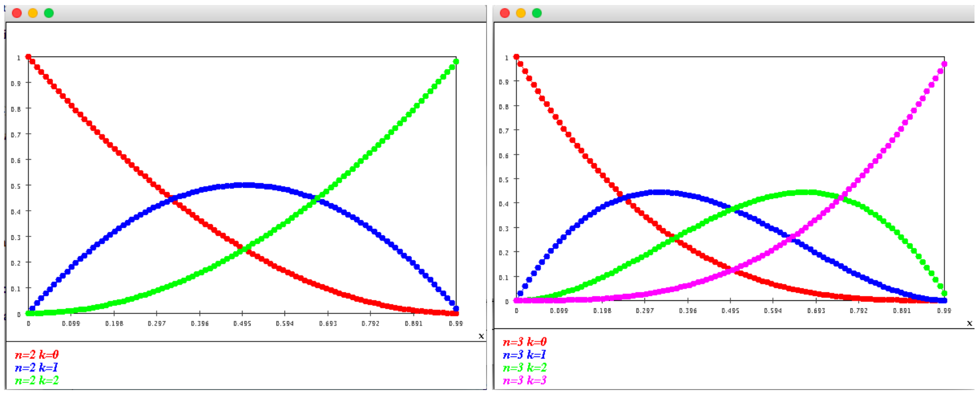
$\blacksquare$

n	numCalls	numCallsRecursive
5	62	58
10	222	2037
20	842	2097131

The return type is double due to the $\frac{n}{k}$ that is in the formula in the iterative method.



13.

14. 

15. For $n = 5$, there were 1800 numCalls and 5800 numRecursiveCalls.

16. Since $a < b$, we have

$$\begin{aligned} (1-t)a + tb &> (1-t)a + ta \\ &= a(1 - t + t) \\ &= a \end{aligned}$$

and we have

$$\begin{aligned} (1-t)a + tb &< (1-t)b + tb \\ &= b(1 - t + t) \\ &= b \end{aligned}$$

. So, $a < (1-t)a + tb < b$. Therefore, $(1-t)a + tb \in [a, b]$.

17. $B_1 = \left\{\binom{1}{0}(1-t), \binom{1}{1}(t)\right\}$
$B_2 = \left\{\binom{2}{0}(1-t)^2, \binom{2}{1}(t)(1-t), \binom{2}{2}t^2\right\}$
$B_3 = \left\{\binom{3}{0}(1-t)^3, \binom{3}{1}(t)(1-t)^2, \binom{3}{2}t^2(1-t), \binom{3}{3}t^3\right\}$

18. Suppose $B_n^{'}$ is a basis. Then,
$p(t) = a_0(1-t)^n + a_1 t(1-t)^{n-1} + \cdots + a_n t^n$.
Now, we can say that $a_k = b_k\binom{n}{k}$ for $k \leq n$ since each of these combinations is a constant, and b_k is a constant for $k \leq n$. So, we now have
$p(t) = b_0\binom{n}{0}(1-t)^n + b_1\binom{n}{1}t(1-t)^{n-1} + \cdots + b_n\binom{n}{n}t^n$.
Therefore, B_n is also a basis. ∎

19.

$$\begin{aligned} RHS &= t^k(1-t)^{n+1-k} + t^{k+1}(1-t)^{n-k} \\ &= t^k((1-t)^{n+1-k} + t(1-t)^{n-k}) \\ &= t^k(1-t)^{n-k}(1-t+t) \\ &= t^k(1-t)^{n-k} \end{aligned}$$

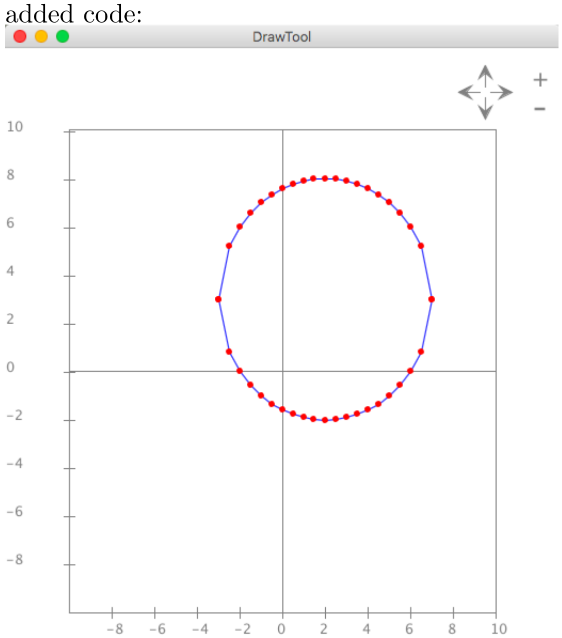
∎

20. We want to show that $p_1 = at + b = \alpha(1-t) + \beta(t)$. So,

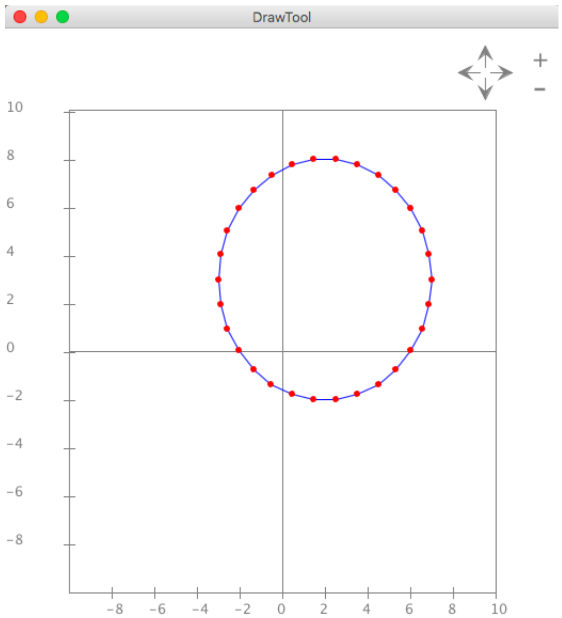
$$
\begin{aligned}
at + b &= \alpha - \alpha t + \beta t \\
at + b &= \alpha - (\alpha - \beta)t \\
a &= -\alpha + \beta \\
b &= \alpha
\end{aligned}
$$

Therefore, we have found integer coefficients such that $(1-t)$ and t are a linear combination for any polynomial of degree 1.

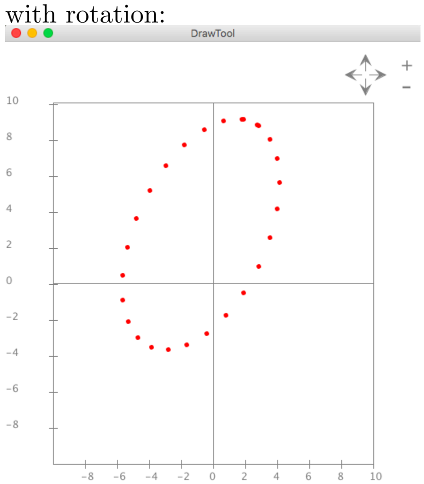
21. added code:



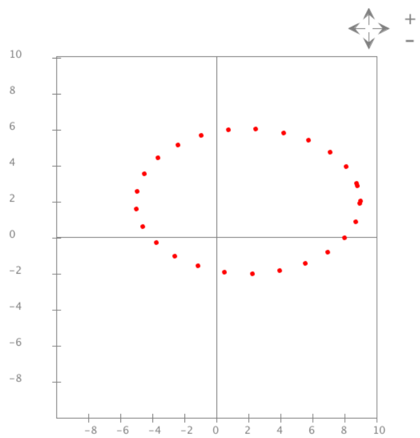
22. You can complete the circle by saving the initial calculated x and y values, and then drawing the curve between the last calculated values and these first calculated values:



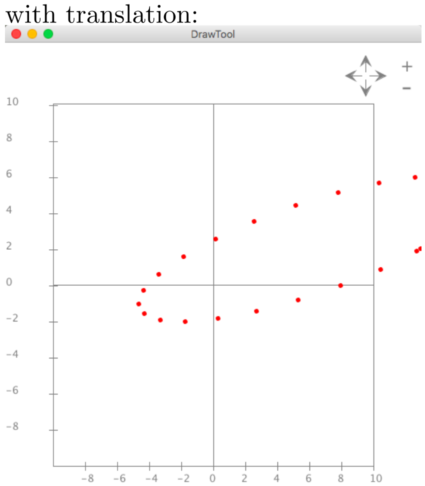
23. with rotation:



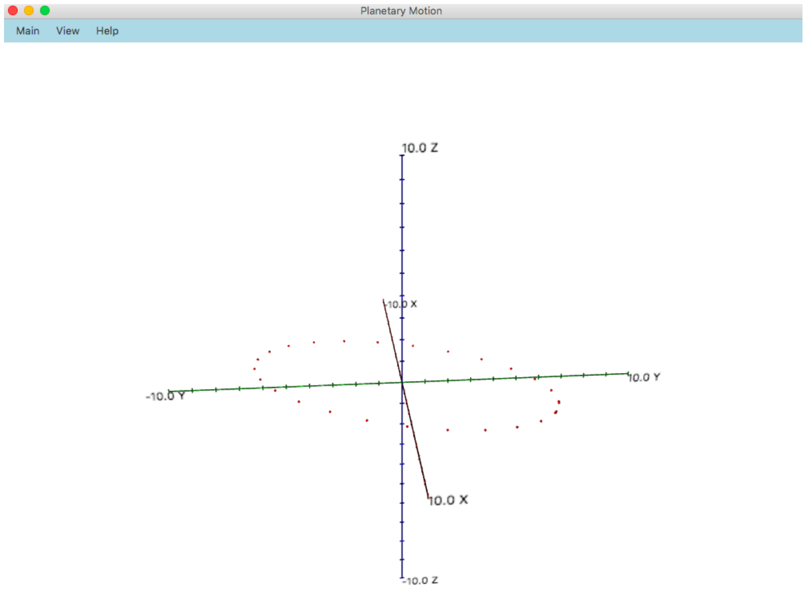
without rotation:



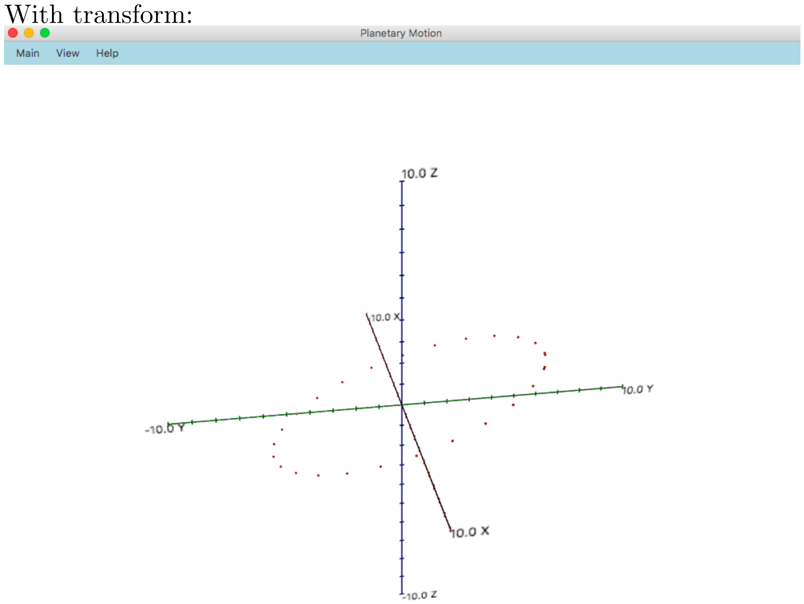
with translation:



24. Without transform:



With transform:



25. (a) Since $x_0 < x_1$,

$$\begin{aligned} tx_0 + (1-t)x_1 &> tx_0 + (1-t)x_0 \\ &= x_0(t+1-t) \\ &= x_0 \end{aligned}$$

and

$$\begin{aligned} tx_0 + (1-t)x_1 &< tx_1 + (1-t)x_1 \\ &= x_1(t+1-t) \\ &= x_1 \end{aligned}$$

The same argument can be made for when $x_1 < x_0$, $x_1 \leq x(t) \leq x_0$.

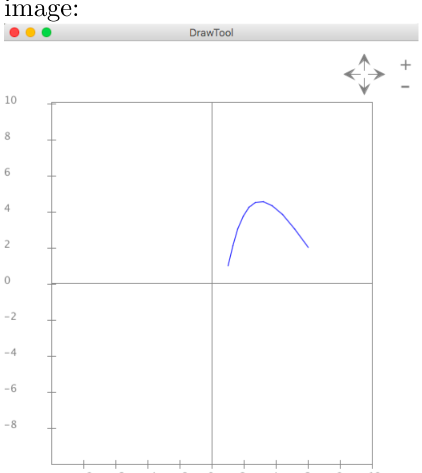
(b) The slope between the endpoints $(2,3)$ and $(5,9)$ is 2. So, now if we plug this into the point-slope form, and use $x(t)$ and $y(t)$ as our (x,y) point that we are looking at, we have:

$$\begin{aligned} y - 3 &= 2(x-2) \\ y(t) - 3 &= 2(x(t)-2) \\ (3t + 9(1-t)) - 3 &= 2((2t+5(1-t))-2) \\ 6 - 6t &= 2(3-3t) \\ 6 - 6t &= 6 - 6t \end{aligned}$$

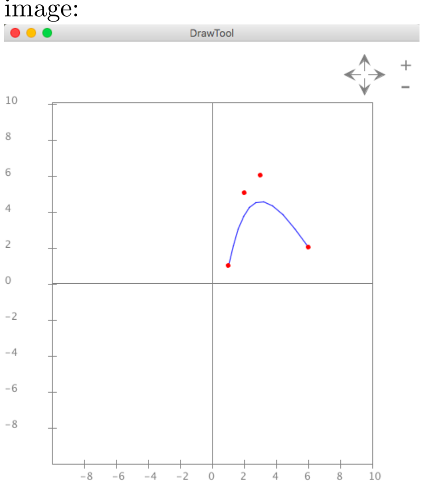
Therefore, each point $(x(t), y(t))$ is on the line segment between the end points.

(c) The poitns fall on the same line as the line segment, but they do not fall on the actual line segment between the two endpoints.

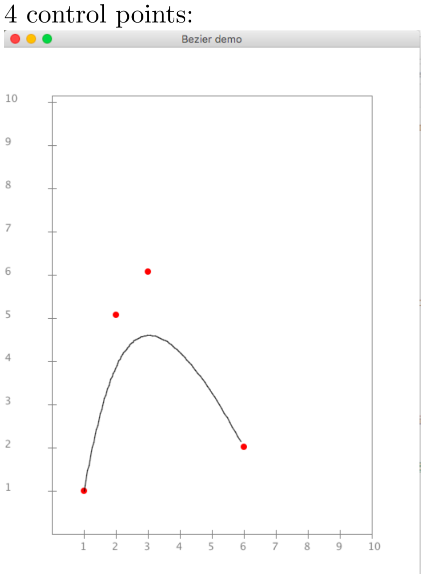
26. image:



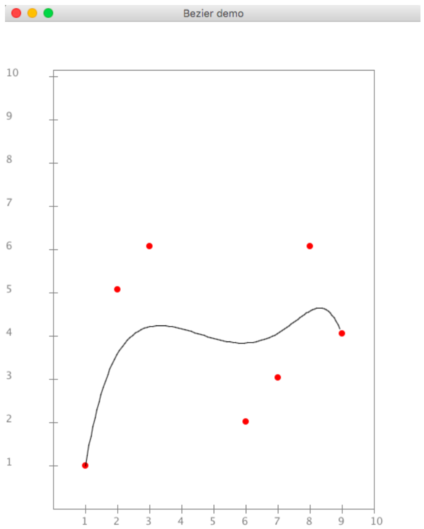
27. image:



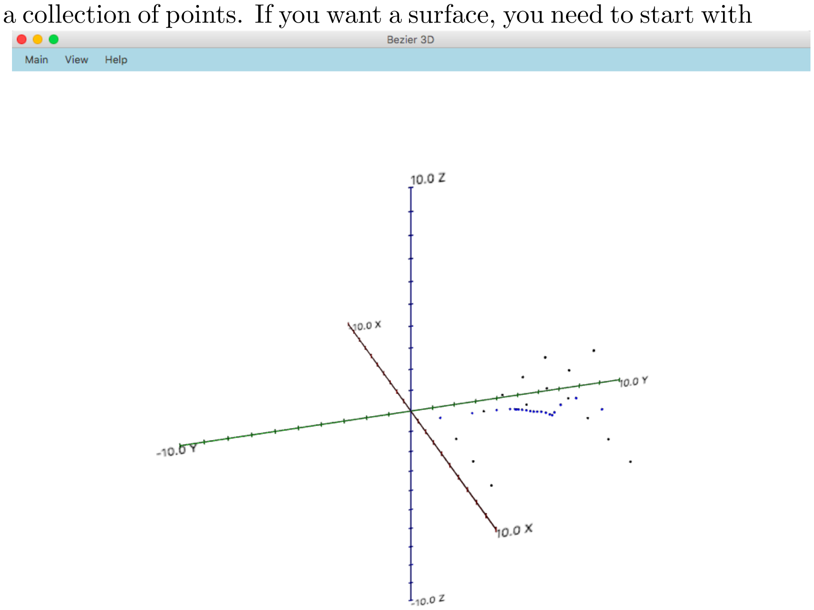
28. 4 control points:



10 control points:

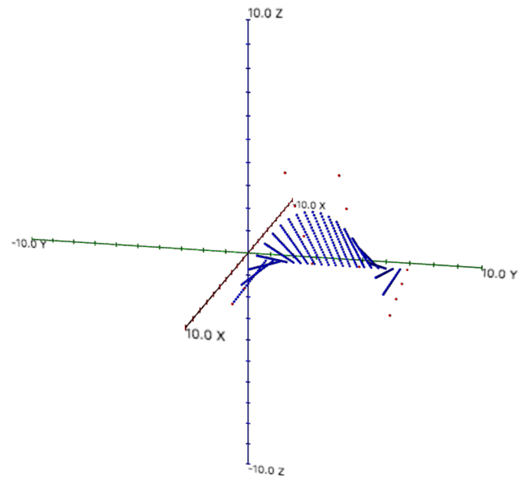


29. You don't get the surface, you get a line instead. A linear combination will give you a collection of points. If you want a surface, you need to start with

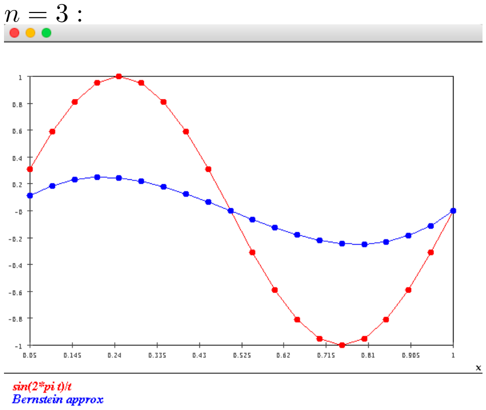


surfaces.

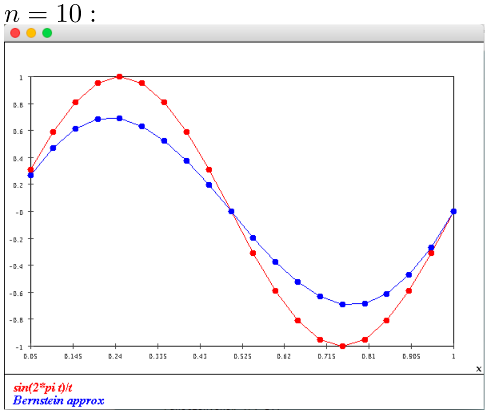
30. Bezier3D:


10.0 Z
-10.0 Y
10.0 X
10.0 X
10.0 Y
-10.0 Z

31. $n = 3$:


x
sin(2*pi t)/t
Bernstein approx

$n = 10$:


x
sin(2*pi t)/t
Bernstein approx