Applied Stochastic Processes (FIN 514) Midterm Exam

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BM stands for Brownian motion. Assume that B_t, W_t, X_t and Z_t are standard BMs unless stated otherwise. **RN** and **RV** stand for random number and random variable, respectively. $P(A)$ denotes the probability of the event A.

1. (6 points) We are going to derive Simpson's rule for numerical integration. We need to integral $f(x)$ from -1 to $+1$ and we know the values of $f(x)$ at $x = -1$, 0, and $+1$:

$$f(-1) = A, \quad f(0) = B, \quad f(+1) = C.$$

We are going to find the quadratic approximation of $f(x)$:

$$g(x) = Ax^2 + Bx + C \quad \approx f(x)$$

that has same values at $x = -1$, 0, and $+1$. Then, we can approximate the integral as

$$\int_{-1}^{1} f(x)dx \approx \int_{-1}^{1} g(x)dx.$$

(a) We are going find $g(x)$ via

$$g(x) = A\, g_{-1}(x) + B\, g_0(x) + C\, g_{+1}(x),$$

where $g_{-1}(x)$, $g_0(x)$, and $g_{+1}(x)$ are quadratic functions satisfying

$$\begin{aligned} g_{-1}(x): &\quad g_{-1}(-1) = 1, \quad g_{-1}(0) = 0, \quad g_{-1}(+1) = 0 \\ g_0(x): &\quad g_0(-1) = 0, \quad g_0(0) = 1, \quad g_0(+1) = 0 \\ g_{+1}(x): &\quad g_{+1}(-1) = 0, \quad g_{+1}(0) = 0, \quad g_{+1}(+1) = 1. \end{aligned}$$

Find three quadratic functions: $g_{-1}(x)$, $g_0(x)$, and $g_{+1}(x)$.

(b) From (a), calculate

$$G_{-1} = \int_{-1}^{1} g_{-1}(x)dx, \quad G_0 = \int_{-1}^{1} g_0(x)dx, \quad G_{+1} = \int_{-1}^{1} g_{+1}(x)dx,$$

Then, finally express the integral:

$$\int_{-1}^{1} f(x)dx \approx \int_{-1}^{1} g(x)dx = A\, G_{-1} + B\, G_0 + C\, G_{+1}.$$

Solution:

(a) The three basis functions are

$$g_{-1}(x) = \frac{1}{2}x(x-1), \quad g_0(x) = -(x+1)(x-1), \quad \text{and} \quad g_{+1}(x) = \frac{1}{2}x(x+1).$$

(b) The integral of the three basis functions are

$$G_{-1} = \int_{-1}^{1} \frac{1}{2}x(x-1)dx = \frac{1}{3},$$
$$G_0 = \int_{-1}^{1} -(x+1)(x-1)dx = \frac{4}{3},$$
$$G_{+1} = \int_{-1}^{1} \frac{1}{2}x(x+1)dx = \frac{1}{3}.$$

Therefore, we obtain Simpson's rule:

$$\int_{-1}^{1} f(x)dx \approx \frac{1}{3}(A + 4B + C).$$

In general, Simpson's rule is given by

$$\int_{a}^{b} f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

2. (6 points) Poisson RV, $N \sim \text{Pois}(\lambda)$, takes non-negative integer values with probability

$$P(N = k) = \lambda^k \frac{e^{-\lambda}}{k!}.$$

We are going to sample N.

(a) Thanks to Taylor's expansion of exponential function,

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!},$$

the probability satisfies $\sum_{k=0}^{\infty} P(k) = 1$. With this information, how can you sample N?

(b) The Poisson RV, N, expresses the number of arrivals of the Poisson-type events with intensity λ within unit time ($t = 1$). The arrival time T of the Poisson-type event follows the exponential distribution whose CDF is given by

$$F_T(t) = 1 - e^{-\lambda t}.$$

Given these knowledge, write another algorithm, different from (a), for sampling N.

Assume that you can sample the uniform RN U as much as you want.

Solution: This question is from this Stack Exchange question.

(a) From the discrete probability distribution, we sample N by

$$N = \min\left\{ n = 0, 1, \cdots \;\middle|\; U \leq e^{-\lambda} \sum_{k=0}^{n} \frac{\lambda^k}{k!} \right\},$$

where U is the uniform RN.

(b) The arrival time T for the event can be sampled by

$$T \sim -\frac{\log U}{\lambda} \quad \text{for uniform RN } U.$$

Therefore, N can be sampled by

$$N = \min\left\{ n = 0, 1, \cdots \;\Big|\; -\sum_{k=1}^{n+1} \frac{\log U_k}{\lambda} > 1 \right\},$$

$$\text{or} \quad \max\left\{ n = 1, \cdots \;\Big|\; -\sum_{k=1}^{n} \frac{\log U_k}{\lambda} \le 1 \right\} - 1.$$

where $\{U_k\}$ are sequentially generated uniform RN's.

3. (6 points) Assume that Y is a non-negative RV with mean μ, variance v, and skewness s. We are going to sample Y approximately with inverse Gaussian (IG) RV. Let $X \sim \mathrm{IG}(\sigma)$ be a standard (mean 1) IG RV with parameter σ. The PDF of X is given by

$$f_\sigma(x) = \frac{\sigma}{\sqrt{2\pi x^3}} \, \exp\left(-\frac{\sigma^2 (x-1)^2}{2x} \right),$$

and its variance and skewness are given by $1/\sigma^2$ and $3/\sigma$, respectively.

(a) Using X, how can we sample Y approximately to match the mean μ and variance v?

(b) Using X, how can we sample Y approximately to match the mean μ, variance v, and skeness s?

Assume that you can sample X.

Solution:

(a) To match μ, we let $Y = \mu X$. We determine σ by

$$v = \frac{\mu^2}{\sigma^2} \quad \Rightarrow \quad \sigma = \frac{\mu}{\sqrt{v}}.$$

(b) To match skewness, we let

$$Y = (1-\eta)\mu + \eta\,\mu\, X.$$

This form match μ because $E(Y) = \mu$. We determine σ and η by matching s and v by

$$s = \frac{3}{\sigma} \quad \text{and} \quad v = \frac{\eta^2 \mu^2}{\sigma^2}.$$

Here, note that the skewness is not affected by linear scaling. We obtain

$$\sigma = \frac{3}{s} \quad \text{and} \quad \eta = \frac{\sigma\sqrt{v}}{\mu} = \frac{3\sqrt{v}}{s\mu}.$$

4. (12 points) The variance process of the GARCH diffusion model is given by

$$dv_t = \kappa(\theta - v_t)dt + \nu v_t \, dZ_t.$$

In a 2019 exam question, we derived the Milstein scheme for v_t as

$$v_{t+\Delta t} = v_t + \kappa(\theta - v_t)\Delta t + \nu v_t Z \sqrt{\Delta t} + \frac{\nu^2}{2} v_t (Z^2 - 1)\Delta t \quad \text{for} \quad Z \sim N(0,1). \tag{1}$$

But we are going to improve it in this question.

(a) Define

$$y_t := (v_t - \theta)e^{\kappa t}.$$

Obtain the SDE for y_t.

(b) From (a), obtain $E(v_{t+h} \,|\, v_t)$ for $h \geq 0$.

(c) From (a), derive the Milstein scheme for y_t. Then, convert it to the scheme for v_t.

(d) Find $E(v_{t+\Delta t} \,|\, v_t)$ from the scheme in (c). Why the scheme in (c) is better than the scheme in Eq. (1)?

Solution:

(a) The SDE for y_t is

$$dy_t = e^{\kappa t} dv_t + \kappa\, e^{\kappa t}(v_t - \theta)dt = \nu v_t\, e^{\kappa t} dZ_t = \nu(y_t + \theta e^{\kappa t})dZ_t.$$

(b) Since y_t is a martingale, $E(y_{t+h}|y_t) = y_t$.

$$\begin{aligned} e^{\kappa(t+h)} E(v_{t+h} - \theta \,|\, v_t) &= e^{\kappa t}(v_t - \theta) \\ E(v_{t+h} \,|\, v_t) &= \theta + (v_t - \theta)e^{-\kappa h}. \end{aligned}$$

(c) Applying Milstein scheme for y_t, we obtain

$$y_{t+\Delta t} = y_t + \nu(y_t + \theta e^{\kappa t}) Z\sqrt{\Delta t} + \frac{\nu^2}{2}(y_t + \theta e^{\kappa t})(Z^2 - 1)\Delta t.$$

From this, we obtain the scheme for v_t:

$$\begin{aligned} (v_{t+\Delta t} - \theta)e^{\kappa \Delta t} &= (v_t - \theta) + \nu v_t\, e^{\kappa \Delta t} Z\sqrt{\Delta t} + \frac{\nu^2}{2} v_t\, e^{\kappa \Delta t}(Z^2 - 1)\Delta t, \\ v_{t+\Delta t} &= \theta + (v_t - \theta)e^{-\kappa \Delta t} + \nu v_t \left(Z\sqrt{\Delta t} + \frac{\nu}{2}(Z^2 - 1)\Delta t \right) \end{aligned}$$

(d) Since $E(Z^2 - 1) = 0$,

$$E(v_{t+\Delta t} \,|\, v_t) = \theta + (v_t - \theta)e^{-\kappa \Delta t}.$$

The new Milstein scheme in (c) is better the scheme in Eq. (1) because it preserves the mean from (b).