Efficient simulation of the SABR model
Applied Stochastic Processes (FIN 514)

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Stochastic-alpha-beta-rho (SABR) Model

$$\frac{dF_t}{F_t^{\beta}} = \sigma_t \, dW_t \quad \text{and} \quad \frac{d\sigma_t}{\sigma_t} = \xi \, dZ_t \; (\sigma_0 = \alpha) \quad \text{for} \quad dW_t \, dZ_t = \rho \, h$$

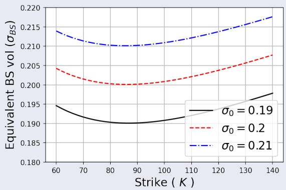
$$\text{or} \quad \frac{dF_t}{F_t^{\beta}} = \sigma_t \, (\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2}, \; dX_t \, dZ_t = 0$$

- One of the most popular stochastic volatility (SV) models used in financial engineering (Hagan et al., 2002).
- A flexible choice of *backbone* (volatility v.s. spot change): $0 \le \beta \le 1$
- Parsimonious and intuitive parameters: $\sigma_0 = \alpha$, ρ and ξ (and β)
- Asymptotic approximation (HKLW formula) for the equivalent BS volatility ($\xi\sqrt{T} \ll 1$)

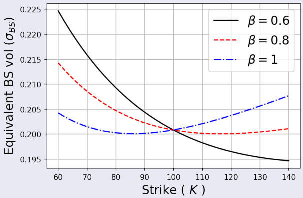
The impact of parameters

Base parameters: $\sigma_0 = 0.2,\ \xi = 0.2,\ \rho = 0.1,\ \beta = 1.$

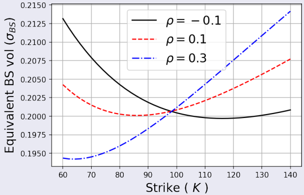
Initial vol σ_0 (level)



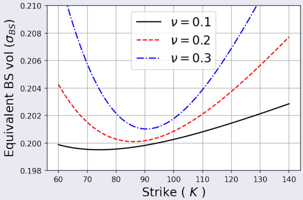
Backbone β (skew)



Correlation ρ (skew)



Vol-of-vol ν (smile)



Black-Scholes (BS) Model

- Black and Scholes (1973); Merton (1973).
- Dynamics (geometric BM):

$$\frac{dF_t}{F_t} = \sigma_{\text{BS}}\, dW_t$$

- Distribution:

$$F_T \sim \mathsf{LN}(F_0, \sigma_{\text{BS}}^2 T).$$

- Option Price (Black, 1976):

$$C_{\text{BS}} = F_0\, N(d_+) - K\, N(d_-) \quad \text{where} \quad d_\pm = \frac{\log(F_0/K)}{\sigma_{\text{BS}}\sqrt{T}} \pm \frac{1}{2}\sigma_{\text{BS}}\sqrt{T}$$

- Asset class: equity, FX, etc.
- As $\xi \downarrow 0$, SABR $\to$ BS model.

Bachelier (Normal) Model

- The birth of Brownian motion (Bachelier, 1900)
- Dynamics (arithmetic BM):

$$dF_t = \sigma_{\rm N}\, dW_t$$

- Distribution:

$$F_T \sim N(F_0, \sigma_{\rm N}^2 T).$$

- Option Price:

$$C_{\rm N} = (F_0 - K)N(d_{\rm N}) + \sigma\sqrt{T}\, n(d_{\rm N}) \quad \text{where} \quad d_{\rm N} = \frac{(F_0 - K)}{\sigma_{\rm N}\sqrt{T}}$$

- Daily price changes may not be proportional to the price level. Underlying price can be negative.
- Asset class: interest rate, inflation, spread $(X - Y)$, WTI futures
- Fast implied volatility inversion (Choi et al., 2009) and review (Choi et al., 2022).
- As $\xi \downarrow 0$, SABR $\rightarrow$ Bachelier model.

Constant-Elasticity-of-Variance (CEV) Model

- Dynamics (CEV):

$$\frac{dF_t}{F_t^\beta} = \sigma_{\text{CEV}}\, dW_t \quad (\beta_* = 1-\beta)$$

- Distribution:

$$F_T :\sim \text{CEV}(\beta, F_0, \sigma_{\text{CEV}}^2 T).$$

- Option Price (Schroder, 1989):

$$q_t = F_t^{\beta_*}/\beta_* \quad \text{and} \quad q_K = K^{\beta_*}/\beta_*$$

$$C_{\text{CEV}} = F_0\, \bar{F}_{\chi^2}\left(\frac{q_K^2}{\sigma_{\text{CEV}}^2 T};\, 2+\frac{1}{\beta_*},\, \frac{q_0^2}{\sigma_{\text{CEV}}^2 T}\right) - K\, \underbrace{F_{\chi^2}\left(\frac{q_0^2}{\sigma_{\text{CEV}}^2 T};\, \frac{1}{\beta_*},\, \frac{q^2(K)}{\sigma_{\text{CEV}}^2 T}\right)}_{=\text{Prob}(F_T>K)=1-F_{\text{CEV}}(K)},$$

where $F_{\chi^2}(x; r, x_0)$ ($\bar{F}_{\chi^2}$) is the CDF (1−CDF) of noncentral χ^2 with degrees of freedom r and noncentrality x_0.

- F_T is not a χ^2 RV.
- Mass at zero exists:

$$\text{Prob}(F_T = 0) = \bar{F}_{\chi^2}\left(\frac{q_0^2}{\sigma_{\text{CEV}}^2 T};\, \frac{1}{\beta_*},\, 0\right) = \bar{F}_\Gamma\left(\frac{q_0^2}{2\sigma_{\text{CEV}}^2 T};\, \frac{1}{2\beta_*}\right)$$

- As $\xi \downarrow 0$, SABR $\to$ CEV (β). (Motivation for this study and Choi and Wu (2021a))

SABR Papers

Normal SABR ($\beta = 0$)

- **Closed-form Monte-Carlo simulation of the normal SABR:** Choi, Liu and Seo (2019). Hyperbolic normal stochastic volatility model. *Journal of Futures Markets*, 39:186–204. arXiv:1809.04035
- Choi and Seo (2024). Option pricing under the normal SABR model with quadratic approximation. *Working paper*

CEV SABR ($0 < \beta < 1$)

- Choi and Wu (2021a). The equivalent CEV volatility of the SABR model. *Journal of Economic Dynamics and Control*, 128:104143. arXiv:1911.13123
- **Option pricing for the uncorrelated ($\rho = 0$) case:** Choi and Wu (2021b) A note on the option price and 'Mass at zero in the uncorrelated SABR model and implied volatility asymptotics.' *Quantitative Finance*, 21:1083–1086. arXiv:2011.00557
- Choi, Hu, and Kwok (2024). Efficient simulation of the SABR model. *Working paper*

The equivalent CEV volatility of the SABR model

Choi, J., & Wu, L. (2021). Journal of Economic Dynamics and Control, 128:104143. arXiv:1911.13123.

Equivalent CEV volatility ($0 < \beta < 1$)

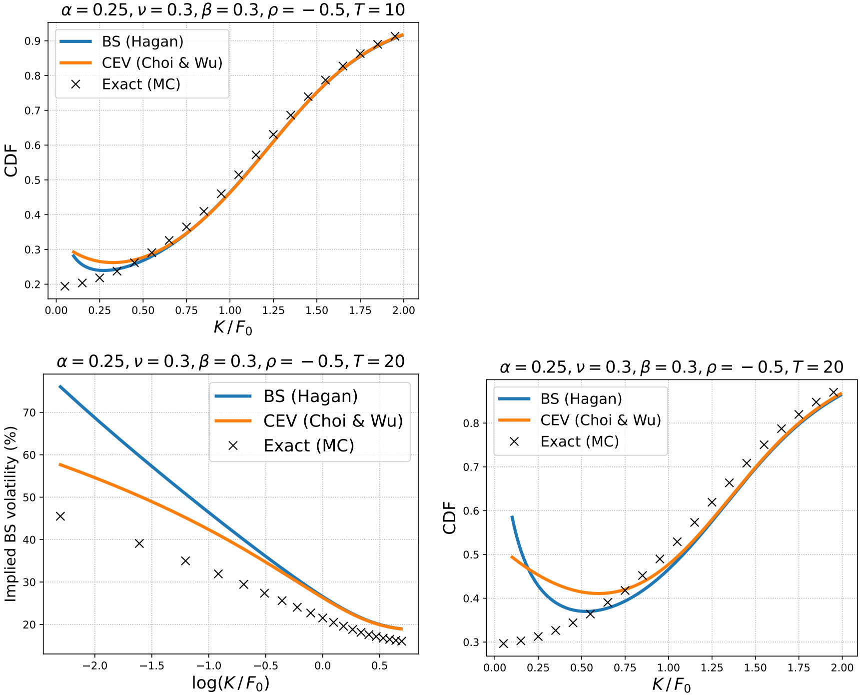
- SABR is the CEV model (backbone) with stochastic vol.
- When $\xi \downarrow 0$, SABR $\to$ CEV (β) model with the fixed vol $\sigma_0 = \alpha$:

$$\sigma_{\text{BS}} = \alpha \frac{H(0) \to 1}{K^{\beta_*/2}} \frac{1 + \left(\frac{\beta_*^2}{24\, K^{\beta_*}} \alpha^2 + \frac{\rho\beta}{4\, K^{\beta_*/2}} \alpha \cdot 0 + \frac{2-3\rho^2}{24} 0^2 \right) T}{1 + \frac{\beta_*^2}{24} \log^2 K + \frac{\beta_*^4}{1920} \log^4 K} \qquad (F_0 = 1)$$

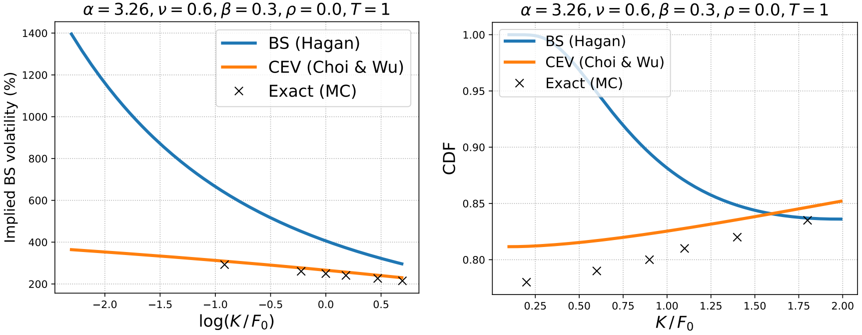
- The blue terms are for the approximate CEV-to-BS vol conversion, which has nothing to do with the stochastic vol (ξ).
- Instead, derive an equivalent CEV vol; (i) the conversion error is avoided, (ii) $\sigma_{\text{CEV}} \to \alpha$ as $\xi \downarrow 0$, and (iii) expressions are simpler.

$$\sigma_{\text{CEV}} = \alpha H(z) \left(1 + \left[\frac{\rho\beta}{4\, K^{\beta_*/2}} \alpha\xi + \frac{2 - 3\rho^2}{24} \xi^2 \right] T \right) \quad \text{for} \quad z = \frac{\xi}{\alpha\beta_*} (K^{\beta_*} - 1)$$

- The BS vol can be converted from the analytic CEV option price.
- Equivalent CEV vol allows less arbitrage in low strike.



$\alpha=0.25, \nu=0.3, \beta=0.3, \rho=-0.5, T=10$
BS (Hagan)
CEV (Choi & Wu)
Exact (MC)
CDF
K/F_0
$\alpha=0.25, \nu=0.3, \beta=0.3, \rho=-0.5, T=20$
BS (Hagan)
CEV (Choi & Wu)
Exact (MC)
Implied BS volatility (%)
$\log(K/F_0)$
$\alpha=0.25, \nu=0.3, \beta=0.3, \rho=-0.5, T=20$
BS (Hagan)
CEV (Choi & Wu)
Exact (MC)
CDF
K/F_0



$\alpha = 3.26, \nu = 0.6, \beta = 0.3, \rho = 0.0, T = 1$
Implied BS volatility (%)
$\log(K/F_0)$
BS (Hagan)
CEV (Choi & Wu)
Exact (MC)
$\alpha = 3.26, \nu = 0.6, \beta = 0.3, \rho = 0.0, T = 1$
CDF
K/F_0
BS (Hagan)
CEV (Choi & Wu)
Exact (MC)

Option pricing under the normal SABR model with quadratic approximation.

Choi & Seo (2024). Working Paper.

Normal SABR model ($\beta = 0$)

- The integral representations for option value has been investigated by many (Henry-Labordère, 2008; Korn and Tang, 2013; Antonov et al., 2015).
- Antonov et al. (2015)'s representation is the simplest: single integral with the elliptic integral of the 2nd kind:

$$C_{\text{SABR}} = [F_0 - K]^+ + \frac{\sigma_0\sqrt{T}}{\pi}\int_{u_0}^{\infty} F(u)\, w(u) du,$$

where

$$u_0 = \frac{1}{2\xi}\text{acosh}\left(\frac{V-\rho k}{\rho_*^2}\right) \quad \text{where} \quad \xi = \frac{\xi\sqrt{T}}{2},\ k = z_{\text{N}} + \rho,\ V = \sqrt{k^2+\rho_*^2},$$

$$w(u) = e^{-\frac{\xi^2}{2}} n(u)\left[R_{+\xi}(u) + R_{-\xi}(u)\right] \quad \text{with} \quad R_\Delta(u) := \frac{N(-u-\Delta)}{n(u+\Delta)}$$

$$F(u) = \sqrt{\alpha_+(u)}\ \text{EllipticE}\left(\frac{\alpha_+(u)-\alpha_-(u)}{\alpha_+(u)}\right) \quad (F(u_0)=0),$$

$$\alpha_\pm(u) = \rho k + \rho_*^2\cosh(2\xi u) \pm \rho_*\sqrt{\sinh^2(2\xi u) - (k-\rho\cosh(2\xi u))^2}.$$

Approximation for option value

- Analytic integrals are available for

$$\int_{u_0}^{\infty} (u-u_0)^n \, w(u) du \quad \text{for} \quad n = 0, 1, \text{ and } 2.$$

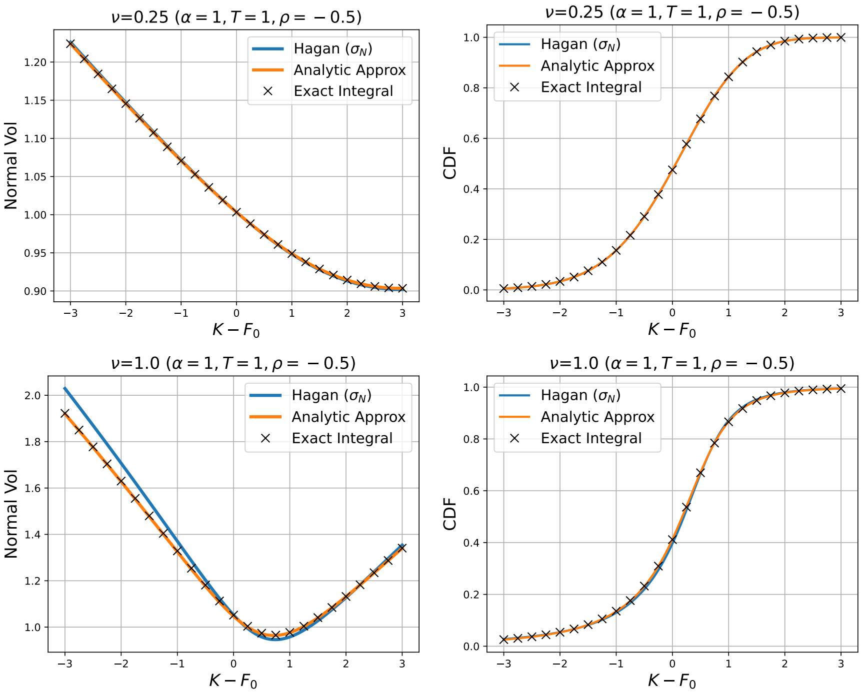
- Approximate $F(u)$ as a quadratic function near $u = u_0$:

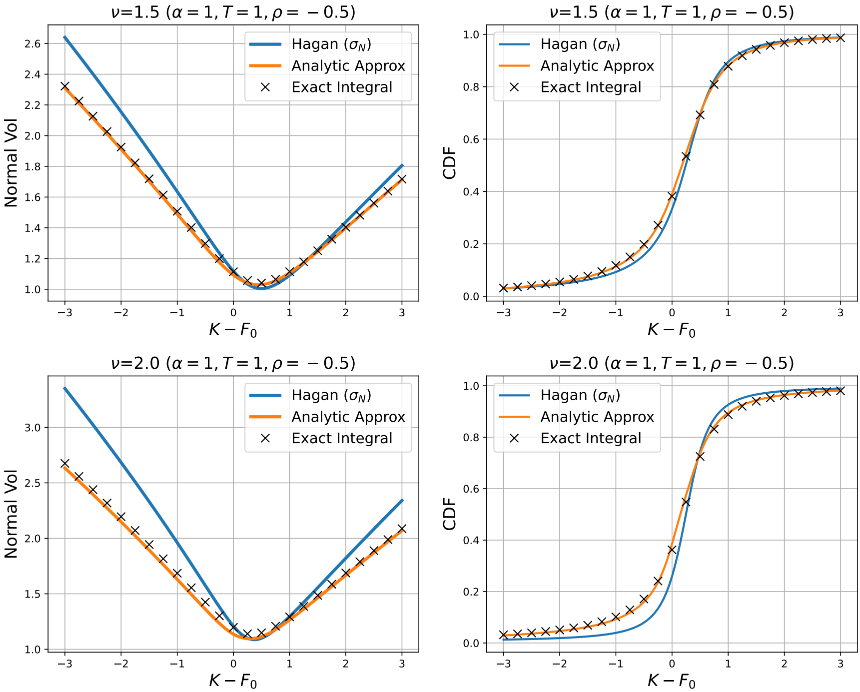
$$F(u) \approx \frac{\pi}{2}\sqrt{V}\left[1 + A\,\xi(u-u_0) + B\,\xi^2(u-u_0)^2\right],$$

$$A = \frac{3}{4}\left|\rho - \frac{k}{V}\right| \quad \text{and} \quad B = \frac{3}{4}\left(1 - \frac{\rho k}{V}\right) - \frac{15}{64}\left|\rho - \frac{k}{V}\right|^2$$

Approximate option value under the normal SABR

$$C \approx [F_0 - K]^+ + \frac{\sigma_0\sqrt{V}\,n(u_0)}{\xi\, e^{\frac{\xi^2}{2}}}\left[\begin{array}{c} [R_{-\xi}(u_0) - R_{+\xi}(u_0)] \\ +A\,[R_{+\xi}(u_0) + R_{-\xi}(u_0) - 2R_0(u_0)] \\ +2B\,[R_{-\xi}(u_0) - R_{+\xi}(u_0) - 2\xi(1 - u_0 R_0(u_0))] \end{array}\right]$$


ν=0.25 (α = 1, T = 1, ρ = −0.5)
Hagan (σ_N)
Analytic Approx
Exact Integral
Normal Vol
CDF
K − F_0
ν=1.0 (α = 1, T = 1, ρ = −0.5)



ν=1.5 (α = 1, T = 1, ρ = −0.5)
Normal Vol
CDF
K − F0
ν=2.0 (α = 1, T = 1, ρ = −0.5)
Hagan (σN)
Analytic Approx
Exact Integral

Stochastic Integral

The volatility process is a geometric BM:

$$\frac{d\sigma_t}{\sigma_t} = \xi\, dZ_t, \quad \sigma_T = \sigma_0 \exp\left(\xi Z_T - \frac{\xi^2 T}{2}\right) \quad \Rightarrow \quad \int_0^T \sigma_t dZ_t = \frac{1}{\xi}(\sigma_T - \sigma_0).$$

The price process is integrated to

$$\begin{aligned}\int_0^T \frac{dF_t}{F_t^\beta} &= \int_0^T \sigma_t \left(\rho dZ_t + \rho_* dX_t\right) = \frac{\rho}{\xi}(\sigma_T - \sigma_0) + \rho_* \int_0^T \sigma_t dX_t \\ &\sim \underbrace{\frac{\rho}{\xi}(\sigma_T - \sigma_0)}_{\text{skew}} + \underbrace{\rho_* X_{V_{0,T}}}_{\text{smile}} \sim \underbrace{\frac{\rho}{\xi}(\sigma_T - \sigma_0)}_{\text{adjustment to } F_0} + \underbrace{\rho_* \sigma_0 \sqrt{T I_{0,T}}}_{\text{volatility}}\, X, \quad (X \sim N(0,1))\end{aligned}$$

where $V_{0,T}$ ($I_{0,T}$) is the (normalized) conditional integrated variance or stochastic time clock:

$$V_{0,T}(\sigma_0, \sigma_T) = \int_0^T \sigma_t^2 dt \,\bigg|\, \sigma_0, \sigma_T \quad \text{and} \quad I_{0,T}(\sigma_0, \sigma_T) = \frac{V_{0,T}}{\sigma_0^2 T} \left(\lim_{\xi \downarrow 0} I_{0,T} \to 1\right).$$

The joint sampling of $(\sigma_T, I_{0,T})$ is critical for understanding the SABR model.

Simulation Steps

- 1. Simulate σ_{t+h} from σ_t (trivial):

$$\sigma_{t+h} = \sigma_t \exp\left(\xi\sqrt{h}Z - \frac{\xi^2 h}{2}\right), \quad Z \sim N(0,1).$$

- 2. Simulate $I_{t,t+h}$ given σ_t and σ_{t+h}.
- 3. Simulate F_{t+h} given F_t, σ_{t+h}, and $I_{t,t+h}$.
  - A. For $0 < \beta < 1$, how to approximate F_{t+h}? A CEV distribution?
  - B. How to sample F_{t+h} from the approximated distribution?

- We propose better methods for (2), (3A), and (3B).
- Our new methods independent.
- Applying all methods yield best results. However, they can be applied selectively to the existing methods.

(2) Simulation of $I_{t,t+h}$: existing studies

- Chen et al. (2012) used the lognormal sampling, but based on the short-time expansion:

$$\mu_1 = E(I_{t,t+h}) \approx 1 + \xi\sqrt{h}\,Z + \cdots \quad \text{and} \quad \mu_2 = E(I_{t,t+h}^2) \approx \mu_1^2 + \frac{\xi^2 h}{3},$$

  where Z is the normal variate used in σ_{t+h}.

- Cai et al. (2017) used the Laplace transform of $1/I_{t,t+h}$ (Matsumoto and Yor, 2005):

$$E\left(e^{-s/I_{t,t+h}}\right) = \exp\left(-\frac{\phi_x^2(\xi^2 h s) - x^2}{2\xi^2 h}\right)$$

$$\text{for} \quad x = \log(\sigma_{t+h}/\sigma_t) \quad \text{and} \quad \phi_x(\lambda) = \operatorname{acosh}(s e^{-x} + \cosh(x)).$$

  Then, they sample $1/I_{t,t+h}$ from the numerical inversion.

- Leitao et al. (2017a) use Fourier transform of $1/I_{t,t+h}$.

(2) Simulation of $I_{t,t+h}$: (shifted) lognormal approximation

- Trapezoidal rule:

$$\text{TZ} : I_{t,t+h} \approx \frac{1}{2}\left(1 + \frac{\sigma_{t+h}^2}{\sigma_t^2}\right) \quad \text{(deterministic)}$$

- We sample $I_{t,t+h}$ from the lognormal approximation with moments μ_1 and μ_2:

$$\text{LN} : I_{t,t+h} \approx \mu_1 \exp\left(\lambda Z - \lambda^2/2\right), \quad \lambda = \sqrt{\ln\left(\mu_2/\mu_1^2\right)},\ Z \sim N(0,1).$$

Conditional moments of $I_{t,t+h}$ are available from Kennedy et al. (2012):

$$\mu_1 = E(I_{t,t+h}) = \frac{N(d_{+1}) - N(d_{-1})}{2\xi\sqrt{h}\, n(d_{+1})}, \quad \left(d_k = \frac{\ln(\sigma_{t+h}/\sigma_t)}{\xi\sqrt{h}} + k\,\xi\sqrt{h}\right)$$

$$\mu_2 = E(I_{t,t+h}^2) = \frac{1}{(\xi\sqrt{h})^{1.5}}\left[\frac{N(d_{+2}) - N(d_{-2})}{4\, n(d_{+2})} - \frac{\sigma_t^2 + \sigma_{t+h}^2}{2\sigma_0^2}\frac{N(d_{+1}) - N(d_{-1})}{2\, n(d_{+1})}\right].$$

- We also sample from the shifted lognormal distribution matched to μ_1, μ_2, and μ_3:

$$\text{S-LN} : I_{t,t+h} \approx (1-\eta)\mu_1 + \eta\mu_1 \exp\left(\lambda Z - \lambda^2/2\right).$$

(3AB) Conditional Simulation for $\beta = 0$ or 1

- SABR ($\beta = 0$) is simulated with conditional normal distribution:

$$F_T = F_0 + \frac{\rho}{\xi}(\sigma_T - \sigma_0) + \rho_* \sigma_0 \sqrt{T I_{0,T}}\, X.$$

  Conditional on σ_T and $I_{0,T}$,

$$F_T \sim N(\bar{F}_T, \rho_*^2 \sigma_0^2 T I_{0,T}) \quad \text{where} \quad \bar{F}_T = F_0 + \frac{\rho}{\xi}(\sigma_T - \sigma_0).$$

  But, closed-form simulation method is available in Choi et al. (2019).
- SABR ($\beta = 1$) is simulated with conditional BS distribution:

$$\log\left(\frac{F_T}{F_0}\right) \sim \frac{\rho}{\xi}(\sigma_T - \sigma_0) - \frac{\sigma_0^2 T I_{0,T}}{2} + \rho_* \sigma_0 \sqrt{T I_{0,T}}\, X.$$

  Conditional on σ_T and $I_{0,T}$,

$$F_T \sim \text{LN}(\bar{F}_T, \rho_*^2 \sigma_0^2 T I_{0,T}) \text{ where } \boxed{\bar{F}_T = F_0 \exp\left(\frac{\rho}{\xi}(\sigma_T - \sigma_0) - \frac{\rho^2 \sigma_0^2 T I_{0,T}}{2}\right)}$$

- Note $F_0 = E(\bar{F}_T)$ for any initial $\sigma_0 > 0$.

Closed-form Simulation for $\beta = 0$

Choi, Liu and Seo (2019). Hyperbolic normal stochastic volatility model. *Journal of Futures Markets*, 39:186–204. arXiv:1809.04035

$$\sigma_T = \sigma_0 \exp\left(\xi Z'_T\right), \quad Z'_T = Z_T + \mu T$$

$$F_T \sim F_0 + \frac{\sigma_0 \rho}{\xi} \left(e^{\xi Z'_T} - e^{\lambda \xi^2 T/2}\right) + \frac{\sigma_0 \rho_*}{\xi} \cos\theta\, \phi\left(\xi Z'_T, \xi\sqrt{R_T^2 + (Z'_T)^2}\right),$$

- $\phi(Z, D) = e^{Z/2}\sqrt{2\cosh D - 2\cosh Z} \quad (Z \le D)$
- R_T is the 2-d squared Bessel process (i.e., $R_T = X_T^2 + Y_T^2$ for two independent BMs, X_T and Y_T)
- $\theta \in [0, \pi]$ is a uniformly distributed random angle.
- 1.5 normal variates for sampling one pair of σ_T and F_T.

(3AB) Conditional Simulation for $\rho = 0$ and $0 < \beta < 1$

- F_t follows the CEV process with stochastic time:

$$\frac{dF_\tau}{F_\tau^\beta} = dW_\tau \quad \text{where} \quad \tau = \int_0^t \sigma_s^2 ds = V_{0,t}.$$

Conditional on σ_T and $I_{0,T}$,

$$F_T \sim \text{CEV}(\beta, F_0, \rho_*^2 \sigma_0^2 T I_{0,T}). \quad (\rho_* = 1)$$

- Approximate option price is available in Choi and Wu (2021b).
- How to sample F_T from the CEV distribution?
- However, no exact distribution for F_T if $\rho \neq 0$.

Euler/Milstein scheme

$$F_{t+h} \approx F_t + \sigma_t F_T^\beta \sqrt{h} W + \frac{\beta \sigma_t^2 F_t^{2\beta-1}}{2} h(W^2 - 1),$$

where W is normal variate with $E(WZ) = \rho$.

Option pricing for the uncorrelated ($\rho = 0$) case

Choi and Wu (2021b) A note on the option price and 'Mass at zero in the uncorrelated SABR model and implied volatility asymptotics.' *Quantitative Finance*, 21:1083–1086. arXiv:2011.00557

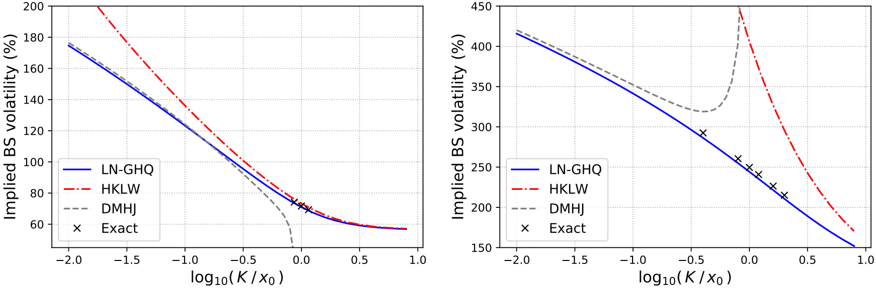
$$C_{\text{SABR}}(K) = E\left(C_{\text{CEV}}(K, \beta, F_0, \sigma_0^2 T I_{0,T})\right) = \int_0^\infty C_{\text{CEV}}(K, \beta, F_0, \sigma_0^2 T y) f(y) dy$$
$$= \sum_{k=1}^{N} C_{\text{CEV}}(K, \beta, F_0, \sigma_0^2 T y_k) w_k$$

- Approximate $I_{0,T}$ with moment-matched (μ_1 and μ_2) lognormal distribution.

$$y_k = \mu_1 \exp\left(\lambda z_k - \lambda^2/2\right), \quad \lambda = \sqrt{\ln\left(\mu_2/\mu_1^2\right)},$$

- (z_k, w_k) is the point and weights of the Gauss-Hermite quadrature.

Figure: The Black-Scholes volatility smile as a function of log strike price for two parameter sets: $(x_0, y_0, \xi, \beta, T) = (0.5, 0.5, 0.4, 0.5, 2)$ (left) and $(x_0, y_0, \xi, \beta, T) = (0.05, 0.4, 0.6, 0.3, 1)$ (right). The mass at zero, estimated by the method of this study, are $m_{\text{SABR}} = 0.1657$ and 0.7624, respectively.



(3A) Approximation for CEV SABR: Islah (2009)

- All SABR MC literature (Chen et al., 2012; Leitao et al., 2017a,b; Cai et al., 2017) use Islah (2009)'s approximation.
- For $q_t = F_t^{\beta_*}/\beta_*$,

$$q_{t+h} = q_t + \frac{\rho}{\xi}(\sigma_{t+h} - \sigma_t) + \int_t^{t+h} \rho_* \sigma_s \, dq_s - \frac{\beta}{2\rho_*^2 \beta_*} \int_t^{t+h} \frac{(\rho_* \sigma_s)^2}{q_s} dt$$

$$\frac{\beta'}{\beta'_*} = \frac{\beta}{\rho_*^2 \beta_*} \quad \Rightarrow \quad \beta' = \frac{\beta}{\beta + \beta_* \rho_*^2}$$

$$\text{Prob}(F_T \le K) \approx \bar{F}_{\chi^2}\left(\frac{\left| q_0 + \frac{\rho}{\xi}(\sigma_T - \sigma_0) \right|^2}{\sigma_0^2 T I_{0,T}}; 1 + \frac{\beta}{\beta_* \rho_*^2}, \frac{q_K^2}{\sigma_0^2 T I_{0,T}} \right).$$

- As $\xi \downarrow 0$, distribution does not converge to CEV (β). $\beta' \neq \beta$.
- Only exact when $\rho = 0$.
- $$\bar{F}_T = \left(\beta'_* \left| q_0 + \frac{\rho}{\xi}(\sigma_T - \sigma_0) \right| \right)^{1/\beta'_*}. \quad E(\bar{F}_T) \neq F_0$$

(3B) Simulation of χ^2 RV: Existing studies

It is not simple to sample F_T from Islah (2009)'s noncentral χ^2 approximation.

- Cai et al. (2017): For a uniform U, solve x with numerical root-finding:

$$\text{Prob}(F_T \le x) = U.$$

- Chen et al. (2012); Leitao et al. (2017a,b): moment-matched quadratic Gaussian Andersen (2008):

$$q_T^2 \sim d(e+Z)^2 \quad \text{for} \quad Z \sim N(0,1).$$

- Grzelak et al. (2019) used stochastic collocation: efficient interpolation using orthogonal polynomial.

(3A) Approximation for CEV SABR: New method

- Conditional on F_t, σ_t, and $I_{t,t+h}$, assume F_{t+h} follows a CEV distribution with β:

$$F_{t+h} \sim \text{CEV}(\beta, \underbrace{\bar{F}_{t+h}}_{\textbf{???}}, \underbrace{\rho_*^2 \sigma_t^2 h I_{t,t+h}}_{\text{Same as other cases}}).$$

- Approximate dynamics with GBM with $\sigma_t := \sigma_t / F_0^{\beta_*}$:

$$\frac{dF_t}{F_t} = \frac{\sigma_t}{F_t^{\beta_*}} dW_t \approx \frac{\sigma_t}{F_0^{\beta_*}} dW_t.$$

- Borrow $\bar{F}_T$ from the $\beta = 1$ case:

$$\bar{F}_{t+h} \approx F_t \exp\left(\frac{\rho}{\xi} \frac{(\sigma_{t+h} - \sigma_t)}{F_t^{\beta_*}} - \frac{\rho^2 \sigma_t^2 h \, I_{t,t+h}}{2 \, F_t^{2\beta_*}} \right)$$

- Exact when $\rho = 0$: $\bar{F}_T = F_t$.
- Preserves martingale: $E(F_T) = E(\bar{F}_T) = F_0$.

(3B) Simulation of the CEV process: Kang (2014)

- Kang (2014). Simulation of the shifted Poisson distribution with an application to the CEV model. *Management Science and Financial Engineering* 20:27–32. 10.7737/MSFE.2014.20.1.027
- Poisson variate, $N \sim$ Pois(λ):

$$1 = e^{\lambda} e^{-\lambda} = \sum_{k=0}^{\infty} \boxed{P_k := \frac{\lambda^k}{k!} e^{-\lambda}}.$$

- Let $F_\Gamma(x;\nu)$ be the CDF of standard gamma distribution (lower incomplete gamma fn):

$$F_\Gamma(x;\nu) = \frac{1}{\Gamma(\nu)} \int_0^x t^{\nu-1} e^{-t} dt = \sum_{k=0}^{\infty} \frac{x^{k+\nu} e^{-x}}{\Gamma(k+\nu+1)}$$

- Shifted Poisson variate, $N \sim$ dPois(λ,ν):

$$1 = \frac{F_\Gamma(\lambda;\nu)}{F_\Gamma(\lambda;\nu)} = \frac{1}{\Gamma(\nu)} \int_0^x t^{\nu-1} e^{-t} dt = \sum_{k=0}^{\infty} \boxed{P_k^\nu := \frac{1}{F_\Gamma(\lambda;\nu)} \frac{\lambda^{k+\nu} e^{-\lambda}}{\Gamma(k+\nu+1)}},$$

- dPois$(\lambda, \nu = 0) =$ Pois(λ).
- Kang (2014) samples as Gamma-mixed Poisson variate:

$$\text{dPois}(\lambda,\nu) \sim \text{Pois}\,(\lambda - X) \quad \text{conditional on} \quad X \sim \Gamma(\nu) \geq \lambda.$$

- Let

$$z_t = \frac{q_t^2}{\sigma_{\text{CEV}}^2 T} = \frac{F_t^{2\beta_*}}{\beta_*^2 \sigma_{\text{CEV}}^2 T} \quad \text{and} \quad \nu = \frac{1}{2\beta_*}$$

- Kang (2014) also shows the transition probability from z_0 to z_T follows the shifted-Poisson-mixed Gamma distribution.

$$f(z_T \,|\, z_0) = \frac{1}{2}\left(\frac{\sqrt{z_0}}{\sqrt{z_T}}\right)^{\nu} I_\nu(\sqrt{z_0\, z})\, e^{-\frac{z+z_0}{2}} = \frac{z_0^\nu}{2^\nu} \sum_{k=0}^{\infty} \frac{(z_0/2)^k e^{-\frac{z_0}{2}}}{\Gamma(k+\nu+1)} \frac{(z_T/2)^k}{2\,k!}\, e^{-\frac{z_T}{2}}$$

$$= \sum_{k=0}^{\infty} \frac{(z_0/2)^{k+\nu} e^{-\frac{z_0}{2}}}{\Gamma(k+\nu+1)} f_\Gamma(z_T; k+1, 1/2) = \sum_{k=0}^{\infty} P_k^\nu \, f_\Gamma(z_T; k+1, 1/2).$$

Simulation algorithm for $F_T \sim \text{CEV}(\beta, F_0, \sigma_{\text{CEV}}^2 T)$ from Kang (2014):

1. Sample $X \sim 2\Gamma\left(\frac{1}{2\beta_*}\right)$.
2. If $X \geq z_0 = \frac{F_0^{2\beta_*}}{\beta_*^2 \sigma_{\text{CEV}}^2 T}$, $F_T = 0$ and terminate.
3. Else, sample $z_T \sim 2\Gamma\left(\text{Pois}\left(\frac{z_0 - X}{2}\right) + 1\right)$.
4. Obtain F_T from $z_T = \frac{F_T^{2\beta_*}}{\beta_*^2 \sigma_{\text{CEV}}^2 T}$.

- Let

$$z_t = \frac{q_t^2}{\sigma_{\rm CEV}^2 T} = \frac{F_t^{2\beta_*}}{\beta_*^2 \sigma_{\rm CEV}^2 T} \quad \text{and} \quad \nu = \frac{1}{2\beta_*}$$

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$$= \sum_{k=0}^\infty \frac{(z_0/2)^{k+\nu} e^{-\frac{z_0}{2}}}{\Gamma(k+\nu+1)} f_\Gamma(z_T; k+1, 1/2) = \sum_{k=0}^\infty P_k^\nu\, f_\Gamma(z_T; k+1, 1/2).$$

Simulation algorithm for $F_{t+h} \sim \mathrm{CEV}(\beta, F_t, \rho_*^2\sigma_t^2 h I_{t,t+h})$ from Kang (2014):

1. Sample $X \sim 2\Gamma\left(\frac{1}{2\beta_*}\right)$.
2. If $X \geq z_t = \frac{F_t^{2\beta_*}}{\rho_*^2\sigma_t^2 h I_{t,t+h}}$, $F_{t+dt} = 0$ and terminate.
3. Else, sample $z_{t+dt} \sim 2\Gamma\left(\text{Pois}\left(\frac{z_t - X}{2}\right) + 1\right)$.
4. Obtain F_{t+dt} from $z_{t+dt} = \frac{F_{t+dt}^{2\beta_*}}{\rho_*^2\sigma_t^2 h I_{t,t+h}}$.

Three simulation step

- Simulate σ_{t+h} from σ_t (trivial):

$$\sigma_{t+h} = \sigma_t \exp\left(\xi\sqrt{h}Z - \xi^2 h/2\right), \quad Z \sim N(0,1).$$

- (1) Simulate $I_{t,t+h}$ given σ_t and σ_{t+h}.
  - Trapezoidal:
  - Lognormal:
  - Shifted lognormal:
- Simulate F_{t+h} given F_t, σ_{t+h}, and $I_{0,T}$.
  - For $0 < \beta < 1$, approximate F_{t+h} with a martingale-preserving CEV distribution.
  - Sample F_{t+h} based on Kang (2014)'s algorithm.

Numerical Results: Case 4. Accuracy of $I_{t,t+h}$

Cases	F_0	σ_0	ξ	ρ	β	T	K
Case IV	0.05	0.4	0.6	0	0.3	1	[0.02,0.1]

- With $\rho = 0$, CEV approximation is accurate.
- This case validates our LN approximation of $I_{t,t+h}$.
- With one jump ($h = 1$), our method is more accurate than existing methods.

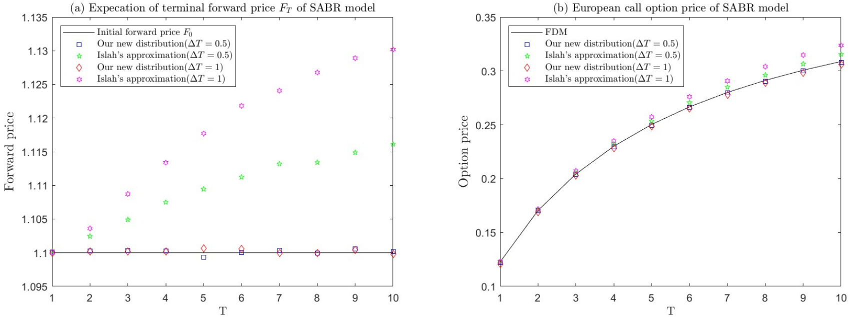
K/F_0	h	40%	80%	100%	120%	160%	200%	
Exact Price								
FDM		0.0456	0.0414	0.0394	0.0375	0.0339	0.0306	
Price Error ($\times 10^{-3}$)								CPU
Euler	1/400	1.60	1.50	1.50	1.40	1.30	1.20	49.40
	1/800	0.70	0.60	0.50	0.50	0.40	0.30	99.10
	1/1600	-0.30	-0.30	-0.30	-0.30	-0.30	-0.30	194.00
Low-bias (Chen et al., 2012)	1/4	0.50	0.50	0.50	0.40	0.40	0.40	78.40
	1/8	0.40	0.40	0.40	0.30	0.30	0.20	176.00
Exact (Cai et al., 2017)	1	0.10	0.20	0.20	0.20	0.20	0.20	98.30
LN	1	-0.01	-0.01	0.00	-0.02	-0.03	-0.04	0.03

Note: "Euler", "Low-bias", and "Exact" results are from other papers.

Numerical Results: Case 6. Martingale preservation

Cases	F_0	σ_0	ξ	ρ	β	T	K
Case VI	1.1	0.3	0.5	-0.8	0.4	[1,10]	1.1

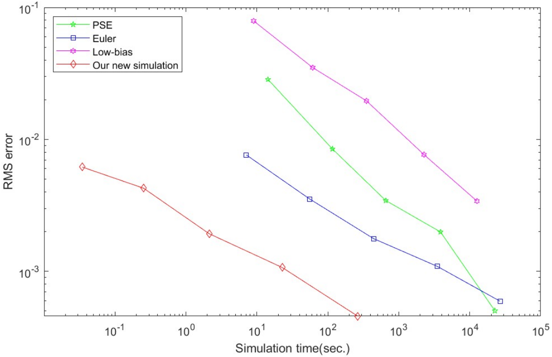
- We measure $E(F_T)$ and $E((F_T - K)^+)$ as functions of T.
- Our CEV approximation for F_{t+h} preserves martingale ($E(F_T) = F_0$).
- Naturally, option price is more accurate.



Numerical Results: Case 5. RMS v.s. CPU time

Cases	F_0	σ_0	ξ	ρ	β	T	K
Case V	1.1	0.4	0.8	-0.3	0.3	4	1.1

- We plot RMSE vs CPU time.
- For the same level of error, our new method is at least 100 times faster than existing methods.



Numerical Results: Case 1

Cases	F_0	σ_0	ξ	ρ	β	T	K
Case I	1	0.25	0.3	-0.8	0.3	10	[0.2,2]

- Our methods (LN, S-LN, TZ) accurately price challenging cases ($\xi = 0.3$ and $T = 10$.)

K	0.20	0.40	0.80	1.00	1.20	1.60	2.00
FDM	0.843	0.689	0.406	0.285	0.183	0.053	0.011
Error ($\times 10^{-3}$) for $h = 1$							
LN	-1.17	-1.45	-0.35	0.49	1.23	1.60	1.22
S-LN	-1.22	-1.49	-0.37	0.49	1.28	1.72	1.32
TZ	-3.13	-3.41	-2.15	-1.11	-0.12	0.79	0.65
Stdev	1.91	1.73	1.33	1.13	0.94	0.52	0.28
Error ($\times 10^{-3}$) for $h = 1/2$							
LN	-0.17	-0.09	0.55	0.91	1.22	1.15	0.80
S-LN	-0.01	0.10	0.78	1.16	1.43	1.36	0.98
TZ	-1.26	-1.10	-0.25	0.26	0.66	0.75	0.54
Stdev	1.83	1.65	1.26	1.06	0.85	0.50	0.27
Error ($\times 10^{-3}$) for $h = 1/4$							
LN	-0.35	-0.15	0.32	0.53	0.66	0.59	0.51
S-LN	-0.46	-0.24	0.22	0.42	0.56	0.56	0.48
TZ	-0.60	-0.36	0.12	0.31	0.41	0.41	0.33
Stdev	2.34	2.12	1.63	1.36	1.10	0.61	0.31

Numerical Results: Case 2

Cases	F_0	σ_0	ξ	ρ	β	T	K
Case II	1	0.25	0.3	-0.5	0.6	10	[0.2,2]

- Our methods (LN, S-LN, TZ) accurately price challenging cases ($\xi = 0.3$ and $T = 10$.)

K	0.20	0.40	0.80	1.00	1.20	1.60	2.00
FDM	0.829	0.670	0.398	0.291	0.207	0.100	0.050
Error ($\times 10^{-3}$) for $h = 1$							
LN	-0.04	-0.22	-0.30	-0.29	-0.30	-0.33	-0.25
S-LN	-0.14	-0.30	-0.42	-0.43	-0.43	-0.40	-0.30
TZ	-0.24	-0.38	-0.43	-0.37	-0.32	-0.30	-0.27
Stdev	2.11	1.98	1.73	1.61	1.44	1.11	0.91
Error ($\times 10^{-3}$) for $h = 1/2$							
LN	-0.34	-0.43	-0.45	-0.43	-0.39	-0.31	-0.21
S-LN	0.04	-0.05	-0.09	-0.09	-0.10	-0.11	-0.04
TZ	-0.57	-0.64	-0.59	-0.53	-0.48	-0.40	-0.27
Stdev	2.36	2.21	1.83	1.62	1.42	1.06	0.82
Error ($\times 10^{-3}$) for $h = 1/4$							
LN	0.37	0.27	0.15	0.12	0.07	-0.02	-0.02
S-LN	0.45	0.37	0.27	0.20	0.10	-0.02	0.00
TZ	0.08	-0.01	-0.05	-0.04	-0.05	-0.12	-0.11
Stdev	2.44	2.31	1.98	1.76	1.53	1.12	0.86

Numerical Results: Case 3

Cases	F_0	σ_0	ξ	ρ	β	T	K
Case III	1	0.25	0.3	-0.2	0.9	10	[0.2,2]

- Our methods (LN, S-LN, TZ) accurately price challenging cases ($\xi = 0.3$ and $T = 10$.)

K	0.20	0.40	0.80	1.00	1.20	1.60	2.00
FDM	0.816	0.652	0.393	0.303	0.236	0.151	0.105
Error ($\times 10^{-3}$) for $h = 1$							
LN	-0.83	-0.85	-0.79	-0.77	-0.75	-0.68	-0.65
S-LN	0.78	0.79	0.79	0.81	0.82	0.85	0.91
TZ	-0.63	-0.68	-0.63	-0.54	-0.50	-0.45	-0.45
Stdev	3.80	3.78	3.66	3.57	3.47	3.25	3.07
Error ($\times 10^{-3}$) for $h = 1/2$							
LN	-0.73	-0.77	-0.90	-0.89	-0.88	-0.83	-0.81
S-LN	0.50	0.48	0.42	0.39	0.38	0.44	0.47
TZ	-0.66	-0.66	-0.66	-0.62	-0.65	-0.70	-0.67
Stdev	4.94	4.89	4.72	4.61	4.50	4.30	4.17
Error ($\times 10^{-3}$) for $h = 1/4$							
LN	0.19	0.16	0.06	0.06	0.10	0.14	0.14
S-LN	-0.82	-0.85	-0.90	-0.92	-0.94	-0.99	-1.00
TZ	5.74	5.72	5.74	5.81	5.91	5.97	6.01
Stdev	4.31	4.24	4.04	3.96	3.88	3.75	3.63

Summary and Conclusion

- The SABR model is an SV model widely used in industry and academia.
- We propose an efficient simulation scheme for the SABR model.
  - Use the moment-matched lognormal sampling for the integrated variacne ($I_{t,t+h}$).
  - Adopt Kang (2014)'s CEV simulation algorithm in the context of the SABR model.
- Our new algorithm outperforms the existing algorithm in terms of error-vs-speed tradeoff.

Python implementation: PyFENG

PyFENG (Python Financial ENGineering)

- Implements standard quant finance methods: Black–Scholes, Bachelier (normal), CEV, SABR, Heston models and etc.
- PyPI package: `pip install pyfeng`
- Source: https://github.com/PyFE/PyFENG/
- Implemented in pure Python (no C/C++ extensions).

PyFengForPapers

- Inspired by PapersWithCode project.
- https://github.com/PyFE/PyfengForPapers/
- A collection of Jupyter Notebooks reproducing quant finance papers (mostly in derivative pricing and stochastic volatility) using the methods implemented in PyFENG.
- The code for this talk is also available in PyFengForPapers.

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