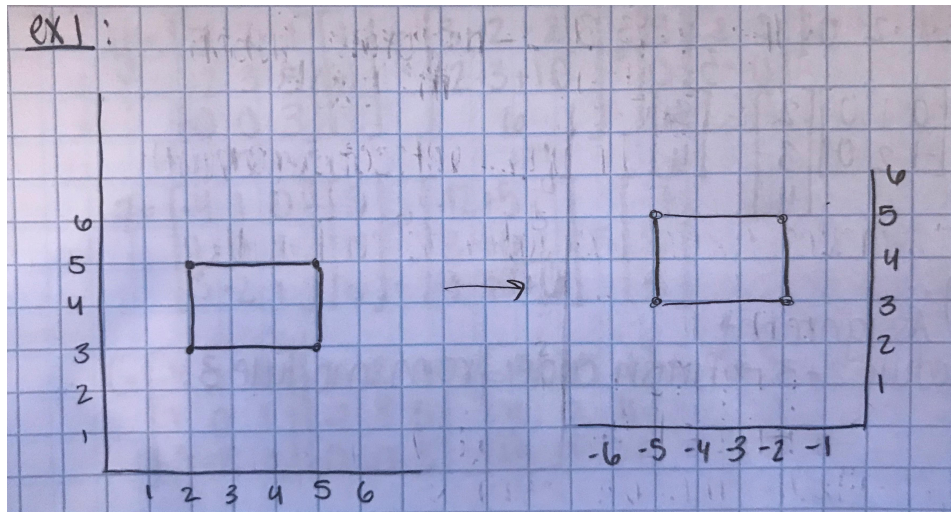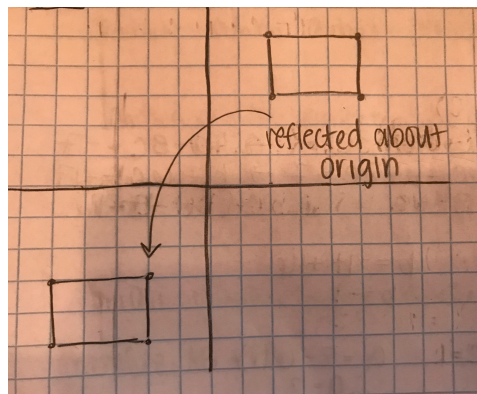# Computational Linear Algebra, Module 4

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Due: February 21st, 2018

1. image:

2. See code.

3. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ would result in reflecting a shape about the x-axis.

4. image:

5. See code.

6. $BA = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This is a reflection about the x-axis, which is what we would expect since we are first reflecting about the y-axis and then rotating about the origin.

7. We need the associativity property.

8. When we change the order, nothing changes about the result. We have $AB = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This makes it seem like the multiplication of matrices is commutative. Geometrically, this just means that the two transformations we performed do commute with each other.

9. This transformation looks like a rotation about some point other than the origin.

10. In this case, the order of the transformations does matter - this is because there is some rotation involved.

11. This transformation effectively puts the rectangle where it was originally. This means that the matrix B is just rotating the rectangle by some amount when used alone. If we apply C to a vector v, we will see that v will remain unchanged.

12. Prove $\mathbf{AB} \neq \mathbf{BA}$:

Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1i} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2i} & ... & a_{n2} \\ ... & ... & ... & ... & ... & ... \\ a_{j1} & a_{j2} & ... & a_{ji} & ... & a_{nj} \\ ... & ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mi} & ... & a_{mn} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1i} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2i} & ... & b_{n2} \\ ... & ... & ... & ... & ... & ... \\ b_{j1} & b_{j2} & ... & b_{ji} & ... & b_{nj} \\ ... & ... & ... & ... & ... & ... \\ b_{m1} & b_{m2} & ... & b_{mi} & ... & b_{mn} \end{bmatrix}$

Then,

$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1i} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2i} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{j1} & a_{j2} & ... & a_{ji} & ... & a_{jn} \\ ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{ni} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & ... & b_{1i} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2i} & ... & b_{2n} \\ ... & ... & ... & ... & ... & ... \\ b_{j1} & b_{j2} & ... & b_{ji} & ... & b_{jn} \\ ... & ... & ... & ... & ... & ... \\ b_{n1} & b_{n2} & ... & b_{ni} & ... & b_{nn} \end{bmatrix}$

$= \begin{bmatrix} a_{11}b_{11} + ... + a_{1n}b_{n1} & a_{11}b_{12} + ... + a_{1n}b_{n2} & ... & a_{11}b_{1i} + ... + a_{1n}b_{ni} & ... & a_{11}b_{1n} + ... + a_{1n}b_{nn} \\ a_{21}b_{11} + ... + a_{2n}b_{n1} & a_{21}b_{12} + ... + a_{2n}b_{n2} & ... & a_{21}b_{2i} + ... + a_{2n}b_{ni} & ... & a_{21}b_{2n} + ... + a_{2n}b_{nn} \\ ... & ... & ... & ... & ... & ... \\ a_{j1}b_{11} + ... + a_{jn}b_{n1} & a_{j1}b_{12} + ... + a_{jn}b_{n2} & ... & a_{j1}b_{1i} + ... + a_{jn}b_{ni} & ... & a_{j1}b_{1n} + ... + a_{jn}b_{nn} \\ ... & ... & ... & ... & ... & ... \\ a_{n1}b_{11} + ... + a_{nn}b_{n1} & a_{n1}b_{12} + ... + a_{nn}b_{n2} & ... & a_{n1}b_{1i} + ... + a_{nn}b_{ni} & ... & a_{n1}b_{1n} + ... + a_{nn}b_{nn} \end{bmatrix}$

and

$$\mathbf{BA} = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1i} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2i} & ... & b_{2n} \\ ... & ... & ... & ... & ... & ... \\ b_{j1} & b_{j2} & ... & b_{ji} & ... & b_{jn} \\ ... & ... & ... & ... & ... & ... \\ b_{n1} & b_{n2} & ... & b_{ni} & ... & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & ... & a_{1i} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2i} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{j1} & a_{j2} & ... & a_{ji} & ... & a_{jn} \\ ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{ni} & ... & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}a_{11} + ... + b_{1n}a_{n1} & b_{11}a_{12} + ... + b_{1n}a_{n2} & ... & b_{11}a_{1i} + ... + b_{1n}a_{ni} & ... & b_{11}a_{1n} + ... + b_{1n}a_{nn} \\ b_{21}a_{11} + ... + b_{2n}a_{n1} & b_{21}a_{12} + ... + b_{2n}a_{n2} & ... & b_{21}a_{2i} + ... + b_{2n}a_{ni} & ... & b_{21}a_{2n} + ... + b_{2n}a_{nn} \\ ... & ... & ... & ... & ... & ... \\ b_{j1}a_{11} + ... + b_{jn}a_{n1} & b_{j1}a_{12} + ... + b_{jn}a_{n2} & ... & b_{j1}a_{1i} + ... + b_{jn}a_{ni} & ... & b_{j1}a_{1n} + ... + b_{jn}a_{nn} \\ ... & ... & ... & ... & ... & ... \\ b_{n1}a_{11} + ... + b_{nn}a_{n1} & b_{n1}a_{12} + ... + b_{nn}a_{n2} & ... & b_{n1}a_{1i} + ... + b_{nn}a_{ni} & ... & b_{n1}a_{1n} + ... + b_{nn}a_{nn} \end{bmatrix}$$

Since, $b_{j1}a_{12} + ... + b_{jn}a_{n2} \neq a_{j1}b_{12} + ... + a_{jn}b_{n2}$ therefore $\mathbf{AB} \neq \mathbf{BA}$ ∎

Prove $\mathbf{A(BC)} = \mathbf{(AB)C}$:

Let $\mathbf{A} = \begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & ... & b_{1n} \\ b_{21} & ... & b_{2n} \\ ... & ... & ... \\ b_{m1} & ... & b_{mn} \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} c_{11} & ... & c_{1n} \\ c_{21} & ... & c_{2n} \\ ... & ... & ... \\ c_{m1} & ... & c_{mn} \end{bmatrix}$

Then,

$$\mathbf{A(BC)} = \begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix} \left(\begin{bmatrix} b_{11} & ... & b_{1n} \\ b_{21} & ... & b_{2n} \\ ... & ... & ... \\ b_{m1} & ... & b_{mn} \end{bmatrix} \begin{bmatrix} c_{11} & ... & c_{1n} \\ c_{21} & ... & c_{2n} \\ ... & ... & ... \\ c_{m1} & ... & c_{mn} \end{bmatrix} \right)$$

$$= \begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix} \left(\begin{bmatrix} b_{11}c_{11} + ... + b_{1n}c_{m1} & ... & b_{11}c_{1n} + ... + b_{1n}c_{mn} \\ b_{21}c_{11} + ... + b_{2n}c_{m1} & ... & b_{21}c_{1n} + ... + b_{2n}c_{mn} \\ ... & ... & ... \\ b_{m1}c_{11} + ... + b_{mn}c_{m1} & ... & b_{m1}c_{1n} + ... + b_{mn}c_{mn} \end{bmatrix} \right)$$

$$= \begin{bmatrix} a_{11}(b_{11}c_{11} + ... + b_{1n}c_{m1}) + ... & ... & a_{11}(b_{m1}c_{11} + ... + b_{mn}c_{m1}) + ... \\ ... & ... & ... \\ a_{m1}(b_{11}c_{11} + ... + b_{1n}c_{m1}) + ... & ... & a_{m1}(b_{m1}c_{11} + ... + b_{mn}c_{m1}) + ... \end{bmatrix}$$

$$= \left(\begin{bmatrix} a_{11}b_{11} + ... + a_{1n}b_{m1} & ... & a_{11}b_{1n} + ... + a_{1n}b_{mn} \\ a_{21}b_{11} + ... + a_{2n}b_{m1} & ... & a_{21}b_{1n} + ... + a_{2n}b_{mn} \\ ... & ... & ... \\ a_{m1}b_{11} + ... + a_{mn}b_{m1} & ... & a_{m1}b_{1n} + ... + a_{mn}b_{mn} \end{bmatrix} \right) \begin{bmatrix} c_{11} & ... & c_{1n} \\ c_{21} & ... & c_{2n} \\ ... & ... & ... \\ c_{m1} & ... & c_{mn} \end{bmatrix}$$

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$$
= \left(\begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & ... & b_{1n} \\ b_{21} & ... & b_{2n} \\ ... & ... & ... \\ b_{m1} & ... & b_{mn} \end{bmatrix} \right) \begin{bmatrix} c_{11} & ... & c_{1n} \\ c_{21} & ... & c_{2n} \\ ... & ... & ... \\ c_{m1} & ... & c_{mn} \end{bmatrix}
$$

Therefore, $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ ∎

Prove $\mathbf{A}(\mathbf{B} + \mathbf{C}) = (\mathbf{AB}) + (\mathbf{CB})$:

Let $\mathbf{A} = \begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & ... & b_{1n} \\ b_{21} & ... & b_{2n} \\ ... & ... & ... \\ b_{m1} & ... & b_{mn} \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} c_{11} & ... & c_{1n} \\ c_{21} & ... & c_{2n} \\ ... & ... & ... \\ c_{m1} & ... & c_{mn} \end{bmatrix}$

Then,

$$
\mathbf{A(B+C)} = \begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix} \left(\begin{bmatrix} b_{11} & ... & b_{1n} \\ b_{21} & ... & b_{2n} \\ ... & ... & ... \\ b_{m1} & ... & b_{mn} \end{bmatrix} + \begin{bmatrix} c_{11} & ... & c_{1n} \\ c_{21} & ... & c_{2n} \\ ... & ... & ... \\ c_{m1} & ... & c_{mn} \end{bmatrix} \right)
$$

$$
= \begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix} \left(\begin{bmatrix} b_{11} + c_{11} & ... & b_{1n} + c_{1n} \\ b_{21} + c_{21} & ... & b_{2n} + c_{2n} \\ ... & ... & ... \\ b_{m1} + c_{m1} & ... & b_{mn} + c_{mn} \end{bmatrix} \right)
$$

$$
= \begin{bmatrix} a_{11}(b_{11} + c_{11}) + ... + a_{1n}(b_{m1} + c_{m1}) & ... & a_{11}(b_{1n} + c_{1n}) + ... + a_{1n}(b_{mn} + c_{mn}) \\ a_{21}(b_{11} + c_{11}) + ... + a_{2n}(b_{m1} + c_{m1}) & ... & a_{21}(b_{1n} + c_{1n}) + ... + a_{2n}(b_{mn} + c_{mn}) \\ ... & ... & ... \\ a_{m1}(b_{11} + c_{11}) + ... + a_{mn}(b_{m1} + c_{m1}) & ... & a_{m1}(b_{1n} + c_{1n}) + ... + a_{mn}(b_{mn} + c_{mn}) \end{bmatrix}
$$

$$
= \begin{bmatrix} (a_{11}b_{11} + a_{11}c_{11}) + ... + (a_{1n}b_{m1} + a_{1n}c_{m1}) & ... & (a_{11}b_{1n} + a_{11}c_{1n}) + ... + (a_{1n}b_{mn} + a_{1n}c_{mn}) \\ (a_{21}b_{11} + a_{21}c_{11}) + ... + (a_{2n}b_{m1} + a_{2n}c_{m1}) & ... & (a_{21}b_{1n} + a_{21}c_{1n}) + ... + (a_{2n}b_{mn} + a_{2n}c_{mn}) \\ ... & ... & ... \\ (a_{m1}b_{11} + a_{m1}c_{11}) + ... + (a_{mn}b_{m1} + a_{mn}c_{m1}) & ... & (a_{m1}b_{1n} + a_{m1}c_{1n}) + ... + (a_{mn}b_{mn} + a_{mn}c_{mn}) \end{bmatrix}
$$

$$
= \begin{bmatrix} a_{11}b_{11} + ... + a_{1n}b_{m1} & ... & a_{11}b_{1n} + ... + a_{1n}b_{mn} \\ a_{21}b_{11} + ... + a_{2n}b_{m1} & ... & a_{21}b_{1n} + ... + a_{2n}b_{mn} \\ ... & ... & ... \\ a_{m1}b_{11} + ... + a_{mn}b_{m1} & ... & a_{m1}b_{1n} + ... + a_{mn}b_{mn} \end{bmatrix}
$$

$$
+ \begin{bmatrix} a_{11}c_{11} + ... + a_{1n}c_{m1} & ... & a_{11}c_{1n} + ... + a_{1n}c_{mn} \\ a_{21}c_{11} + ... + a_{2n}c_{m1} & ... & a_{21}c_{1n} + ... + a_{2n}c_{mn} \\ ... & ... & ... \\ a_{m1}c_{11} + ... + a_{mn}c_{m1} & ... & a_{m1}c_{1n} + ... + a_{mn}c_{mn} \end{bmatrix}
$$

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Therefore, $\mathbf{A}(\mathbf{B}+\mathbf{C}) = (\mathbf{AB}) + (\mathbf{CB})$ ∎

Prove $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B}$:

$$\text{Let } \mathbf{A} = \begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_{11} & ... & b_{1n} \\ b_{21} & ... & b_{2n} \\ ... & ... & ... \\ b_{m1} & ... & b_{mn} \end{bmatrix}$$

Then,

$$\alpha(\mathbf{AB}) = \alpha \left(\begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & ... & b_{1n} \\ b_{21} & ... & b_{2n} \\ ... & ... & ... \\ b_{m1} & ... & b_{mn} \end{bmatrix} \right)$$

$$= \alpha \begin{bmatrix} a_{11}b_{11} + ... + a_{1n}b_{m1} & ... & a_{11}b_{1n} + ... + a_{1n}b_{mn} \\ a_{21}b_{11} + ... + a_{2n}b_{m1} & ... & a_{21}b_{1n} + ... + a_{2n}b_{mn} \\ ... & ... & ... \\ a_{m1}b_{11} + ... + a_{mn}b_{m1} & ... & a_{m1}b_{1n} + ... + a_{mn}b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(a_{11}b_{11} + ... + a_{1n}b_{m1}) & ... & \alpha(a_{11}b_{1n} + ... + a_{1n}b_{mn}) \\ \alpha(a_{21}b_{11} + ... + a_{2n}b_{m1}) & ... & \alpha(a_{21}b_{1n} + ... + a_{2n}b_{mn}) \\ ... & ... & ... \\ \alpha(a_{m1}b_{11} + ... + a_{mn}b_{m1}) & ... & \alpha(a_{m1}b_{1n} + ... + a_{mn}b_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha a_{11})b_{11} + ... + (\alpha a_{1n})b_{m1} & ... & (\alpha a_{11})b_{1n} + ... + (\alpha a_{1n})b_{mn} \\ (\alpha a_{21})b_{11} + ... + (\alpha a_{2n})b_{m1} & ... & (\alpha a_{21})b_{1n} + ... + (\alpha a_{2n})b_{mn} \\ ... & ... & ... \\ (\alpha a_{m1})b_{11} + ... + (\alpha a_{mn})b_{m1} & ... & (\alpha a_{m1})b_{1n} + ... + (\alpha a_{mn})b_{mn} \end{bmatrix}$$

$$= \left(\alpha \begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix} \right) \begin{bmatrix} b_{11} & ... & b_{1n} \\ b_{21} & ... & b_{2n} \\ ... & ... & ... \\ b_{m1} & ... & b_{mn} \end{bmatrix}$$

Therefore, $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B}$ ∎

13. I think more properties are needed for this proof, because none of the given properties say anything about the direction of multiplication, and we would need either some property stating something about direction of multiplication, or some sort of commutative property (which we know not to be true).

14. See code.

15. The vectors $(0, 1)$ and $(0, 2)$ do not span the whole space of 2D vectors. This is because they are not linearly independent vectors. In 3D, an example of vectors that do not span all of 3D space is $(0, 1, 0)$, $(1, 0, 0)$ and $(1, 2, 0)$ since all of these vectors are in the $z = 0$ plane.

16. No, you always need n vectors to span n-D space.

17. $(1, 4) = (1, 0) + 4(0, 1) = e_1 + 4e_2$
    $(2, 3) = 2(1, 0) + 3(0, 1) = 2e_1 + 3e_2$
    Yes, e_1 and e_2 span 2D space.

18. In standard coordinates, we see that $\cos\theta = \frac{x}{h}$ and $\sin\theta = \frac{u}{h}$. Similarly, we can get the coordinates for the other new axis.

19. Clockwise rotation by θ: $\begin{bmatrix} \cos\theta & \cos\theta \\ -\sin\theta & \sin\theta \end{bmatrix}$

    Reflection about the x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

20. You can't derive a multiplicative matrix to do translations by using the above step-by-step approach.

21. Need to do still.

22. When C is applied to a 2D vector, we get $\begin{bmatrix} c_{11}x + c_{12}y \\ c_{21}x + c_{22}y \end{bmatrix}$. When the affine matrix for C is applied to the 3D affine vector for the original vector, we get $\begin{bmatrix} c_{11}x + c_{12}y \\ c_{21}x + c_{22}y \\ 1 \end{bmatrix}$

23. Need to do still.

24. We will show this using 2x2 matrices (3x3 affine matrices).
    $affine(B) = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $affine(A) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $affine(u) = \begin{bmatrix} u_1 \\ u_2 \\ 1 \end{bmatrix}$. So, $affine(B)affine(A) = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{12} & b_{11}a_{12} + b_{12}a_{22} & 0 \\ b_{21}a_{11} + b_{22}a_{12} & b_{21}a_{12} + b_{22}a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 1 \end{bmatrix} = \begin{bmatrix} u_1(b_{11}a_{11} + b_{12}a_{12}) & u_2(b_{11}a_{12} + b_{12}a_{22}) & 0 \\ u_1(b_{21}a_{11} + b_{22}a_{12}) & u_2(b_{21}a_{12} + b_{22}a_{22}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Now, the projection of this is $\begin{bmatrix} u_1(b_{11}a_{11} + b_{12}a_{12}) & u_2(b_{11}a_{12} + b_{12}a_{22}) \\ u_1(b_{21}a_{11} + b_{22}a_{12}) & u_2(b_{21}a_{12} + b_{22}a_{22}) \end{bmatrix}$ and this is the same as BAu.

25. See code.

26. $AI = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$
So, $AI = A$. For an m x n matrix, the identity matrix has to be an m x m square matrix.

27. This is true because of how the matrix multiplication works. If we look at Ab_i, we will see that each of these sums (rows of A multiplied by the vector b_i) will equate to an entry in the vector c_i.

28. The undo matrix for reflection about the y-axis is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. If we multiply the undo matrix with its original reflection matrix, we have $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

29. The undo matrix for translation by p, q is $\begin{bmatrix} 1 & 0 & -p \\ 0 & 1 & -q \\ 0 & 0 & 1 \end{bmatrix}$. If we multiple the undo matrix with the original translation matrix, we have $\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -p \\ 0 & 1 & -q \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

30. The affine-extended matrix for $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

31. See code. By using the inverse, it moves the point to (3, 1, 1).

32. $A^{-1}B^{-1} \neq B^{-1}A^{-1}$. There is a contribution from every sum of products when doing the multiplication. However, when the order of the multiplication is changed, the contribution of certain cell-products is different, causing the order to matter in this multiplication.

33. the cuboid is in the $(+x, -y, +z)$ coordinate space when the eye is moved to the origin.

34. Yes, we do get the desired result. See code.

35. $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \alpha(v_1) \\ \alpha(v_2) \end{bmatrix} = \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Thus this stretches the length by α.

36. When $\mathbf{v} = \mathbf{u}$, we see that $|v| = |u|$, so we end up with the statement $u \cdot u = u \cdot u$.

37. See code.

38. For the reflection matrix, in the 2 x 2 case, it is easy to see that the columns are orthogonal since when we do their dot product, we get 0. We can also see that they are orthonormal because the magnitude of each column is 1 and they are orthogonal to each other. For the rotation matrix, we defined it in such a way that the columns are all vectors that are perpendicular to each other since they are defining a new set of axes. Also the rotation matrix is orthonormal since the magnitude of the vectors will always be 1 by the trig identity that $(\sin\theta)^2 + (\cos\theta)^2 = 1$.

39. $A^T A = I$ when A is orthogonal, and we know that $A^{-1}A = I$. Therefore, $A^T = A^{-1}$.

40. See code.

41. $(1 + 2i, i, -3 + 4i, 5) + (-2 + 1, 2, 4, 1 + i) = (-1 + 3i, 2 + i, 1 + 4i, 6 + i)$

42. $(1 - 2i)(1 + 2i, i, -3 + 4i, 5) = (5, 2 + i, 5 + 10i, 5 - 10i)$

43. $(a + bi)(a + bi) = a^2 + 2abi - b^2$
    $(a + bi)(a - bi) = a^2 + b^2$

44. $z \cdot z = (1 + 2i, i, 5)(1 - 2i, -i, 5) = (5, 1, 25)$
    $(1 + 2i)^2 + (i)^2 + (5)^2 = (1 + 4i - 4) + (-1) + 25 = 21 + 4i$

45. The matrix rotates the vectors, but we can see that it does not change their lengths.

46. There is a point $\theta = 2\pi$ where it starts over.

47. The points of intersection are at π and 2π.

48. See code.

49. The vectors converge overtime because the angle decreases over time.

50. There are approximately 28 fixed points, 7 per quadrant, that emerge - these fixed points are called eigenvectors.