Stochastic Volatility Models and Simulation
Applied Stochastic Processes (FIN 514)

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Stochastic Volatility (SV) Models

- The price process (martingale):

$$\frac{dS_t}{S_t^\beta} = \sigma_t dW_t = \sigma_t(\rho dZ_t + \rho_* dX_t), \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2}.$$

  BSM-base: $\beta = 1$, normal-base: $\beta = 0$.
  For the models except SABR, the base model is BSM (i.e., $\beta = 1$).

- The (stochastic) volatility process may vary:

$$d\sigma_t = a(t, \sigma_t)dt + b(t, \sigma_t)dZ_t,$$

- The correlation between the two Brownian motions:

$$dW_t\, dZ_t = \rho\, dt.$$

  The correlation explains the *leverage effect*: equity volatility increases as price goes down.

Various SV models

- SABR model (Hagan et al, 2002):

$$d\sigma_t = \nu\sigma_t\, dZ_t.$$

- Heston (1993) model (Cox et al, 1985, CIR process):

$$dv_t = \kappa(\theta - v_t)dt + \nu\, \sqrt{v_t} dZ_t.$$

- 3/2 model (Heston, 1997; Lewis, 2000):

$$dv_t = \kappa v_t(\theta - v_t)dt + \nu\, v_t^{3/2}\, dZ_t.$$

- Ornstein–Uhlenbeck-driven SV model (Stein and Stein, 1991):

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu\, dZ_t.$$

- GARCH diffusion model (relatively new):

$$dv_t = \kappa(\theta - v_t)dt + \nu\, v_t\, dZ_t.$$

- Rough volatility: use a factional BM Z_t^H instead.

Integrated variance

In all SV models, the integrated variance V_T plays an important role:

$$V_T = \int_0^T \sigma_t^2 dt = \int_0^T v_t dt$$

Conditional on V_T (and other variables), S_T has lognormal distribution. Therefore, we can use BS model. From Itô's isometry,

$$\int_0^T \rho_* \sigma_t dX_t \sim \rho_* \sqrt{V_T} X_1 \sim N(0, \rho_*^2 V_T)$$

So the volatility between 0 and T is

$$\sigma = \rho_* \sqrt{V_T / T}$$

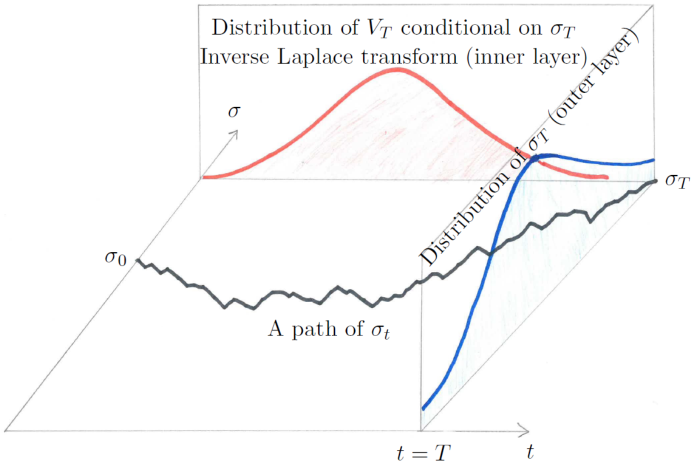
Simulation scheme for the SV models

- Time discretization (Euler/Milstein):
  - Easy to implement, but possible bias and computationally expensive.
- Conditional MC:
  - Can skip the simulation of price S_t. (Simulate volatility σ_t only)
  - The final price S_T should be expressed by σ_T and $V_T = \int_0^T \sigma_t^2 dt$.
- Exact Simulation:
  - No need for time-discretization: jump from $t = 0$ to T.
  - σ_T follows a well-known distribution.
  - Conditional Laplace transform of $V_T \,|\, v_T$ is analytically available (Heston, 3/2, SABR, etc)

    $$E(e^{-sV_T} | v_T = x) = f(s, x)$$

    The CDF of $V_T \,|\, v_T$ can be obtained by (numerical) Laplace inverse transform although computationally expensive.
  - No Laplace inversion required in some special cases: Normal SABR (Choi et al, 2019, NSVh model), OUSV (working paper)

Exact MC scheme



Simulation Procedures

Conditional MC:

1) Simulate the path of σ_t for $(0 \le t_1 \le t_2 \le \cdots \le T)$. Obtain σ_T.
2) Obtain V_T with time-integral (trapezoidal / Simpson's rule).

Exact MC:

1) Sample v_T from the (well-known) distribution
2) Sample V_T from the (numerical) CDF of $V_T \,|\, v_T$.

In Common:

3) Obtain $E(S_T|v_T, V_T)$ and effective volatility (usually $\rho_* \sqrt{V_T/T}$)

4-1) Price sampling: draw normal / log-normal distribution

4-2) Option price: Bachelier / BSM option price formula with $S_0 := E(S_T|v_T)$ and $\sigma := \rho_* \sqrt{V_T/T}$.
Then, average over the simulations.

Heston model (conditional MC)

Integrating v_t,

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dZ_t \quad (v_t = \sigma_t^2)$$

$$v_T - v_0 = \kappa(\theta T - V_T) + \nu\int_0^T \sqrt{v_t}dZ_t$$

$$\int_0^T \sqrt{v_t}\,dZ_t = \frac{1}{\nu}\Big(v_T - v_0 + \kappa(V_T - \theta T)\Big).$$

You can also express S_T by v_T and V_T (conditional MC possible!).

$$d\,\log S_t = \sqrt{v_t}(\rho dZ_t + \rho_* dX_t) - \frac{1}{2}v_t dt$$

$$\log(S_T/S_0) = \int_0^T \sqrt{v_t}(\rho dZ_t + \rho_* dX_t) - \int_0^T \frac{1}{2}v_t dt$$

$$\log\left(S_T/S_0\right) = \frac{\rho}{\nu}\Big(v_T - v_0 + \kappa(V_T - \theta T)\Big) + \rho_*\sqrt{V_T}\,X_1 - \frac{1}{2}V_T$$

Therefore, we can sample S_T as

$$S_T = S_0 \exp\left(\frac{\rho}{\nu}\big(v_T - v_0 + \kappa(V_T - \theta T)\big) - \frac{1}{2}V_T + \rho_* \sqrt{V_T}\, X_1\right).$$

But, instead of sampling S_T, we use the BS model. For BS, we need to spot and volatility.

$$\begin{aligned} E(S_T \mid v_T, V_T) &= S_0 \exp\left(\frac{\rho}{\nu}\big(v_T - v_0 + \kappa(V_T - \theta T)\big) - \frac{1}{2}V_T + \frac{\rho_*^2}{2}V_T\right) \\ &= S_0 \exp\left(\frac{\rho}{\nu}\big(v_T - v_0 + \kappa(V_T - \theta T)\big) - \frac{\rho^2}{2}V_T\right) \end{aligned}$$

$$\sigma_{\mathrm{BS}} = \rho_* \sqrt{V_T/T}$$

Heston model (exact MC)

Broadie and Kaya (2006) pioneered the exact MC scheme:

- v_T is distributed as a noncentral chi-square distribution, $\chi^2(\delta,\lambda)$ (see **2017ME Bessel process** problem):

$$v_T \;=\; \frac{\nu^2(1-e^{-\kappa T})}{4\kappa}\chi^2(\delta,\lambda) = \frac{e^{-\kappa T/2}}{\phi(\kappa)}\chi^2(\delta,\lambda),$$

  where the degrees of freedom δ and the noncentrality λ are

$$\delta = \frac{4\kappa\theta}{\nu^2},\quad \lambda = \frac{4v_0\kappa\, e^{-\kappa T}}{\nu^2(1-e^{-\kappa T})} = v_0 e^{-\kappa T/2}\phi(\kappa),\quad \phi(\kappa) = \frac{2\kappa/\nu^2}{\sinh(\kappa T/2)}.$$

  Standard library is available for drawing χ^2 random number.

- The conditional Laplace transform of V_T (Pitman and Yor, 1982):

$$E\left(e^{-aV_T}\middle| v_T\right) = \frac{\phi(\gamma(a))}{\phi(\kappa)}\frac{\exp\left(-\frac{v_0+v_T}{2}\cosh(\frac{\gamma(a)T}{2})\phi(\gamma(a))\right)}{\exp\left(-\frac{v_0+v_T}{2}\cosh(\frac{\kappa T}{2})\phi(\kappa)\right)}\frac{I_\nu\left(\sqrt{v_0\,v_T}\phi(\gamma(a))\right)}{I_\nu\left(\sqrt{v_0\,v_T}\phi(\kappa)\right)}$$

- See Glasserman and Kim (2011) for improvement.

SABR model (conditional MC)

$$\frac{d\sigma_t}{\sigma_t} = \nu\, dZ_t \quad \Rightarrow \quad \sigma_T = \sigma_0 \exp\left(-\frac{1}{2}\nu^2 T + \nu Z_T\right)$$

Integrating σ_t,

$$\nu \int_0^T \sigma_t dZ_t = \sigma_T - \sigma_0 = \sigma_0 \exp\left(-\frac{1}{2}\nu^2 T + \nu Z_T\right) - \sigma_0$$

S_T is expressed by σ_T and V_T (conditional MC possible) !

$$\text{Normal SABR}(\beta = 0): S_T = S_0 + \frac{\rho}{\nu}(\sigma_T - \sigma_0) + \rho_* \sqrt{V_T}\, X_1$$

$$\text{BS SABR}(\beta = 1): \log\left(\frac{S_T}{S_0}\right) = \frac{\rho}{\nu}(\sigma_T - \sigma_0) - \frac{1}{2}V_T + \rho_* \sqrt{V_T}\, X_1$$

See the **SABR Model** slides for detail.

SABR Model (exact MC)

- σ_T is distributed by a log-normal distribution. Sampling is trivial.
- The conditional Laplace transform of $1/V_T$ is also known:

$$E\left(e^{-s/V_T}\middle| v_T\right) = \exp\left(-\frac{\phi_x(s)^2 - x^2}{2T}\right)$$

  where $\phi_x(s) = \text{acosh}(se^{-x} + \cosh(x))$ and $v_T = \exp(\nu x)$
- From above, we can sample $1/V_T$ and get V_T.
- Reference: Cai et al (2017)

3/2 model (conditional MC)

$$dv_t = \kappa v_t(\theta - v_t)dt + \nu\, v_t^{3/2}\, dZ_t.$$

The change of variable, $x_t = 1/v_t$ yields (a good Itô calculus exercise!)

$$dx_t = -\frac{dv_t}{v_t^2} + \frac{(dv_t)^2}{v_t^3} = (\kappa + \nu^2 - \kappa\theta\, x_t)dt - \nu\sqrt{x_t}\, dZ_t.$$

This is same as v_t in Heston model with new parameters:

$$\nu' = -\nu, \quad \kappa' = \kappa\theta, \quad \text{and} \quad \theta' = (\kappa + \nu^2)/\kappa\theta$$

$$(\nu = -\nu', \quad \kappa = \theta'\kappa' - \nu'^2, \quad \text{and} \quad \nu' = \kappa'/(\kappa'\theta' - \nu'^2))$$

We can express S_T as a function of V_T and v_T (conditional MC possible)!

$$d\log(x_t) = \left(\frac{\kappa + \nu^2/2}{x_t} - \kappa\theta\right) dt - \frac{\nu}{\sqrt{x_t}}\, dZ_t$$

$$\int_0^T \frac{1}{\sqrt{x_t}}\, dZ_t = \frac{1}{\nu}\left(\log\left(\frac{x_0}{x_T}\right) + (\kappa + \nu^2/2)V_T - \kappa\theta T\right),$$

$$\log\left(\frac{S_T}{S_0}\right) = \frac{\rho}{\nu}\left(\log\left(\frac{v_T}{v_0}\right) + (\kappa + \nu^2/2)V_T - \kappa\theta T\right) - \frac{1}{2}V_T + \rho_*\sqrt{V_T}\, X_1$$

3/2 model (exact MC)

- From the Heston model, $1/v_T$ is distributed as a noncentral chi-square distribution, $\chi^2(\delta', \lambda')$ where the degrees of freedom δ' and the noncentrality λ' are

$$\delta' = \frac{4\kappa'\theta'}{\nu^2}, \quad \lambda = \frac{4\kappa' \, e^{-\kappa' T}}{v_0 \nu^2 (1 - e^{-\kappa' T})}.$$

  Standard library is available for drawing χ^2 random number.
- The conditional Laplace transform of V_T is also known.
- Reference: Baldeaux (2012)

OUSV model (conditional MC). 2019ME Question

Let $U_T = \int_0^T \sigma_t\, dt$ and $V_T = \int_0^T \sigma_t^2\, dt$.

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu\, dZ_t.$$

$$d\sigma_t^2 = 2\sigma_t d\sigma_t + (d\sigma_t)^2 = (\nu^2 + 2\kappa(\theta\sigma_t - \sigma_t^2))dt + 2\nu\sigma_t dZ_t$$

$$\sigma_T^2 - \sigma_0^2 = \nu^2 T + 2\kappa(\theta U_T - V_T) + 2\nu \int_0^T \sigma_t dZ_t$$

$$\int_0^T \sigma_t dZ_t = \frac{1}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\nu}{2}T - \frac{\kappa\theta}{\nu}U_T + \frac{\kappa}{\nu}V_T$$

S_T is expressed by v_T and V_T (conditional MC possible)!

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t - \frac{1}{2}V_T \\ &= \frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{1}{2}\right)V_T + \rho_*\sqrt{V_T}\, X_1\end{aligned}$$

$$S_0 := E(S_T) = S_0 \exp\left(\frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{\rho^2}{2}\right)V_T\right)$$

$$\sigma_{\rm BS} := \rho_*\sqrt{V_T/T}.$$

GARCH model (conditional MC): 2020ME Question

$$dv_t = \kappa(\theta - v_t)dt + \nu v_t\, dZ_t$$

We derive the SDE for $\sigma_t = \sqrt{v_t}$,

$$\begin{aligned} d\sigma_t = d\sqrt{v_t} &= \frac{1}{2}\frac{dv_t}{\sqrt{v_t}} - \frac{1}{8}\frac{(dv_t)^2}{v_t\sqrt{v_t}} = \frac{1}{2}\kappa\left(\frac{\theta}{\sigma_t} - \sigma_t\right)dt + \frac{\nu}{2}\sigma_t dZ_t - \frac{\nu^2}{8}\sigma_t dt \\ &= \frac{1}{2}\left(\frac{\kappa\theta}{\sigma_t} - \left(\kappa + \frac{\nu^2}{4}\right)\sigma_t\right)dt + \frac{\nu}{2}\sigma_t dZ_t. \end{aligned}$$

Integrating above,

$$\begin{aligned} \sigma_T - \sigma_0 &= \frac{1}{2}\left(\kappa\theta\, Y_T - \left(\kappa + \frac{\nu^2}{4}\right)U_T\right) + \frac{\nu}{2}\int_0^T \sigma_t dZ_t \\ \Rightarrow \int_0^T \sigma_t dZ_t &= \frac{2}{\nu}(\sigma_T - \sigma_0) - \left(\frac{\kappa\theta}{\nu}\, Y_T - \left(\frac{\kappa}{\nu} + \frac{\nu}{4}\right)U_T\right) \end{aligned}$$

where $\quad Y_T = \int_0^T \frac{1}{\sigma_t}\, dt, \quad U_T = \int_0^T \sigma_t\, dt \quad$ and $\quad V_T = \int_0^T \sigma_t^2\, dt.$

Therefore,

$$\log\left(\frac{S_T}{S_0}\right) = \rho\int_0^T \sigma_t dZ_t + \rho_*\int_0^T \sigma_t dX_t - \frac{1}{2}V_T$$
$$= \frac{2\rho}{\nu}(\sigma_T - \sigma_0) - \frac{\rho\kappa\theta}{\nu}Y_T + \rho\left(\frac{\kappa}{\nu}+\frac{\nu}{4}\right)U_T - \frac{1}{2}V_T + \rho_*\sqrt{V_T}\,X_1.$$

Finally,

$$E(S_T|\sigma_T, Y_T, U_T, V_T) = S_0\,\exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2}V_T\right)$$
$$= S_0\exp\left(\frac{2\rho}{\nu}(\sigma_T - \sigma_0) - \frac{\rho\kappa\theta}{\nu}Y_T + \rho\left(\frac{\kappa}{\nu}+\frac{\nu}{4}\right)U_T - \frac{\rho^2}{2}V_T\right)$$
$$\sigma_{\text{BS}} = \rho_*\sqrt{V_T/T}.$$

- For the Euler/Milstein/Log schemes, see **2019ME** problem.
- Analytic approximations are available only for $\rho = 0$ (Barone-Adesi et al, 2005).

Project Suggestion: *Almost* Exact MC (by Choi)

- General scheme:
  - Implementing existing paper is OK:
  - Improving Euler / Milstein scheme? or exact simulation?
  - Or try something new (see below):
- Simulation for GARCH diffusion:

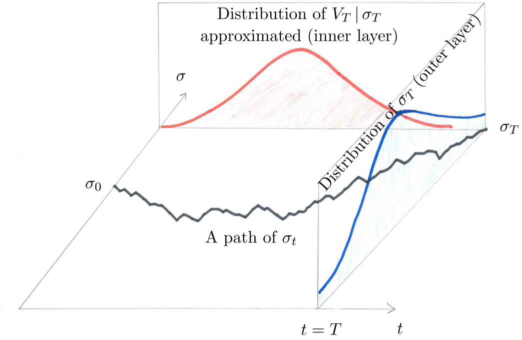
$$dv_t = \kappa(\theta - v_t)dt + \nu v_t\, dZ_t.$$

  - Currently, there is no easy way to solve the SDE.
  - Conditional MC possible? Option pricing with conditional MC?
  - How to express S_T as a function of v_T and V_T (or something else)?
  - Exact simulation possible?
- *Almost* Exact MC (by Choi)

Project Suggestion: *Almost* Exact MC (by Choi)

The illustration of the proposed double layer approximation method:

1. The outer layer distribution of σ_T (in blue) is typically known
2. The inner layer distribution of $V_T|\sigma_T$ (in red) is approximated as well-known distributions such as log-normal or inverse Gaussian.



Project Suggestion: *Almost* Exact MC (by Choi)

Drawback of the exact simulation methods:

- Inverse of the Laplace transform, $E(e^{-sV_T}|v_T) = f(s, v_T)$, is complicated.
- Drawing random number from the numerical CDF is also slow.

How can we simplify this step with some approximation?

- Approximate $V_T|v_T$ with a well-known distribution by matching the first two moments , $M_1 = E(V_T|v_T)$ and $M_2 = E(V_T^2|v_T)$.
- The RN sampling should be easy from the approximate distribution.

Almost Exact MC: Candidates for distributions

- Log-normal (LN):

$$Y \sim \mu \exp(\sigma Z - \sigma^2/2) \quad \text{for} \quad Z \sim N(0,1)$$

The parameters (μ, σ) can be obtained from the two moments:

$$\mu = M_1 \quad \text{and} \quad \sigma = \sqrt{\log(M_2/M_1^2)}.$$

- Inverse-Gaussian (IG):

$$f_{\mathsf{IG}}(x \,|\, \gamma, \delta) = \frac{\delta}{\sqrt{2\pi x^3}} \exp\left(-\frac{(\gamma x - \delta)^2}{2x}\right) \quad \text{for} \quad \gamma \geq 0,\ \delta > 0.$$

How to determine (γ, μ) from M_1 and M_2?

- The sampling methods for LN and IG are available. See Michael et al (1976) (WIKIPEDIA) for IG.

Almost Exact MC: How to obtain M_1 and M_2?

Keep in mind for a random variable $X \geq 0$, the MGF and Laplace transform are same:

$$M_X(-s) = E(e^{-sX}) = \int_{x=0}^{\infty} e^{-sX} f_X(x) dx = f(s)$$

$$f(s) = 1 - M_1 s + \frac{1}{2} M_2 s^2 + \cdots,$$

where $M_1 = E(V_T | v_T)$ and $M_2 = E(V_T^2 | v_T)$.

- Numerical method: Choudhury and Lucantoni (1996)
- Analytic method (from Taylor's expansion or etc):
  - SABR: available in Kennedy et al (2012).
  - Heston, 3/2, OUSV?

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