Applied Stochastic Processes (FIN 514)
Midterm Exams and Solutions

Instructor: Jaehyuk Choi

2017-18 M1, 2018-19 M1, 2019-20 M1, 2020-21 M3, 2021-22 M3, 2022-23 M1

- **BM** stands for Brownian motion. Assume that B_t, W_t, X_t and Z_t are standard BMs if unless stated otherwise.
- **RN** and **RV** stand for random number and random variable, respectively.
- **MC** stands for Monte–Carlo.
- **SV** stands for stochastic volatility.
- $P(\cdot)$ and $E(\cdot)$ are probability and expectation, respectively.
- The PDF and CDF of the standard normal distribution are denoted by $n(z)$ and $N(z)$, respectively.
- Assume the interest rate and dividend rate are zero (i.e., $r = q = 0$) in option pricing.

1. **[2016(StoFin), Generating RNs for correlated BMs]** Throughout this problem, assume that X_t and Y_t are two independent standard BMs.

   (a) Other than the examples we covered in the class, there are many ways to create standard BMs. A linear combination of the two BMs with the coefficients a and b,

   $$W_t = aX_t + bY_t$$

   is also a BM. (No need to prove it.) What is the condition for a and b under which W_t is a **standard** BM.

   (b) What is the correlation between X_t and W_t? We have not defined the correlation of two BMs yet, so simply compute the correlation of the two distributions of the BMs at $t = 1$, i.e, X_1 and W_1. (In fact, the correlation is same for any time t.) You do not have to use the answer of (a).

   (c) Assume that $\{z_k\}$ for $k = 1, 2, \cdots$ is a sequence of standard normal RVs, i.e., $N(0, 1)$, which are generated from computer (e.g., using Box–Muller algorithm). Use $\{z_k\}$ to generate RNs for X_t for a fixed time t.

   (d) Assume that we have two standard BMs, X_t and W_t, which have correlation ρ. How can you generate the pairs of RNs for X_t and W_t for a fixed time t?

Solution:

(a) $\mathrm{Var}(W_t) = a^2\mathrm{Var}(X_t) + b^2\mathrm{Var}(Y_t) = (a^2+b^2)t$ should be t. Therefore, $a^2+b^2=1$.

(b)

$$\mathrm{Corr}(W_t, X_t) = \frac{\mathrm{Cov}(X_t, W_t)}{\sqrt{\mathrm{Var}(X_t)\mathrm{Var}(W_t)}} = \frac{at}{\sqrt{t\cdot(a^2+b^2)t}} = \frac{a}{\sqrt{a^2+b^2}}$$

(c) $\{\sqrt{t}\,z_k\}$ is the RNs for X_t.

(d) We can rewrite W_t as $W_t = \rho X_t + \sqrt{1-\rho^2}Y_t$. Therefore, the RNs for X_t and W_t can be generated as

$$\begin{gathered}(\sqrt{t}\,z_1,\ \rho\sqrt{t}\,z_1 + \sqrt{1-\rho^2}\sqrt{t}\,z_2)\\ (\sqrt{t}\,z_3,\ \rho\sqrt{t}\,z_3 + \sqrt{1-\rho^2}\sqrt{t}\,z_4)\\ \cdots\\ (\sqrt{t}\,z_{2k-1},\ \rho\sqrt{t}\,z_{2k-1} + \sqrt{1-\rho^2}\sqrt{t}\,z_{2k})\end{gathered}$$

2. **[2017(StoFin), Box–Muller algorithm for generating normal RN]** The probability and cumulative distribution functions (PDF and CDF) of exponential RV, Z, are given respectively as

$$f(z) = \lambda e^{-\lambda z}, \quad P(z) = 1 - e^{-\lambda z} \quad \text{for} \quad \lambda > 0, z \geq 0.$$

(a) If U is a uniform RV, how can you generate the RNs of Z?

(b) Let X and Y be two independent standard normal RVs. Show that the squared radius, $Z = X^2+Y^2$, follows an exponential distribution by computing $P(X^2+Y^2<z)$. What is λ?

(c) How can you generate the RNs of X and Y from uniform RNs? Hint: introduce another uniform RV, V, and consider the random angle $2\pi V$.

Solution:

(a) The RN can be generated from the inverse CDF:

$$Z = P^{-1}(U) = -\frac{1}{\lambda}\log(1-U) \quad \text{or} \quad Z = -\frac{1}{\lambda}\log U,$$

where we use that $1-U$ is also a uniform RV.

(b) With the change of variable $r^2 = x^2+y^2$ and radial symmetry,

$$P(X^2+Y^2<z) = \frac{1}{2\pi}\int_{x^2+y^2<z} e^{-(x^2+y^2)/2}dxdy = \frac{1}{2\pi}\int_{r=0}^{\sqrt{z}} e^{-r^2/2}2\pi r\,dr = 1-e^{-z/2}.$$

Therefore Z follows an exponential distribution with $\lambda = 1/2$.

(c) The RVs, X and Y, can be thought as x- and y-components of $\sqrt{Z}$ with a random angle $2\pi V$. Also, from the results of (a) and (b), the pair (X,Y) is generated by

$$(X,Y) = \sqrt{Z}(\cos(2\pi V), \sin(2\pi V)) = \sqrt{-2\log U}(\cos(2\pi V), \sin(2\pi V))$$

3. **[2017, Poisson process]** In Poisson process, the CDF for the arrival time t is given as $F(t) = 1 - e^{-\lambda t}$ for the arrival rate λ.

   (a) From a uniform RV, U, generate RN for the **conditional** arrival time t conditional on that the next arrival is after some time t_0, (i.e., $t > t_0$)

   Solution: The RV for unconditional arrival time t can be simulated as

   $$t = -(1/\lambda) \log U,$$

   where U is a uniform RV. From the memoryless property, t conditional on $t \geq t_0$ can be simulated as

   $$t = t_0 - (1/\lambda) \log U.$$

   (b) Assume that the default of a company follows the Poisson process with the arrival rate λ. In the credit default swap (CDS) on the company, party A pays (to B) premium continuously at the rate p (i.e., pays $p\,dt$ during a time period dt) until the maturity T or the company's default whichever comes first, and party B pays (to A) \$1 when the company defaults. What is the fair premium rate p (which makes the NPVs of both parties equal)? Assume that the risk-free rate is zero, i.e., $r = 0$ (although the problem becomes more interesting if $r > 0$).

   Solution:

   $$\begin{aligned}
   \text{NPV of party A} &= \text{NPV of party B} \\
   \int_0^T 1 \cdot \lambda e^{-\lambda t} dt &= \int_0^T pt \cdot \lambda e^{-\lambda t} dt + pT \cdot e^{-\lambda T} \\
   1 - e^{-\lambda T} &= p \left[-t\, e^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t} \right]_{t=0}^{T} + pT \cdot e^{-\lambda T} \\
   1 - e^{-\lambda T} &= \frac{p}{\lambda} (1 - e^{-\lambda T})
   \end{aligned}$$

   Therefore the fair premium value is $p = \lambda$.

4. **[2019, RN generation]** Pareto distribution is defined by the survival function:

   $$S(x) = P(X > x) = \begin{cases} \left(\frac{\lambda}{x}\right)^\alpha & (x \geq \lambda) \\ 1 & (x < \lambda). \end{cases}$$

   (a) Find the mean and variance of the distribution. Clearly state the condition that the mean and variance are finite (i.e., not infinite).

   (b) How can you generate the RN following the Pareto distribution from a uniform RN, U ?

   Solution:

(a) Based on the PDF of X,

$$f(x) = \frac{\alpha\lambda^{\alpha}}{x^{\alpha+1}} \quad \text{for} \quad x \geq \lambda \quad (0 \quad \text{otherwise}),$$

the mean and variance are computed as

$$E(X) = \frac{\alpha\lambda}{\alpha - 1} \quad \text{for} \quad \alpha > 1 \quad (\infty \quad \text{otherwise}),$$

$$\text{Var}(X) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} \quad \text{for} \quad \alpha > 2 \quad (\infty \quad \text{otherwise}).$$

(b) The CDF is easily invertible. From

$$U = 1 - \left(\frac{\lambda}{X}\right)^{\alpha} \quad \Rightarrow \quad X = \frac{\lambda}{(1-U)^{1/\alpha}} \quad \text{or} \quad \frac{\lambda}{U^{1/\alpha}}$$

Reference: Pareto Distribution (WIKIPEDIA)

5. **[2020, Gamma RN generation]** A gamma RV, $X \sim \text{Gamma}(k, \beta)$, is distrusted by the PDF,

$$f_X(x) = \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} \quad \text{for} \quad \Gamma(k) = (k-1)\cdots 2 \cdot 1 \; (\Gamma(1) = 1),$$

where k is a positive integer and $X \geq 0$.

(a) Find the mean and variance of X. **Hint:** $\int_0^\infty f_X(x)dx = 1$ for any k.

(b) How can you generate the RV of $X \sim \text{Gamma}(1, \beta)$?

(c) If $X \sim \text{Gamma}(1, \beta)$, $X' \sim \text{Gamma}(k, \beta)$, and X and X' are independent, find the PDF of $Y = X + X'$.

(d) How can we generate the RV of $\text{Gamma}(k, \beta)$?

Solution:

(a)

$$E(X) = \int_0^\infty x \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} dx = \frac{k}{\beta} \int_0^\infty \frac{\beta^{k+1}}{\Gamma(k+1)} x^k e^{-\beta x} dx = \frac{k}{\beta}$$

$$E(X^2) = \int_0^\infty x^2 \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} dx = \frac{k(k+1)}{\beta^2} \int_0^\infty \frac{\beta^{k+2}}{\Gamma(k+2)} x^{k+1} e^{-\beta x} dx = \frac{k(k+1)}{\beta^2}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{k}{\beta^2}$$

(b) When $k = 1$, X has the same PDF as the exponential distribution with $\lambda = \beta$:

$$f_X(x) = \beta e^{-\beta x}.$$

Therefore, we can generate X by

$$X = -\frac{1}{\beta}\log U \quad \text{or} \quad -\frac{1}{\beta}\log(1-U),$$

where U is a uniform RV.

(c) **Method 1:**

$$\begin{aligned} f_Y(y) &= \int_{x=0}^{y} f_X(y-x) f_{X'}(x)dx = \int_{x=0}^{y} \beta e^{-\beta(y-x)} \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} dx \\ &= \frac{\beta^{k+1}}{\Gamma(k)} e^{-\beta y} \int_{x=0}^{y} x^{k-1} dx = \frac{\beta^{k+1}}{\Gamma(k)} e^{-\beta y} \frac{y^k}{k} = \frac{\beta^{k+1}}{\Gamma(k+1)} y^k e^{-\beta y}. \end{aligned}$$

Therefore, Y follows Gamma$(k+1, \beta)$.

Method 2: The MGF of X' is

$$E\left(e^{-tX'}\right) = \int_0^\infty \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-(\beta+t)x} = \frac{\beta^k}{(\beta+t)^k} = (1+t/\beta)^{-k},$$

where we used the hint of (a) for $\beta' = \beta + t$. It follows that the MGF of X is $(1+t/\beta)^{-1}$. Since X and X' are independent,

$$E\left(e^{-tY}\right) = E\left(e^{-tX}\right) E\left(e^{-tX'}\right) = (1+t/\beta)^{-1}(1+t/\beta)^{-k} = (1+t/\beta)^{-(k+1)}.$$

Therefore, we know that $Y \sim$ Gamma$(k+1, \beta)$.

Method 3: $X' \sim$ Gamma(k, β) is the RV for the k-th arrival time of the Poisson-type events with intensity β. Because the events are memory-less, $X' + X$ is the $(k+1)$-th arrival time and it is Gamma$(k+1, \beta)$.

(d) From (b), Gamma$(k, \beta) \sim X_1 + \cdots X_k$, where X_i's are independent Gamma variables following Gamma$(1, \beta)$. Therefore,

$$X = -\frac{1}{\beta}\log(U_1 \cdots U_k),$$

where U_k are the sequence of uniform RVs.

6. **[2021, Acceptance-rejection sampling]** We want to generate the RNs of X with PDF $f(x)$ and CDF $F(x)$. Suppose that it is <u>not</u> possible to draw X by the inversion, $Y = F^{-1}(U)$, for a uniform RN U (probably because $F^{-1}(u)$ is not analytically available). Instead, we are going to sample X by taking advantage of another RV Y with PDF $g(x)$ and CDF $G(x)$, whose RNs we can easily generate. Suppose that Y is *similar* to X in the sense that the ratio of the two PDFs are bounded everywhere by $C > 0$:

$$\frac{f(x)}{g(x)} \le C \quad \text{for all} \quad x. \tag{1}$$

Now let us consider an RV, Y', obtained as a result of the following algorithm:

Step 1 Independently draw Y and a uniform RN U.

Step 2 If Y and U satisfy the condition,

$$U \le \frac{f(Y)}{Cg(Y)}, \tag{2}$$

accept $Y' = Y$. Otherwise, reject Y and repeat **Step 1** until you get an accepted Y'.

In this question, we are going to prove that the above algorithm actually draws the RNs of X by showing that

$$P(Y' \le x) = F(x) = P(X \le x).$$

For the proof, let us define two events:

$$A_x = \{Y \le x\} \quad \text{(for a given value } x\text{)} \quad \text{and} \quad B = \left\{U \le \frac{f(Y)}{Cg(Y)}\right\}$$

(a) What is $P(A_x)$? What is $P\left(U \le \frac{f(x)}{Cg(x)}\right)$? Hint: x is a given number, not an RV.

(b) What are $P(A_x \cap B)$ and $P(B)$? Hint: work on $P(A_x \cap B)$ first because $P(B) = \lim_{x\to\infty} P(A_x \cap B)$.

(c) The probability $P(Y' \le y)$ can be written as the conditional probability:

$$P(Y' \le x) = P(A_x|B).$$

Using the conditional probability law and the results from (b), verify that $P(A_x|B) = F(x)$ (and complete the proof).

Solution: This algorithm is called rejection sampling (WIKIPEDIA) or acceptance-rejection method. It is a powerful method to sample RVs.

(a)

$$P(A_x) = P(Y \le x) = G(x) \quad \text{and} \quad P\left(U \le \frac{f(x)}{Cg(x)}\right) = \frac{f(x)}{Cg(x)} \ (\le 1).$$

(b)

$$\begin{aligned} P(A_x \cap B) &= P\left(U \le \frac{f(Y)}{Cg(Y)} \cap Y \le x\right) = \int_{-\infty}^{x} P\left(U \le \frac{f(y)}{Cg(y)} \cap Y \in (y, y+dy)\right) \\ &= \int_{-\infty}^{x} P\left(U \le \frac{f(y)}{Cg(y)}\right) g(y)dy = \int_{-\infty}^{x} \frac{f(y)}{Cg(y)} g(y)dy = \frac{F(x)}{C}. \end{aligned}$$

It follows that

$$P(B) = \lim_{x\to\infty} P(A_x \cap B) = \frac{F(\infty)}{C} = \frac{1}{C}.$$

(c)

$$P(Y' \le x) = P(A_x|B) = \frac{P(A_x \cap B)}{P(B)} = \frac{F(x)/C}{1/C} = F(x).$$

7. **[2021, Normal RN generation]** We want to sample standard normal RNs using the algorithm from 2021 question above. We will draw $X = |Z|$ for a standard normal RV, Z, and use an exponential RV with $\lambda = 1$ as Y. Reminded that the two PDFs are given by

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad g(x) = e^{-x} \quad (x \geq 0),$$

and that you can draw $Y = -\log U'$ for a uniform RV U'. (You can solve this problem even though you did not answer 2021 question.)

(a) Prove that Equation (1) holds between X and Y. What is C?

(b) Express the acceptance condition, Equation (2), using the two uniform RVs, U and U'.

(c) For the final step, how can you draw Z from X?

Solution:

(a)
$$\frac{f(x)}{g(x)} = \frac{2}{\sqrt{2\pi}} e^{-x^2/2+x} = \frac{2}{\sqrt{2\pi}} e^{-(x-1)^2/2+1/2} = \sqrt{\frac{2e}{\pi}} e^{-(x-1)^2/2} \leq \sqrt{\frac{2e}{\pi}}.$$
Therefore, $C = \sqrt{2e/\pi} \approx 1.315$ and the maximum occurs at $x = 1$.

(b) Therefore, the condition becomes

$$U \leq e^{-(Y-1)^2/2} = e^{-(-\log U'-1)^2/2}$$

This is further simplified to

$$-2\log U \geq (\log U' + 1)^2.$$

(c) Z is obtained from X by randomly selecting the sign (e.g., $+$ or $-$). To be specific,

$$Z = \begin{cases} X = -\log U' & \text{if} \quad U'' > 0.5 \\ -X = \log U' & \text{if} \quad U'' \leq 0.5. \end{cases}$$

for another independent uniform random variable U''.

8. **[2022, Gamma RV with Acceptance-rejection]** From the 2020 question above, we learn how to sample $\text{Gamma}(k, \beta)$ RV when k is a positive integer. In this problem, we are going to generate $\text{Gamma}(\alpha, \beta)$ when $\alpha > 0$ is any number using the acceptance-rejection sampling from the 2021 question above. To make problem simple, we can assume $\beta = 1$ because $\text{Gamma}(\alpha, \beta) \sim \text{Gamma}(\alpha, 1)/\beta$.

Let $X \sim \text{Gamma}(\alpha, 1)$ for $\alpha > 0$. Reminded that its PDF is given by

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x},$$

where $\Gamma(\alpha)$ is the gamma function.

(a) (2 points) Consider the following RV, Y (≥ 0), whose PDF $g(x)$ is given by

$$g(x) = \begin{cases} A\, x^{\alpha-1} & \text{if} \quad 0 \leq x \leq 1 \\ A\, e^{-x} & \text{if} \quad 1 < x. \end{cases}$$

Determine the constant A so that $g(x)$ is a proper PDF.

(b) Find the CDF of Y, $G(x) = \int_0^x g(s)ds$.

(c) How can we sample Y using $G(x)$? (Give detail.)

(d) **First assume that** $0 < \alpha < 1$. We are going to use Y with $g(x)$ for sampling $X \sim \text{Gamma}(\alpha, 1)$ in the acceptance-rejection method. For this, $f(x)$ and $g(x)$ must satisfy

$$\frac{f(x)}{g(x)} \leq C \quad \text{for all} \quad x \geq 0.$$

Show that the condition is satisfied. (Why do you need $0 < \alpha < 1$?) What is C?

(e) From (d), we can now sample $X \sim \text{Gamma}(\alpha, 1)$ for $0 < \alpha < 1$. Then, how can we sample X when $\alpha \geq 1$?

Solution:

(a) We need to ensure that $\int_0^\infty g(x)dx = 1$.

$$\begin{aligned} \int_0^\infty g(x)dx &= A \int_0^1 x^{\alpha-1}dx + A \int_1^\infty e^{-x}dx \\ &= A\left(\frac{1}{\alpha} + \frac{1}{e}\right) = 1. \end{aligned}$$

Therefore,

$$A = \frac{\alpha\, e}{\alpha + e}.$$

(b) When $0 \leq x \leq 1$,

$$G(x) = \int_0^x g(s)ds = \frac{e}{\alpha + e} x^\alpha.$$

When $1 < x$,

$$G(x) = \int_0^1 g(s)ds + \int_1^x g(s)ds = \frac{e}{\alpha+e} + \frac{\alpha\, e}{\alpha+e}\left(e^{-1} - e^{-x}\right) = 1 - \frac{\alpha\, e}{\alpha + e} e^{-x}.$$

Therefore,

$$G(x) = \begin{cases} \frac{e}{\alpha+e} x^\alpha & \text{if} \quad 0 \leq x \leq 1 \\ 1 - \frac{\alpha\, e}{\alpha+e} e^{-x} & \text{if} \quad 1 \leq x. \end{cases}$$

(c) We can sample Y using the inverse CDF, $Y = G^{-1}(U)$, for a uniform RV, U.

$$Y = \begin{cases} \left[\frac{\alpha+e}{e} U\right]^{1/\alpha} & \text{if } 0 \leq U \leq \frac{e}{\alpha+e} \\ -\log\left(\frac{\alpha+e}{\alpha e}(1-U)\right) = \log\frac{\alpha e}{\alpha+e} - \log(1-U) & \text{if } \frac{e}{\alpha+e} \leq U. \end{cases}$$

(d) When $0 \le x \le 1$,

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \le \frac{1}{\Gamma(\alpha)} x^{\alpha-1} = \frac{\alpha+e}{\alpha e \Gamma(\alpha)} g(x).$$

When $1 < x$, we know $0 < x^{\alpha-1} \le 1$ because $0 < \alpha < 1$. So,

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \le \frac{1}{\Gamma(\alpha)} e^{-x} = \frac{\alpha+e}{\alpha e \Gamma(\alpha)} g(x).$$

From the two cases, $f(x) \le C g(x)$ is satisfied with $C = \dfrac{\alpha+e}{\alpha e \Gamma(\alpha)} \quad \left(= \dfrac{\alpha+e}{e\Gamma(\alpha+1)} \right)$.

(e) If $\alpha > 1$, we separate α into the largest integer and remainder parts:

$$\alpha = k + \alpha_0 \quad \text{where } k \text{ is the integer part of } \alpha \text{ and } 0 \le \alpha_0 < 1.$$

Then, from the 2020 exam, we can separate $\text{Gamma}(\alpha, 1)$ by

$$\text{Gamma}(\alpha, 1) \sim \text{Gamma}(k, 1) + \text{Gamma}(\alpha_0, 1).$$

We can sample $\text{Gamma}(k, 1)$ from the 2020 exam problem and sample $\text{Gamma}(\alpha_0, 1)$ from above.

9. **[2019, Simulation of multidimensional normal RVs]** Suppose that $\boldsymbol{S}_t$ is a column vector of three asset prices at time t and that $\boldsymbol{S}_T$ is distributed as

$$\boldsymbol{S}_T - \boldsymbol{S}_0 = \boldsymbol{L}\,\boldsymbol{Z},$$

where $\boldsymbol{Z}$ is a standard normal RV (column) vector of size 3 and $\boldsymbol{L}$ is given by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

(Hint: L is the lower triangular matrix in Cholesky decomposition.)

(a) Assuming that $T = 5$, what is the normal volatility of each asset?

(b) What is the correlation between the 2nd and 3rd asset?

(c) What is the price of the at-the-money basket call option based on the three assets with equal weight (i.e, 1/3 each)? Assume that the at-the-money option price under the normal volatility σ_N is $0.4\,\sigma_\text{N}\sqrt{T}$.

Solution: The covariance of the price change is

$$\text{Cov}(\boldsymbol{S}_T - \boldsymbol{S}_0) = \boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T = \begin{pmatrix} 1 & -3 & -2 \\ -3 & 25 & 10 \\ -2 & 10 & 9 \end{pmatrix}$$

(a) The diagonal elements are the variances of assets:

$$1 = \sigma_1^2 T, \quad 25 = \sigma_2^2 T, \quad 9 = \sigma_3^2 T.$$

Therefore, the normal volatilities of the assets are

$$\sigma_1 = \sqrt{1/5}, \quad \sigma_2 = \sqrt{5}, \quad \text{and} \quad \sigma_3 = \sqrt{9/5} = 3/\sqrt{5}.$$

(b) $10/(\sqrt{25}\,\sqrt{9}) = 2/3 \approx 66.6\%$.

(c) From

$$\sigma_{\rm N}^2 T = \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} = 5 \quad \text{for} \quad \mathbf{w} = [1/3, 1/3, 1/3]^T,$$

the basket option price is $0.4\sqrt{5}$.

10. **[2018, Simulation of BM path]** Exotic derivatives often depend on the 'path' of the underlying stock price. Assume that we need to generate the MC paths of standard BM W_t at $t = 1, 3, 5$, and 9. We are going to generate the paths using two approaches, which are eventually same. Assume z_k, for $k = 1, \cdots, 4$ are independent standard normal RV.

    (a) Using the incremental property of BM, i.e., $W_t - W_s \sim N(0, t-s)$, generate RNs for W_1, $W_3 - W_1$, $W_5 - W_3$, and $W_9 - W_5$, using z_k's. Finally, how can you generate RNs for W_1, W_3, W_5, and W_9?

    (b) Now we use covariance matrix approach: Let $\mathbf{\Sigma}$ be the covariance matrix of correlated multivariate normal variables and $\boldsymbol{L}$ (lower-triangular matrix) be its Cholesky decomposition, which satisfy $\mathbf{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$. Then, the simulation of the normal variables can obtained as $\boldsymbol{L}\boldsymbol{z}$, where $\boldsymbol{z}$ is the vector of independent standard normal RVs. What is the covariance matrix $\mathbf{\Sigma}$ for our case? (Hint: you may use $\mathrm{Cov}(W_s, W_t) = \min(t, s)$ without proof.)

    (c) From (a) and (b), what is the Cholesky decomposition $\boldsymbol{L}$? Verify that $\mathbf{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$ by direct computation.

Solution:

(a)

$$\begin{aligned} W_1 &= z_1, \\ W_3 - W_1 &= \sqrt{2} z_2 \\ W_5 - W_3 &= \sqrt{2} z_3 \\ W_9 - W_5 &= 2 z_4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} W_1 &= z_1, \\ W_3 &= z_1 + \sqrt{2} z_2 \\ W_5 &= z_1 + \sqrt{2} z_2 + \sqrt{2} z_3 \\ W_9 &= z_1 + \sqrt{2} z_2 + \sqrt{2} z_3 + 2 z_4 \end{aligned}$$

(b)

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix}$$

(c)

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 2 \end{pmatrix}.$$

$$\boldsymbol{L}\boldsymbol{L}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix} = \boldsymbol{\Sigma}$$

11. **[2020, Simulation of correlated normal RVs]** The tri-variate normal variable $\boldsymbol{X}$ has the following mean and covariance. How can you simulate RNs for X?

$$\mu = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 16 & 4 \\ -2 & 4 & 9 \end{pmatrix}$$

Solution: First, we obtain the Cholesky decomposition of Σ. We find a lower triangular matrix L such that $LL^T = \Sigma$. After some algebra, we get

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

Therefore, X is simulated by $X = \mu + LZ$ where Z is the independent standard normal RVs of size 3.

12. **[2018, Spread/switch option]** Compute the price of the call option on the spread between two stocks. The payout at maturity T is given as

$$\text{Payout} = \max(S_1(T) - S_2(T),\, 0).$$

Assume that $S_1(0) = S_2(0) = 100$, $r = q = 0$, $\sigma_1 = 20\%$, $\sigma_2 = 10\%$, and $T = 1$ year. Also assume that the BMs driving the two stocks are correlated by 89%. You may use the following values for $N(z)$.

z	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
$N(z)$	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564

Solution: We use Margrabe's formula:

$$C = S_1(0)N(d_1) - S_2(0)N(d_2),$$

$$\text{where} \quad d_{1,2} = \frac{\log(S_1(0)/S_2(0))}{\sigma_R\sqrt{T}} \pm \frac{1}{2}\sigma_R\sqrt{T} \quad \text{and} \quad \sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2},$$

we get

$$\sigma_R = \frac{1}{100}\sqrt{400 + 100 - 2 \times 0.89 \times 200} = 12\%,$$
$$d_1 = \frac{\sigma_{\mathrm{R}}}{2} = 0.06, \quad d_2 = -0.06,$$
$$C = S_0 N(d_1) - K N(d_2) = 100 N(0.06) + 100(1 - N(0.06)) = 4.8$$

13. **[2017, Bessel process]** The distribution of the following RV,

$$Q = \|(Z_1 + \mu_1, \cdots, Z_n + \mu_n)\|^2 = (Z_1 + \mu_1)^2 + \cdots + (Z_n + \mu_n)^2,$$
$$\text{with} \quad \mu = \mu_1^2 + \cdots \mu_n^2$$

where $Z_1, \cdots, Z_n$ are independent standard normal RVs, is defined as *non-central chi square* (χ^2, pronounced as *kai*) distribution with degree n and non-centrality parameter $\mu \geq 0$, denoted by $Q \sim \chi^2(n, \mu)$. Thanks to radial symmetry, the distribution is completely determined by $\mu = \mu_1^2 + \cdots + \mu_n^2$. The χ^2 distribution is an important subject of statistics, so the PDF and CDF is well-known although the computation is still challenging from some cases. The degree n can be generalized to any positive real number (i.e., not only integers).

On the other hand, the **squared** Bessel process with dimension n is defined as

$$X_t = \|(B_{1t}, \cdots, B_{nt})\|^2 = B_{1t}^2 + \cdots + B_{nt}^2$$

where $B_{1t}, \cdots, B_{nt}$ are n independent standard BMs. Therefore the distribution of X_t given X_s ($s < t$) follows a scaled non-central χ^2 distribution,

$$X_t = (t - s)\, Q \quad \text{where} \quad Q \sim \chi^2\left(n,\, \frac{X_s}{t - s}\right)$$

(No need to prove this for the remaining questions. Just use it.)

(a) Show that the **squared** Bessel process satisfies

$$dX_t = 2\sqrt{X_t}\, dW_t + n\, dt$$

(b) Show that the Bessel process defined as $R_t = \sqrt{X_t}$ satisfies

$$dR_t = dW_t + \frac{n-1}{2}\frac{dt}{R_t}.$$

(c) The SDE for the CEV process for $0 < \beta \leq 1$ is given as

$$dS_t = \sigma\, S_t^\beta\, dW_t.$$

Show that the CEV process can be reduced to the Bessel process defined in (b). Express the distribution of S_t in terms of S_0 and $Q \sim \chi^2(n, \mu)$. Clearly state the corresponding values for n and μ? (If σ makes the problem difficult for you, you may assume $\sigma = 1$ to solve the problem. But you will get a partial credit.)

Solution:

(a) Taking derivative on X_t, we get

$$dX_t = \sum_{k=1}^{n} \left(2B_{kt}\, dB_{kt} + \frac{1}{2} \cdot 2dt \right) = 2\sqrt{X_t}\, dW_t + n\, dt,$$

where we use $\sum_k B_{kt}\, dB_{kt} = \sqrt{\sum_k B_{kt}^2}\, dW_t = \sqrt{X_t}\, dW_t$ for an independent standard BM W_t.

(b) Applying Itô's lemma,

$$dR_t = \frac{d\,X_t}{2\sqrt{X_t}} - \frac{(dX_t)^2}{8X_t\sqrt{X_t}} = \frac{2R_t\, dW_t + n\, dt}{2R_t} - \frac{(2R_t\, dW_t)^2}{8R_t^3} = dW_t + \frac{n-1}{2}\frac{dt}{R_t}.$$

It also imply that the distribution of R_t given R_s $(s < t)$ follows

$$R_t = \sqrt{(t-s)\, Q} \quad \text{where} \quad Q \sim \chi^2 \left(n,\ \frac{R_s^2}{t-s} \right)$$

(c) We apply Itô's lemma to $Y_t = S_t^{1-\beta}/(1-\beta)$:

$$dY_t = S_t^{-\beta} dS_t + \frac{1}{2}(-\beta S_t^{-1-\beta})(dS_t)^2 = \sigma dW_t - \frac{\beta\,\sigma^2}{2(1-\beta)}\frac{dt}{Y_t}.$$

The σ can be absorbed to t by introducing the variance $\tau = \sigma^2 t$,

$$dY_{\tau/\sigma^2} = dW_\tau - \frac{\beta}{2(1-\beta)}\frac{d\tau}{Y_{\tau/\sigma^2}}$$

Therefore $Y_{\tau/\sigma^2}/\tau$ follows χ^2 distribution with $\mu = Y_0^2/\tau$ and

$$n = \frac{1-2\beta}{1-\beta} \quad \text{from} \quad \frac{n-1}{2} = -\frac{\beta}{2(1-\beta)}:$$

$$Y_{\tau/\sigma^2} = \sqrt{\tau Q} \quad \text{where} \quad Q \sim \chi^2 \left(\frac{1-2\beta}{1-\beta},\ \frac{S_0^{2(1-\beta)}}{(1-\beta)^2\,\sigma^2\, t} \right).$$

Finally, replacing $\tau = \sigma^2 t$ and $Y_t = S_t^{1-\beta}/(1-\beta)$,

$$\frac{S_t^{1-\beta}}{(1-\beta)} = \sigma\sqrt{t\, Q} \quad \text{or} \quad S_t = \left((1-\beta)^2\sigma^2 t\, Q \right)^{\frac{1}{2(1-\beta)}}$$

Alternatively, the result of the Itô's lemma can be expressed as below by dividing σ:

$$d(Y_t/\sigma) = dW_t - \frac{\beta\,\sigma^2}{2(1-\beta)}\frac{dt}{Y_t/\sigma},$$

which leads to the same answer:

$$\frac{Y_t}{\sigma} = \frac{S_t^{1-\beta}}{\sigma(1-\beta)} = \sqrt{t\, Q}$$

14. **[2017, CIR process]** The Cox–Ingersoll–Ross (1985, CIR) process given as

$$dX_t = a(X_\infty - X_t)dt + \sigma\sqrt{X_t}\, dB_t$$

was originally proposed to model the dynamics of interest rate by Cox, Ingersoll, and Ross. The process was also used to model the variance v_t in the Heston stochastic volatility model:

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dZ_t.$$

Applying the similar change of variable used in Ornstein–Uhlenbeck (OU) process, show that the CIR process (either in X_t or v_t) can be represented in terms of the **squared** Bessel process in the 2017 question above. Clearly state the corresponding dimension n of the squared Bessel process.

> **Solution:** We apply the change of variable, $Y_t = e^{at}X_t$, from the OU process. Then, Y_t satisfy
>
> $$dY_\tau = aX_\infty e^{at}\, dt + \sqrt{X_t}\,\sigma e^{at}\, dB_t = aX_\infty e^{at}\, dt + 2\sqrt{Y_t}\,\frac{\sigma e^{at/2}}{2}\, dB_t.$$
>
> Now we also introduce a new time variable from the variance of the BM,
>
> $$\tau = \int_0^t \left(\frac{\sigma e^{at/2}}{2}\right)^2 ds = \frac{\sigma^2}{4a}(e^{at} - 1), \quad d\tau = \frac{\sigma^2 e^{at}}{4} dt$$
>
> Define $\bar{Y}_\tau = Y_t$, then the process $\bar{Y}_\tau$ follows
>
> $$d\bar{Y}_\tau = \frac{4aX_\infty}{\sigma^2}\, d\tau + 2\sqrt{Y_t}\, dB_\tau,$$
>
> which is the squared Bessel process with dimension $n = 4aX_\infty/\sigma^2$. Finally the original process X_t can be expressed in terms of the **squared** Bessel process $\bar{Y}_\tau$ with dimension $n = 4aX_\infty/\sigma^2$:
>
> $$X_t = e^{-at}\, \bar{Y}_{\sigma^2(e^{at}-1)/(4a)}.$$

15. **[2018, Euler/Milstein scheme of CIR process]** In the Heston stochastic volatility model, the stochastic variance $v_t = \sigma_t^2$ follows the SDE:

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}\, dZ_t.$$

We want to MC simulate v_T for some T by discretizing time as $t_k = k\Delta t$ with $k = 1, \ldots, N$ and $\Delta t = T/N$.

(a) Write the formula to compute $v_{t+\Delta t}$ from v_t. Assume Z is a standard normal RV.

(b) Instead of simulating V_t, we may consider simulating $\sigma_t = \sqrt{v_t}$. Using Itô's lemma, drive the SDE for σ_t.

(c) From the result of (b), write the formula to update $\sigma_{t+\Delta t}$ from σ_t. After replacing σ_t^2 with v_t, compare the answer to the result from (a). Are they same?

Solution:

(a)

$$v_{t+\Delta t} = v_t + \kappa(\theta - v_t)\Delta t + \nu\sqrt{v_t \Delta t}\, Z$$

(b) Applying Itô's lemma, we get

$$\begin{aligned} d\sigma_t = d\sqrt{v_t} &= \frac{d\, v_t}{2\sigma_t} - \frac{(dv_t)^2}{8\sigma_t^3} \\ &= \frac{\kappa(\theta - \sigma_t^2)dt}{2\sigma_t} + \frac{\nu}{2}dZ_t - \frac{\nu^2 dt}{8\sigma_t} \\ &= \frac{4\kappa(\theta - \sigma_t^2) - \nu^2}{8\sigma_t}dt + \frac{\nu}{2}dZ_t. \end{aligned}$$

(c) The discretization rule for σ_t is given as

$$\sigma_{t+\Delta t} = \sigma_t + \frac{4\kappa(\theta - \sigma_t^2) - \nu^2}{8\sigma(t_t)}\Delta t + \frac{\nu}{2}\sqrt{\Delta t}\, Z.$$

By taking the square of both sides,

$$\begin{aligned} v_{t+\Delta t} = \sigma_{t+\Delta t}^2 &= \left(\sigma_t + \frac{4\kappa(\theta - \sigma_t^2) - \nu^2}{8\sigma_t}\Delta t + \frac{\nu}{2}\sqrt{\Delta t}\, Z\right)^2 \\ &= v_t + \frac{4\kappa(\theta - v_t) - \nu^2}{4}\Delta t + \frac{\nu^2}{4}\Delta t\, Z^2 + \nu\sqrt{v_t\Delta t}\, Z + o(\Delta t) \\ &= v_t + \kappa(\theta - v_t)\Delta t + \nu\sqrt{v_t\Delta t}\, Z + \frac{\nu^2}{4}\Delta t\,(Z^2 - 1), \end{aligned}$$

where $o(\Delta t)$ is the terms smaller than Δt in order.

This result is differ from (a) by the two terms in red above. Even after ignoring $o(\Delta t)$, the term $\nu^2\Delta t\,(Z^2-1)/4$ remains. So the two discretization methods are different. The discretization method we applied to v_t and σ_t (that we learned from class) is called Euler-Maruyama method (WIKIPEDIA). The discretization for v_t derived via σ_t is called Milstein method (WIKIPEDIA). If we apply Milstein method to v_t, we directly get the same result. Milstein method is known to be more accurate than Euler-Maruyama method.

16. **[2022, Andersen's Heston model simulation]** The variance under the Heston model is given by the CIR process:

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}\, dZ_t.$$

It is well known that the Euler/Milstein scheme (see 2018 question) for v_t is not accurate because v_t becomes negative easily. To handle this problem, Andersen (2008) proposed an advanced scheme. First, let m and s^2 be the mean and variance of $v_{t+\Delta t}$ given v_t, and their ratio be $\psi = s^2/m^2$. We know the analytic solution for m and s^2 (Andersen, 2008, (17)–(18)):

$$
\begin{aligned}
m &= E(v_{t+\Delta t}|v_t) = \theta + (v_t - \theta)e^{-\kappa\Delta t} \\
s^2 &= \mathrm{Var}(v_{t+\Delta t}|v_t) = \frac{\nu^2}{\kappa}(1 - e^{-\kappa\Delta t})\left[v_t e^{-\kappa\Delta t} + \frac{\theta}{2}(1 - e^{-\kappa\Delta t})\right].
\end{aligned}
\tag{3}
$$

High ψ implies high probability of v_t hitting zero. Therefore, when $\psi \geq 1$, Andersen proposed to approximate $v_{t+\Delta t}$ by

$$
v_{t+\Delta t} \approx \begin{cases} 0 & \text{with probability } p \\ \mathrm{Exp}(\lambda) & \text{with probability } 1-p, \end{cases}
$$

where $\mathrm{Exp}(\lambda)$ is the exponential RV with rate λ. We are going to determine the two parameters, p and λ.

(a) How can you simulate $v_{t+\Delta t}$ using only one uniform RV U? Give detail.

(b) What are $E(v_{t+\Delta t})$ and $\mathrm{Var}(v_{t+\Delta t})$ of Andersen's approximation? (Hint: the mean and variance of $\mathrm{Exp}(\lambda)$ are $1/\lambda$ and $1/\lambda^2$, respectively.)

(c) Determine p and λ by matching the result of (b) to m and s^2. Why do you need to assume $\psi \geq 1$. (Express your answers with m, s^2, and $\psi = s^2/m^2$. No need to use Eq. (3).)

Solution:

(a) We use $U \leq p$ to determine the probability p. If $U > p$, the re-scaled value $\frac{U-p}{1-p}$ (or $\frac{1-U}{1-p}$) is also a RV, so we use this to sample $\mathrm{Exp}(\lambda)$.

$$
v_{t+\Delta t} \approx \begin{cases} 0 & \text{if} \quad 0 \leq U \leq p \\ \dfrac{1}{\lambda}\log\left(\dfrac{1-p}{U-p}\right), & \text{if} \quad p < U \leq 1. \end{cases}
$$

See Andersen (2008, Eq. (25)).

(b) The two moments are

$$
\begin{aligned}
E(v_{t+\Delta t}) &= 0 \cdot p + \frac{1}{\lambda}(1-p) = \frac{1-p}{\lambda} \\
E(v_{t+\Delta t}^2) &= 0 \cdot p + E\left(\mathrm{Exp}(\lambda)^2\right)(1-p) = \left(E(\mathrm{Exp}(\lambda))^2 + \mathrm{Var}(\lambda)\right)(1-p) = \frac{2(1-p)}{\lambda^2}.
\end{aligned}
$$

Therefore,

$$
\mathrm{Var}(v_{t+\Delta t}) = E(v_{t+\Delta t}^2) - E(v_{t+\Delta t})^2 = \frac{2(1-p)}{\lambda^2} - \frac{(1-p)^2}{\lambda^2} = \frac{1-p^2}{\lambda^2}
$$

(c) Matching,

$$
\frac{1-p}{\lambda} = m \quad \text{and} \quad \frac{1-p^2}{\lambda^2} = s^2,
$$

leads to

$$
\frac{1-p}{\lambda} = m \quad \text{and} \quad \frac{1+p}{\lambda} = \frac{s^2}{m}.
$$

We solve

$$p = \frac{\psi - 1}{\psi + 1} \quad \text{and} \quad \lambda = \frac{1-p}{m} = \frac{2}{m(\psi+1)}.$$

We need $\psi \geq 1$ in order for the probability p to be positive. See Andersen (2008, § 3.2.2).

17. **[2019, Euler/Milstein Schemes of GARCH model]** The variance process for the GARCH diffusion model is given by

$$dv_t = \kappa(\theta - v_t)dt + \nu v_t\, dZ_t$$

and you want to simulate v_t using time-discretization scheme.

(a) What is the Euler and Milstein schemes for v_t? Explicitly write down the expression for $v_{t+\Delta t} - v_t$ using standard normal RV Z_1.

(b) The SDE for v_t tells us that v_t cannot go negative. However, in the MC simulation with the time-discretization scheme, v_t sometimes go negative. To avoid this problem, it is better simulate $w_t = \log v_t$ instead. Derive the SDE for w_t.

(c) What is the Euler and Milstein schemes for w_t?

Solution:

(a) The Euler and Milstein schemes for v_t is given by

$$v_{t+\Delta t} - v_t = \kappa(\theta - v_t)\Delta t + \nu v_t Z_1 \sqrt{\Delta t} + \boxed{\frac{\nu^2}{2} v_t (Z_1^2 - 1)\Delta t}\,,$$

where the boxed term is only for the Milstein scheme.

(b) Applying Itô's lemma, we obtain

$$\begin{aligned} dw_t &= \frac{dv_t}{v_t} - \frac{1}{2}\frac{(dv_t)^2}{v_t^2} = \kappa\left(\frac{\theta}{v_t} - 1\right) dt + \nu dZ_t - \frac{\nu^2}{2} dt \\ &= \left(\kappa\theta e^{-w_t} - \kappa - \nu^2/2\right) dt + \nu dZ_t. \end{aligned}$$

(c) The Euler and Milstein scheme is same for w_t and they are given by

$$w_{t+\Delta t} - w_t = \left(\kappa\theta e^{-w_t} - \kappa - \nu^2/2\right) \Delta t + \nu Z_1 \sqrt{\Delta t}.$$

So it is better to simulate w_t first and obtain $v_t = e^{w_t}$, which is always positive. Also note that the Milstein scheme for v_t in (a) can be recovered by the Taylor expansion of e^x:

$$\begin{aligned} v_{t+\Delta t} &= v_t\ \exp\left(w_{t+\Delta t} - w_t\right) \\ &= v_t\left(1 + \left(\kappa\theta e^{-w_t} - \kappa - \frac{\nu^2}{2}\right)\Delta t + \nu Z_1\sqrt{\Delta t} + \frac{\nu^2}{2} Z_1^2 \Delta t + o(\Delta t)\right) \\ v_{t+\Delta t} - v_t &= \kappa(\theta - v_t)\Delta t + \nu v_t Z_1 \sqrt{\Delta t} + \frac{\nu^2}{2} v_t (Z_1^2 - 1)\Delta t\,, \end{aligned}$$

18. **[2020, Euler/Milstein Schemes of CEV Model]** The stochastic differential equation for the constant-elasticity-of-variance (CEV) model is given by

$$dS_t = \sigma S_t^\beta dW_t \quad (0 \le \beta \le 1).$$

Find the Euler and Milstein schemes for obtaining $S_{t+\Delta t}$ from S_t.

Solution: For a standard normal RV, W_1, the Milstein scheme for the CEV model is given by

$$\begin{aligned} S_{t+\Delta t} &= S_t + \sigma S_t^\beta W_1 \sqrt{\Delta t} + \frac{\sigma S_t^\beta \cdot \sigma \beta S_t^{\beta-1}}{2}(W_1^2 - 1)\Delta t \\ &= S_t + \sigma S_t^\beta W_1 \sqrt{\Delta t} + \frac{1}{2}\sigma^2 \beta S_t^{2\beta-1}(W_1^2 - 1)\Delta t. \end{aligned}$$

19. **[2019, Conditional MC Simulation of OUSV Model]** We are going to formulate the conditional MC simulation for the Ornstein–Uhlenbeck stochastic volatility (OUSV) model. The processes for the price and volatility under the OUSV model are respectively given by

$$\begin{aligned} \frac{dS_t}{S_t} &= \sigma_t dW_t = \sigma_t(\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2}, \\ d\sigma_t &= \kappa(\theta - \sigma_t)dt + \nu\, dZ_t, \end{aligned}$$

where X_t and Z_t are independent standard BMs.

(a) Derive the SDE for σ_t^2.

(b) Based on the answer of (a), express S_T in terms of (σ_T, U_T, V_T) where U_T and V_T are give by

$$U_T = \int_0^T \sigma_t\, dt \quad \text{and} \qquad V_T = \int_0^T \sigma_t^2\, dt.$$

What are $E(S_T)$ and the BS volatility of S_T conditional on the triplet (σ_T, U_T, V_T)?

Solution:

(a) Using Itô's lemma,

$$d\sigma_t^2 = (\nu^2 + 2\kappa(\theta\sigma_t - \sigma_t^2))dt + 2\nu\sigma_t dZ_t.$$

(b) Integrating the result of (a),

$$\sigma_t^2 - \sigma_0^2 = \nu^2 T + 2\kappa(\theta U_T - V_T) + 2\nu \int_0^T \sigma_t dZ_t.$$

Therefore,

$$\begin{aligned} \log\left(\frac{S_T}{S_0}\right) &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t - \frac{1}{2}V_T \\ &= \frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{1}{2}\right)V_T + \rho_* \sqrt{V_T}\, X_1 \end{aligned}$$

and we obtain

$$\begin{aligned}
E(S_T) &= S_0 \exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2}V_T\right) \\
&= S_0 \exp\left(\frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{\rho^2}{2}\right)V_T\right) \\
\text{Vol}(S_T) &= \rho_*\sqrt{V_T/T}.
\end{aligned}$$

Reference: Li and Wu (2019)

20. **[2020, Conditional MC Simulation of Garch Model]** We are going to formulate the conditional MC simulation for the GARCH diffusion model. The SDEs for the price and volatility under the GARCH diffusion model are given by

$$\frac{dS_t}{S_t} = \sqrt{v_t}(\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2},$$
$$dv_t = \kappa(\theta - v_t)dt + \nu v_t\, dZ_t$$

where X_t and Z_t are independent standard BMs.

(a) Derive the SDE for $\sigma_t = \sqrt{v_t}$.

(b) Based on the answer of (a), express S_T in terms of σ_T, Y_T, U_T, V_T, and a standard normal RV X_1, where Y_T, U_T and V_T are given by

$$Y_T = \int_0^T \frac{1}{\sigma_t}\,dt, \quad U_T = \int_0^T \sigma_t\,dt \quad \text{and} \quad V_T = \int_0^T \sigma_t^2\,dt.$$

(c) What are $E(S_T)$ and the BS volatility of S_T conditional on the quadruplet $(\sigma_T, Y_T, U_T, V_T)$?

Solution:

(a) Using Itô's lemma,

$$\begin{aligned}
d\sigma_t = d\sqrt{v_t} &= \frac{1}{2}\frac{dv_t}{\sqrt{v_t}} - \frac{1}{8}\frac{(dv_t)^2}{v_t\sqrt{v_t}} \\
&= \frac{1}{2}\kappa\left(\frac{\theta}{\sigma_t} - \sigma_t\right)dt + \frac{\nu}{2}\sigma_t dZ_t - \frac{\nu^2}{8}\sigma_t dt \\
&= \frac{1}{2}\left(\frac{\kappa\theta}{\sigma_t} - \left(\kappa + \frac{\nu^2}{4}\right)\sigma_t\right)dt + \frac{\nu}{2}\sigma_t dZ_t
\end{aligned}$$

(b) Integrating the result of (a),

$$\sigma_T - \sigma_0 = \frac{1}{2}\left(\kappa\theta\, Y_T - \left(\kappa + \frac{\nu^2}{4}\right)U_T\right) + \frac{\nu}{2}\int_0^T \sigma_t dZ_t$$
$$\int_0^T \sigma_t dZ_t = \frac{2}{\nu}(\sigma_T - \sigma_0) - \left(\frac{\kappa\theta}{\nu}\,Y_T - \left(\frac{\kappa}{\nu} + \frac{\nu}{4}\right)U_T\right)$$

Therefore,

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \rho\int_0^T \sigma_t dZ_t + \rho_*\int_0^T \sigma_t dX_t - \frac{1}{2}V_T\\ &= \frac{2\rho}{\nu}(\sigma_T-\sigma_0) - \frac{\rho\kappa\theta}{\nu}\,Y_T + \rho\left(\frac{\kappa}{\nu}+\frac{\nu}{4}\right)U_T - \frac{1}{2}V_T + \rho_*\sqrt{V_T}\,X_1.\end{aligned}$$

(c) Accordingly, we obtain

$$\begin{aligned}E(S_T|\sigma_T, Y_T, U_T, V_T) &= S_0\,\exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2}V_T\right)\\ &= S_0 \exp\left(\frac{2\rho}{\nu}(\sigma_T-\sigma_0) - \frac{\rho\kappa\theta}{\nu}\,Y_T + \rho\left(\frac{\kappa}{\nu}+\frac{\nu}{4}\right)U_T - \frac{\rho^2}{2}V_T\right)\\ \sigma_{\text{BS}} &= \rho_*\sqrt{V_T/T}.\end{aligned}$$

21. **[2021, Hull–White SV Model Simulation]** Suppose that an SV model is given by

$$\frac{dS_t}{S_t} = \sqrt{v_t}(\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2},$$
$$\frac{dv_t}{v_t} = \kappa dt + \nu\, dZ_t$$

where X_t and Z_t are independent standard BMs. We are going to formulate the conditional Monte Carlo simulation for this SV model. (Notice that this SV model is different from the SABR model because (i) κdt term exists (ii) $v_t = \sigma_t^2$ is used for the SDE. But what you learned from the SABR would be still useful.)

(a) Solve v_T (i.e., express v_T as a function of v_0, Z_T, and the model parameters). Hint: v_t follows a geometric BM.

(b) From (a), how can you simulate $v_{t+\Delta t}$ from v_t?

(c) Derive the SDE for $\sigma_t = \sqrt{v_t}$. Hint: consider $\log\sigma_t = \frac{1}{2}\log v_t$.

(d) Using the result of (c), express S_T in terms of v_T, V_T, and U_T, and a standard normal RV X_1, where V_T and U_T are respectively the integrated variance and volatility,

$$V_T = \int_0^T v_t\,dt \quad \text{and} \quad U_T = \int_0^T \sigma_t\,dt.$$

(e) What are $E(S_T|v_T, V_T, U_T)$ and the equivalent BS volatility of S_T conditional on v_T, V_T, and U_T?

Solution: The SV model in this problem is from Hull and White (1987). Although it is not popular these days, it was one of the first SV models proposed.

(a) Using Itô's lemma,

$$d\log v_t = \left(\kappa - \frac{\nu^2}{2}\right)dt + \nu dZ_t$$
$$v_T = v_0 \exp\left(\left(\kappa - \frac{\nu^2}{2}\right)T + \nu Z_T\right)$$

(b) $v_{t+\Delta t}$ is obtained from v_t by

$$v_{t+\Delta t} = v_t \exp\left(\left(\kappa - \frac{\nu^2}{2}\right)\Delta t + \nu\sqrt{\Delta t} Z\right),$$

where Z is a standard normal RN.

(c) The SDE for σ_t is derived as

$$d\log\sigma_t = \frac{1}{2}d\log v_t = \left(\frac{\kappa}{2} - \frac{\nu^2}{4}\right)dt + \frac{\nu}{2}dZ_t$$
$$\frac{d\sigma_t}{\sigma_t} = \left(\frac{\kappa}{2} - \frac{\nu^2}{4} + \frac{1}{2}\left(\frac{\nu}{2}\right)^2\right)dt + \frac{\nu}{2}dZ_t = \left(\frac{\kappa}{2} - \frac{\nu^2}{8}\right)dt + \frac{\nu}{2}dZ_t.$$

(d) Integrating the result of (c),

$$\sigma_T - \sigma_0 = \left(\frac{\kappa}{2} - \frac{\nu^2}{8}\right)\int_0^T \sigma_t dt + \frac{\nu}{2}\int_0^T \sigma_t dZ_t$$
$$\int_0^T \sigma_t dZ_t = \frac{2}{\nu}(\sqrt{v_T} - \sqrt{v_0}) + \left(\frac{\nu}{4} - \frac{\kappa}{\nu}\right)U_T$$

Therefore,

$$\log\left(\frac{S_T}{S_0}\right) = \rho\int_0^T \sigma_t dZ_t + \rho_*\int_0^T \sigma_t dX_t - \frac{1}{2}V_T$$
$$= \frac{2\rho}{\nu}(\sigma_T - \sigma_0) + \rho\left(\frac{\nu}{4} - \frac{\kappa}{\nu}\right)U_T - \frac{1}{2}V_T + \rho_*\sqrt{V_T}\,X_1.$$

(e) Accordingly, we obtain the equivalent spot and volatility as

$$E(S_T|\sigma_T, V_T, U_T) = S_0\exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2}V_T\right)$$
$$= S_0\exp\left(\frac{2\rho}{\nu}(\sqrt{v_T} - \sqrt{v_0}) + \rho\left(\frac{\nu}{4} - \frac{\kappa}{\nu}\right)U_T - \frac{\rho^2}{2}V_T\right)$$
$$\sigma_{\rm BS} = \rho_*\sqrt{V_T/T}.$$

22. **[2022, λ–SABR Model Simulation]** The λ–SABR model is given by

$$\frac{dS_t}{S_t^\beta} = \sigma_t(\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2},$$
$$d\sigma_t = \lambda(\theta - \sigma_t)dt + \nu\sigma_t\, dZ_t,$$

where X_t and Z_t are independent standard BMs. This model is an extension of the SABR model because of the additional $\lambda(\theta - \sigma_t)dt$ term. We are going to formulate the conditional Monte Carlo simulation for this model.

(a) How can you simulate $\sigma_{t+\Delta t}$ from σ_t? Write the Euler and Milstein schemes for σ_t. (Hint: the SDE for σ_t is same as the SDE for v_t in the GARCH diffusion model.)

(b) When $\beta = 0$, express S_T in terms of σ_T, V_T, U_T, and a standard normal RV X_1, where V_T and U_T are respectively the integrated variance and volatility,

$$V_T = \int_0^T \sigma_t^2\, dt \quad \text{and} \quad U_T = \int_0^T \sigma_t\, dt.$$

(c) When $\beta = 1$, express S_T in terms of σ_T, V_T, U_T, and X_1. What are $E(S_T|\sigma_T, V_T, U_T)$ and the equivalent BS volatility σ_{BS} conditional on σ_T, V_T, and U_T?

Solution: The λ–SABR model was proposed by Henry-Labordère (2005).

(a) The Euler/Milstein schemes are given by

$$\sigma_{t+\Delta t} = \sigma_t + \lambda(\theta - \sigma_t)\Delta t + \nu\sigma_t\sqrt{\Delta t}\, Z + \frac{\nu^2}{2}\sigma_t \Delta t\, (Z^2 - 1),$$

where Z is standard normal RN. The term in red is for Milstein scheme. The schemes are same as those for v_t in the GARCH diffusion model. See the 2019 question.

(b) By integration,

$$\sigma_T - \sigma_0 = \lambda(\theta T - U_T) + \nu \int_0^T \sigma_t dZ_t$$
$$\int_0^T \sigma_t dZ_t = \frac{1}{\nu}\left(\sigma_T - \sigma_0 - \lambda(\theta T - U_T)\right).$$

Therefore,

$$\begin{aligned} S_T - S_0 &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t \\ &= \frac{\rho}{\nu}\left(\sigma_T - \sigma_0 - \lambda(\theta T - U_T)\right) + \rho_* \sqrt{V_T} X_1. \end{aligned}$$

(c) When $\beta = 1$, we use

$$d \log S_t = \sigma_t(\rho dZ_t + \rho_* dX_t) - \frac{\sigma_t^2}{2} dt.$$

Integrating the equation,

$$\begin{aligned} \log(S_T/S_0) &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t - \int_0^T \frac{\sigma_t^2}{2} dt \\ &= \frac{1}{\nu}\left(\sigma_T - \sigma_0 - \lambda(\theta T - U_T)\right) - \frac{V_T}{2} + \sqrt{V_T} X_1. \end{aligned}$$

Accordingly, we obtain the conditional spot and volatility as

$$\begin{aligned} E(S_T|\sigma_T, V_T, U_T) &= S_0 \exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2} V_T \right) \\ &= S_0 \exp\left(\frac{1}{\nu}\left(\sigma_T - \sigma_0 - \lambda(\theta T - U_T)\right) - \frac{\rho^2}{2} V_T \right) \\ \sigma_{\text{BS}} &= \rho_* \sqrt{V_T/T}. \end{aligned}$$

References

Andersen L (2008) Simple and efficient simulation of the Heston stochastic volatility model. The Journal of Computational Finance 11(3):1–42, doi:10.21314/JCF.2008.189

Cox JC, Ingersoll JE Jr, Ross SA (1985) A Theory of the Term Structure of Interest Rates. Econometrica 53(2):385–407, doi:10.2307/1911242

Henry-Labordère P (2005) A General Asymptotic Implied Volatility for Stochastic Volatility Models. arXiv:cond-mat/0504317 URL `http://arxiv.org/abs/cond-mat/0504317`, `cond-mat/0504317`

Hull J, White A (1987) The Pricing of Options on Assets with Stochastic Volatilities. The Journal of Finance 42(2):281–300, doi:10.1111/j.1540-6261.1987.tb02568.x

Li C, Wu L (2019) Exact simulation of the Ornstein–Uhlenbeck driven stochastic volatility model. European Journal of Operational Research 275(2):768–779, doi:10.1016/j.ejor.2018.11.057