Computational Linear Algebra, Module 11

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Due: April 18th, 2018

1. $AS = A\begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ x_1 & x_2 & \ddots & x_n \\ \vdots & \vdots & \dots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ Ax_1 & Ax_2 & \ddots & Ax_n \\ \vdots & \vdots & \dots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ \lambda x_1 & \lambda x_2 & \ddots & \lambda x_n \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}.$

   Now, $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & \dots \\ 0 & \lambda_2 & \dots & \dots \\ \vdots & \vdots & \ddots & \dots \\ \vdots & \vdots & \dots & \lambda_n \end{bmatrix}$. So,

   $S\Lambda = \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ x_1 & x_2 & \ddots & x_n \\ \vdots & \vdots & \dots & \vdots \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & \dots \\ 0 & \lambda_2 & \dots & \dots \\ \vdots & \vdots & \ddots & \dots \\ \vdots & \vdots & \dots & \lambda_n \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ \lambda x_1 & \lambda x_2 & \ddots & \lambda x_n \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}$

   ■

   However, $AS \neq \Lambda S$ because when you change the order, the λ's would distribute into the rows of S instead of the columns.

2. The dimensions of $\mathbf{C}$ is $m \times n$ or 3×4.
   Example:

   $$\mathbf{A} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix}$$

   Therefore, $\mathbf{AB} = \mathbf{C}$ is:

   $$\mathbf{C} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 & 14 \\ 4 & 12 & 20 & 28 \\ 6 & 18 & 30 & 42 \end{bmatrix}$$

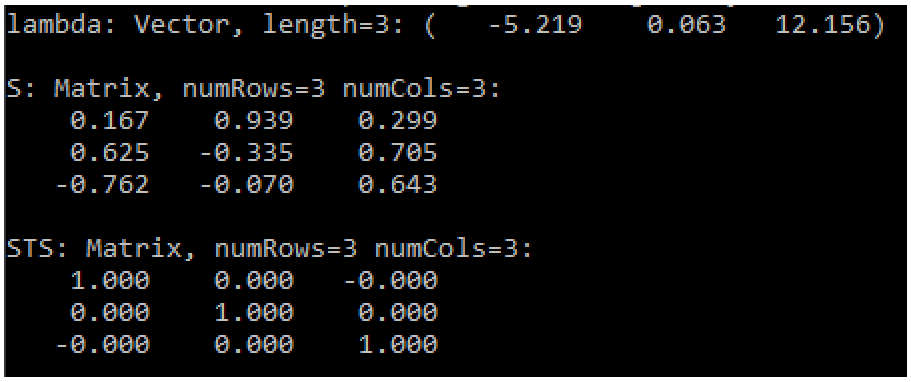
3. $a_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, so, $a_4 a_4^T + 2a_4 a_4^T + 3a_4 a_4^T =$

   $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} =$

$$\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}+\begin{bmatrix}2&2&2&2\\2&2&2&2\\2&2&2&2\\2&2&2&2\end{bmatrix}+\begin{bmatrix}3&3&3&3\\3&3&3&3\\3&3&3&3\\3&3&3&3\end{bmatrix}=\begin{bmatrix}6&6&6&6\\6&6&6&6\\6&6&6&6\\6&6&6&6\end{bmatrix}$$

4. Let $A=\begin{bmatrix}1&3&2\\2&1&3\end{bmatrix}$ and $B=\begin{bmatrix}1&2&3&1\\2&3&2&2\\3&1&1&3\end{bmatrix}$. Then, $AB=\begin{bmatrix}13&13&10&13\\13&10&11&13\end{bmatrix}$.
   Now for the right hand side, we have
   $\sum_{i=1}^{k} a_i b_i^T = \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1&2&3&1\end{bmatrix}+\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}2&3&2&2\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}\begin{bmatrix}3&1&1&3\end{bmatrix}=$
   $\begin{bmatrix}1&2&3&1\\2&4&6&2\end{bmatrix}+\begin{bmatrix}6&9&6&6\\2&3&2&2\end{bmatrix}+\begin{bmatrix}6&2&2&6\\9&3&3&9\end{bmatrix}=\begin{bmatrix}13&13&10&13\\13&10&11&13\end{bmatrix}$.

5.



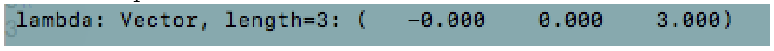
```
lambda: Vector, length=3: (    -5.219     0.063    12.156)

S: Matrix, numRows=3 numCols=3:
    0.167     0.939     0.299
    0.625    -0.335     0.705
   -0.762    -0.070     0.643

STS: Matrix, numRows=3 numCols=3:
    1.000     0.000    -0.000
    0.000     1.000     0.000
   -0.000     0.000     1.000
```

6. The columns of $\mathbf{S}$ are orthogonal.

7. Example matrix: $\begin{bmatrix}\frac{1}{2}&1&\frac{1}{2}\\1&2&1\\\frac{1}{2}&1&\frac{1}{2}\end{bmatrix}$ and the associated eigenvalues and eigenvectors output are:



```
lambda: Vector, length=3: (    -0.000     0.000     3.000)
```

8. $A=\begin{bmatrix}r_1&r_2&\dots&r_n\\\vdots&\vdots&\ddots&\vdots\\\dots&\dots&\dots&\dots\end{bmatrix}$ and $B=\begin{bmatrix}b_{11}&b_{12}&\dots&b_{1k}\\\vdots&\vdots&\ddots&\vdots\\b_{n1}&b_{n2}&\dots&b_{nk}\end{bmatrix}$ so $C=\begin{bmatrix}r_1b_{11}+r_2b_{21}+\dots+r_nb_{n1}&\dots&r_1\\\vdots&&\vdots\\\dots&&\dots\end{bmatrix}$
   So, as we can see from this, the first element of C is a linear combination of the first column of B, the second element of C is a linear combination of the second column of B and so on. So, we can say that the whole first row of C is a linear combination of the rows of B.

9. $A=\begin{bmatrix}r_1&r_2&\dots&r_n\\\vdots&\vdots&\ddots&\vdots\\\dots&\dots&\dots&\dots\end{bmatrix}$ and $B=\begin{bmatrix}b_{11}&b_{12}&\dots&b_{1k}\\\vdots&\vdots&\ddots&\vdots\\b_{n1}&b_{n2}&\dots&b_{nk}\end{bmatrix}$. So,

$$C = \begin{bmatrix} r_1b_{11} + r_2b_{21} + \dots + r_nb_{n1} & \dots & r_1b_{1k} + r_2b_{2k} + \dots r_nb_{nk} \\ a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2n}b_{n1} & \dots & a_{21}b_{1k} + a_{22}b_{2k} + \dots a_{2n}b_{nk} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \dots + a_{mn}b_{n1} & \dots & a_{m1}b_{1k} + a_{m2}b_{2k} + \dots a_{mn}b_{nk} \end{bmatrix}.$$

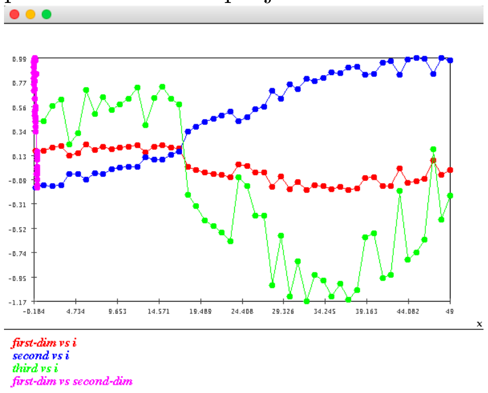
So, we can see that each element in the first column of C is a linear combination of each row of A. So, we can say that the whole first column of C is a linear combination of all of the columns of A.

10. Eigenvalue, eigenvector pairs correspond to the amount of change in each dimension of the space we are working with. A space consists of the linearly independent components, or the span, and we know that the $rank(A)$ is the number of linearly independent dimensions of the space represented by A. Thus, the $rank(A) = rank(\Lambda) =$ number of nonzero eigenvalues.

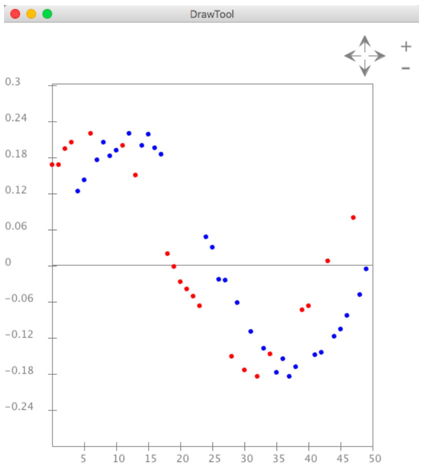
11. When we multiply z^T by x_i, we are going to do x_i with every term in z^T. Since the x_i's are orthogonal, $x_i \cdot x_i = 1$ and $x_i \cdot x_j = 0, i \neq j$. So, the only term that will remain is going to be α_i, and all terms will go to zero.

12. We want to show that A^TA is positive definite for any matrix A. So, we need to show that $z^TA^TAz > 0$. So,
$z^TA^TAz = (Az)^T(Az) = (Az) \cdot (Az) \geq 0$, and when A is non-singular, this dot product is strictly greater than 0. So, A^TA is positive definite. ■

13. plot for DataExample.java:



It does not look like adding the plot of one dimension agains another helps here.
The dimension of the PCA coordinates being plotted is the first dimension:



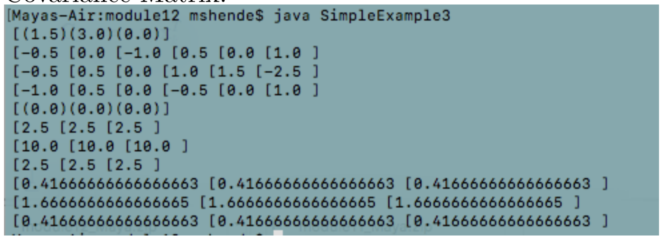
14. We first normalize the data so that the mean is 0 (center the data). Then we compute the covariance matrix and actually find the eigenvectors of this matrix.

15. Mean Vector $= \begin{bmatrix} 1.5 \\ 3 \\ 0 \end{bmatrix}$

16. Mean Vector $= \begin{bmatrix} 1.5 \\ 3 \\ 0 \end{bmatrix}$

    Mean of the Mean Vector $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

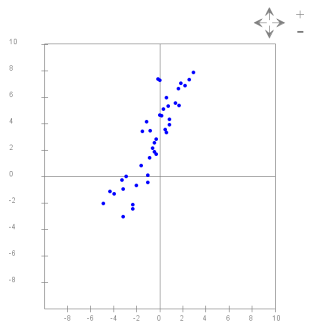
17. Covariance Matrix:



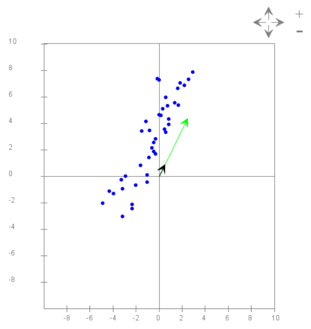
```
[Mayas-Air:module12 mshende$ java SimpleExample3
 [(1.5)(3.0)(0.0)]
 [-0.5 [0.0 [-1.0 [0.5 [0.0 [1.0 ]
 [-0.5 [0.5 [0.0 [1.0 [1.5 [-2.5 ]
 [-1.0 [0.5 [0.0 [-0.5 [0.0 [1.0 ]
 [(0.0)(0.0)(0.0)]
 [2.5 [2.5 [2.5 ]
 [10.0 [10.0 [10.0 ]
 [2.5 [2.5 [2.5 ]
 [0.41666666666666663 [0.41666666666666663 [0.41666666666666663 ]
 [1.6666666666666665 [1.6666666666666665 [1.6666666666666665 ]
 [0.41666666666666663 [0.41666666666666663 [0.41666666666666663 ]
```

18. Based on Useful Fact #1, we know that if $\mathbf{C} = \mathbf{AB}$ then $\mathbf{c}_i = \mathbf{Ab}_i$ for all columns in both $\mathbf{C}$ and $\mathbf{B}$. Using this fact, we can look at each individual column in both $\mathbf{S}$ and $\mathbf{\Lambda}$ and perform the products of them in the desired order. With the resulting vectors, we can re-comply them into a covariance matrix.

19. DataExample3.java:



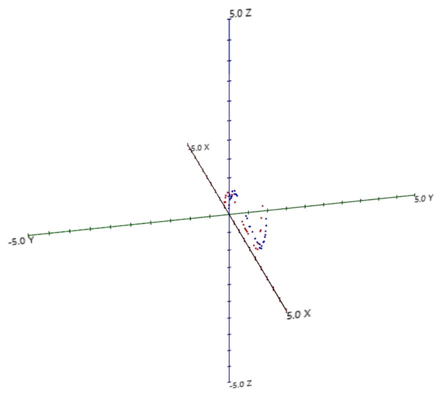
PCAExample2.java:



20. If we use useful fact 1, we can treat S^T as A and X as B, and then distribute the columns in reverse to see that $Y = S^T X$ follows from $y_i = S^T x_i$.

21. First, if we let $z = (1, 1, 1, \ldots, 1)$ and X be a centered data matrix, then when we multiply Xz, we see that for each row of X, the values will just sum to 0 (since they are centered evenly using their mean about 0). So, we will get $Xz = 0$. Now, we want to show that $Yz = 0$. So, substituting $Y = BX$, we have $BXz = B0 = 0$. Therefore, we have shown that $Yz = 0$ and thus Y is centered.

22. The same pattern is observed:

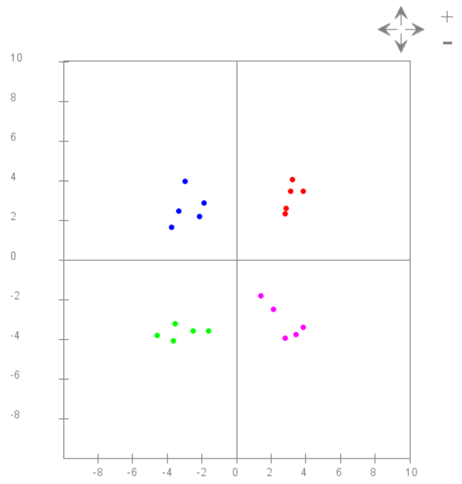


23. Two Applications of Clustering:

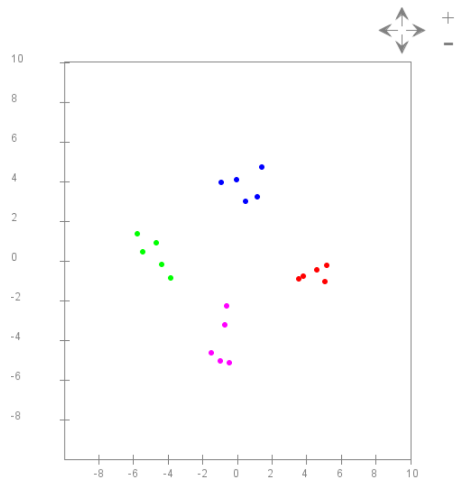
    (a) Targeting Marketing Programs: group people with similar interests together and advertise the same thing to them.

    (b) Land Usage: Similar plots of land can be used for the same thing.

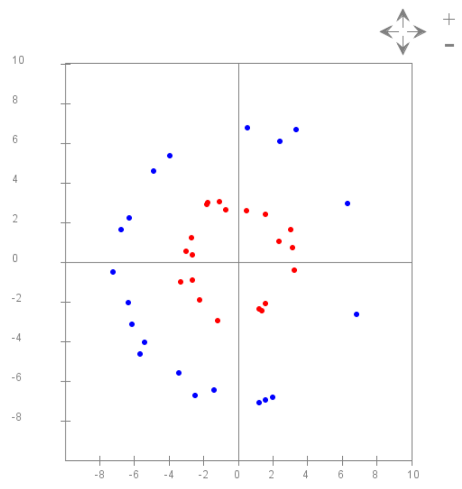
24. DataExample3.java:



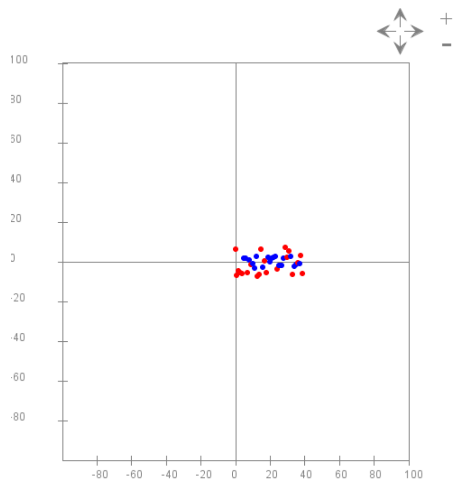
PCAExample3.java:



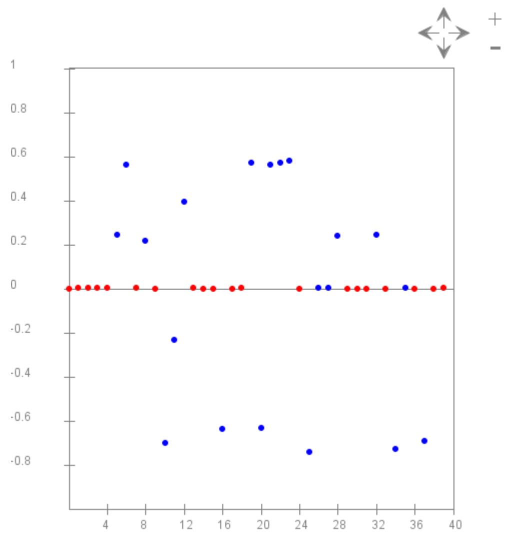
25. Hyperplanes would not work because the data is not clustered into non-overlapping areas. Instead the data is formatted with one type being in the middle and another going around the outside. The K-means algorithm will assign the center of the blue data points and the red data points at the same spot and the planes would overlap.



26. The PCA-transformed points are not easier to cluster because they are tightly mixed together.

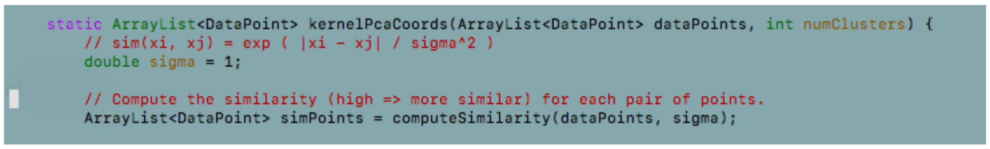


27. They are easier to cluster although it is not a perfect clustering. There will blue points in the red cluster.



28. Suppose we have data that is distributed in two concentric circles about the origin. Then, if we use the first dimension of the transformed data, we would transform the data using the following: $x^2 + y^2$. So, all data would shift to lie in the upper right quadrant, in two 'concentric' arcs. From here, it can be seen that the data is cleanly separable.

29. computation of similarity:



```
static ArrayList<DataPoint> kernelPcaCoords(ArrayList<DataPoint> dataPoints, int numClusters) {
    // sim(xi, xj) = exp ( |xi - xj| / sigma^2 )
    double sigma = 1;

    // Compute the similarity (high => more similar) for each pair of points.
    ArrayList<DataPoint> simPoints = computeSimilarity(dataPoints, sigma);
```