A Guide to Volatility and Variance Swaps

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Volatility swaps are forward contracts on future realized stock volatility. Variance swaps are similar contracts on variance, the square of future volatility. Both these instruments provide an easy way for investors to gain exposure to the future level of volatility.

Unlike a stock option, whose volatility exposure is contaminated by its stock price dependence, these swaps provide pure exposure to volatility alone. You can use these instruments to speculate on future volatility levels, to trade the spread between realized and implied volatility, or to hedge the volatility exposure of other positions or businesses.

In this report we explain the properties and the theory of both variance and volatility swaps, first from an intuitive point of view and then more rigorously. The theory of variance swaps is the more straightforward. We show how a variance swap can be theoretically replicated by a hedged portfolio of standard options with suitably chosen strikes, as long as stock prices evolve without jumps. The fair value of the variance swap is the cost of the replicating portfolio.

We derive an analytic formula for theoretical fair value in the presence of realistic volatility skews. This formula can be used to estimate swap values quickly as the skew changes.

We then examine modifications to these theoretical results when reality intrudes, such as when some necessary strikes are unavailable, or when stock prices undergo jumps. Finally, we point out that volatility swaps can be replicated by dynamically trading the more straightforward variance swap. As a result, the value of the volatility swap depends on the volatility of volatility itself.

A stock's volatility is the simplest measure of its riskiness or uncertainty. Formally, the volatility σ_R is the annualized standard deviation of the stock's returns during the period of interest, where the subscript R denotes the observed or "realized" volatility. This note is concerned with volatility swaps and other instruments suitable for trading volatility.

Why trade volatility? Just as stock investors think they know something about the direction of the stock market, or bond investors think they can foresee the probable direction of interest rates, so you may think you have insight into the level of future volatility. If you think current volatility is low, for the right price you might want to take a position that profits if volatility increases.

Investors who want to obtain pure exposure to the direction of a stock price can buy or short the stock. What do you do if you simply want exposure to a stock's volatility?

Stock options are impure: They provide exposure to both the direction of the stock price and its volatility. If you hedge the options according to the Black-Scholes prescription, you can remove the exposure to the stock price. But delta-hedging is at best inaccurate, because the real world violates many of the Black-Scholes assumptions. Volatility cannot be accurately estimated, stocks cannot be traded continuously, transactions costs cannot be ignored, markets sometimes move discontinuously, and liquidity is often a problem. Nevertheless, imperfect as they are, until recently options were the only volatility vehicle available.

I. VOLATILITY SWAPS

The easy way to trade volatility is to use *volatility swaps*, sometimes called *realized volatility forward contracts*, because they provide pure exposure to volatility (and only to volatility).[1]

A stock volatility swap is a forward contract on annualized volatility. Its payoff at expiration is equal to

$$N(\sigma_R - K_{vol}) \tag{1}$$

where σ_R is the realized stock volatility (quoted in annual terms) over the life of the contract, K_{vol} is the annualized volatility delivery price, and N is the notional amount of the swap in dollars per annualized volatility point.

The holder of a volatility swap at expiration receives N dollars for every point by which the stock's realized volatility σ_R has exceeded the volatility delivery price K_{vol}. He or she is swapping a fixed volatility K_{vol} for the actual (floating) future volatility σ_R.

The delivery price K_{vol} is typically quoted as a volatility, such as 30%. The notional amount is typically quoted in dollars per volatility point, for example, N = $250,000/(volatility point). As with all forward contracts or swaps, the fair value of volatility at any time is the delivery price that makes the swap currently have zero value.

The procedure for calculating the realized volatility should be clearly specified with respect to the several aspects:

- The source and observation frequency of stock or index prices — for example, using daily closing prices of the S&P 500 index.
- The annualization factor in moving from daily or hourly observations to annualized volatilities — for example, using 260 business days per year as a multiplicative factor in computing annualized variances from daily returns.
- Whether the standard deviation of returns is calculated by subtracting the sample mean from each return, or by assuming a zero mean. The zero mean method is theoretically preferable, because it corresponds most closely to the contract that can be replicated by options portfolios. For frequently observed prices, the difference is usually negligible.

Volatility has several characteristics that make trading attractive. It is likely to grow when uncertainty and risk increase. As with interest rates, volatilities appear to revert to the mean: High volatilities will eventually decrease; low ones will likely rise. Finally, volatility is often negatively correlated with stock or index level, and tends to stay high after large downward moves in the market. Given these tendencies, there are several uses for volatility swaps.

Directional Trading of Volatility Levels. Clients who want to speculate on the future levels of stock or index volatility can go long or short realized volatility with a swap. This provides a much more direct method than trading and hedging options.

For example, if you foresee a rapid decline in political and financial turmoil after a forthcoming election, a short position in volatility might be appropriate.

Trading the Spread Between Realized and Implied Volatility Levels. As we show later, the fair delivery price K_{vol} of a volatility swap is a value close to the level of current implied volatilities for options with the same expiration as the swap. Therefore, by unwinding the swap before expiration, you can trade the spread between realized and implied volatility.

Hedging Implicit Volatility Exposure. There are several businesses that are implicitly short volatility:

- Risk arbitrageurs or hedge funds often take positions that assume that the spread between stocks of companies planning mergers will narrow. If overall market volatility increases, the merger may become less likely, and the spread may widen.
- Investors following active benchmarking strategies may require more frequent rebalancing and incur higher transaction expenses during volatile periods.

- Portfolio managers who are judged against a benchmark have tracking error that may increase in periods of higher volatility.
- Equity funds are probably short volatility because of the negative correlation between index level and volatility. As global equity correlations have increased, diversification across countries has become a less effective portfolio hedge. Since volatility is one of the few parameters that tends to increase during global equity declines, a long volatility hedge may be appropriate, especially for financial businesses.

II. VARIANCE SWAPS

Although options market participants talk of volatility, it is variance, or volatility squared, that has more fundamental theoretical significance. This is so because the correct way to value a swap is to value the portfolio that replicates it, and the swap that can be replicated most reliably (by portfolios of options of varying strikes, as we show later) is a *variance swap*.

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$N(\sigma_R^2 - K_{var}) \tag{2}$$

where σ_R^2 is the realized stock variance (quoted in annual terms) over the life of the contract, K_{var} is the delivery price for variance, and N is the notional amount of the swap in dollars per annualized volatility point squared. The holder of a variance swap at expiration receives N dollars for every point by which the stock's realized variance σ_R^2 has exceeded the variance delivery price K_{var}.

Although theoretically simpler, variance swaps are less commonly traded, so their quoting conventions vary. The delivery price K_{var} can be quoted as a volatility squared, for example, $(30\%)^2$. Similarly, for example, the notional amount can be expressed as $\$100{,}000/(\text{one volatility point})^2$. The fair value of variance is the delivery price that makes the swap of zero value.

Most of this article focuses on the theory and properties of variance swaps, which provide similar volatility exposure to straight volatility swaps. Because of its fundamental role, variance can serve as the basic building block for constructing other volatility-dependent instruments.

III. REPLICATING VARIANCE SWAPS: FIRST STEPS

We first explain the replicating strategy that captures realized variance. The cost of implementing that strategy is the fair value of future realized variance.

The Intuitive Approach

We approach variance replication by building on the standard Black-Scholes model. Later we also provide a more general proof that you can replicate variance even when some of the Black-Scholes assumptions fail, as long as the stock price evolves continuously — that is, without jumps.

We ease the development of intuition by assuming here that the riskless interest rate is zero. Suppose at time t you own a standard call option of strike K and expiration T, whose value is given by the Black-Scholes formula $C_{BS}(S, K, \sigma\sqrt{\tau})$, where S is the current stock price, σ is the return volatility of the stock, τ is the time to expiration $(T - t)$, and $v = \sigma^2\tau$ is the total variance of the stock to expiration. (We have written the option value as a function of $\sigma\sqrt{\tau}$ in order to make clear that all its dependence on volatility and time to expiration is expressed through the combined variable $\sigma\sqrt{\tau}$.)

We define the exposure of an option to a stock's variance, $\mathcal{V}$, by

$$\mathcal{V} = \frac{\partial C_{BS}}{\partial \sigma^2} = \frac{S\sqrt{\tau}}{2\sigma}\frac{\exp(-d_1^2/2)}{\sqrt{2\pi}} \tag{3}$$

where

$$d_1 = \frac{\log(S/K) + (\sigma^2\tau)/2}{\sigma\sqrt{\tau}} \tag{4}$$

The variance sensitivity, $\mathcal{V}$, measures the change in value of the position resulting from a change in variance, which we will sometimes refer to as "variance vega." Note that d_1 depends only on the two combinations S/K and $\sigma\sqrt{\tau}$, and that $\mathcal{V}$ decreases extremely rapidly as S leaves the vicinity of the strike K.

Exhibit 1A shows a graph of how $\mathcal{V}$ varies with stock price S, for each of three different options with strikes 80, 100, and 120. For each strike, the variance exposure $\mathcal{V}$ is greatest when the option is at the money, and falls off rapidly as the stock price moves in or out of the money. $\mathcal{V}$ is closely related to the time sensitivity or time decay of the option, because, in the Black-Scholes formula with zero interest rate, options values depend on the total variance $\sigma^2\tau$.

If you want a long position in future realized variance, a single option is an imperfect vehicle; as soon as the stock price moves, your sensitivity to further changes in variance is altered. What you want is a portfolio whose sensitivity to realized variance is independent of the stock price S.

To obtain a portfolio that responds to volatility or variance independent of moves in the stock price, you need to combine options of many strikes. What combination of strikes will give you such undiluted variance exposure?

Exhibit 1B shows the variance exposure for the portfolio consisting of all three option strikes in Exhibit 1A. The dotted line represents the sum of equally weighted strikes; the solid line represents the sum with weights in inverse proportion to the square of their strike.

Exhibits 1C, E, and G show the individual sensitivities to variance of increasing numbers of options, each panel with the options more closely spaced. Exhibits 1D, F, and H show the sensitivity for the equally weighted and strike-weighted portfolios. Clearly, the portfolio with weights inversely proportional to K^2 produces a $\mathcal{V}$ that is virtually independent of stock price S, as long as S lies inside the range of available strikes and far from the edge of the range, and provided the strikes are distributed evenly and closely.

Appendix A provides a mathematical derivation of the requirement that options be weighted inversely proportional to K^2 in order to achieve constant $\mathcal{V}$. You can also understand this intuitively. As the stock price moves up to higher values, each additional option of higher strike in the portfolio will provide an additional contribution to $\mathcal{V}$ proportional to that strike. This follows from Equation (3), and you can observe it in the increasing height of the $\mathcal{V}$-peaks in Exhibit 1A. An option with higher strike will therefore produce a $\mathcal{V}$ contribution that increases with S.

In addition, the contributions of all options overlap at any definite S. Therefore, to offset this accumulation of S-dependence, one needs diminishing amounts of higher-strike options, with weights inversely proportional to K^2.

If you own a portfolio of options of all strikes, weighted in inverse proportion to the square of the strike level, you will obtain an exposure to variance that is independent of stock price, just what is needed to trade variance. What does this portfolio of options look like, and how does trading it capture variance?

Consider the portfolio $\Pi(S, \sigma\sqrt{\tau})$ of options of all strikes K and a single time to expiration T, weighted inversely proportional to K^2. Because out-of-the-money options are generally more liquid, we employ put options $P(S, K, \sigma\sqrt{\tau})$ for strikes K varying continuously from zero up to some reference price S_*, and call options $C(S, K, \sigma\sqrt{\tau})$ for strikes varying continuously from S_* to infinity:

$$\Pi(S, S_*, \sigma\sqrt{\tau}) = \sum_{K>S_*} \frac{1}{K^2} C(S, K, \sigma\sqrt{\tau}) + \sum_{K<S_*} \frac{1}{K^2} P(S, K, \sigma\sqrt{\tau}) \tag{5}$$

You can think of S_* as the approximate at-the-money forward stock level that marks the boundary between liquid puts and liquid calls.

At expiration, when t = T, one can show that the sum of all the payoff values of the options in the portfolio is simply

$$\Pi(S_T, S_*, 0) = \frac{S_T - S_*}{S_*} - \log\left(\frac{S_T}{S_*}\right) \tag{6}$$

where log( ) denotes the natural logarithm function, and S_T is the terminal stock price.

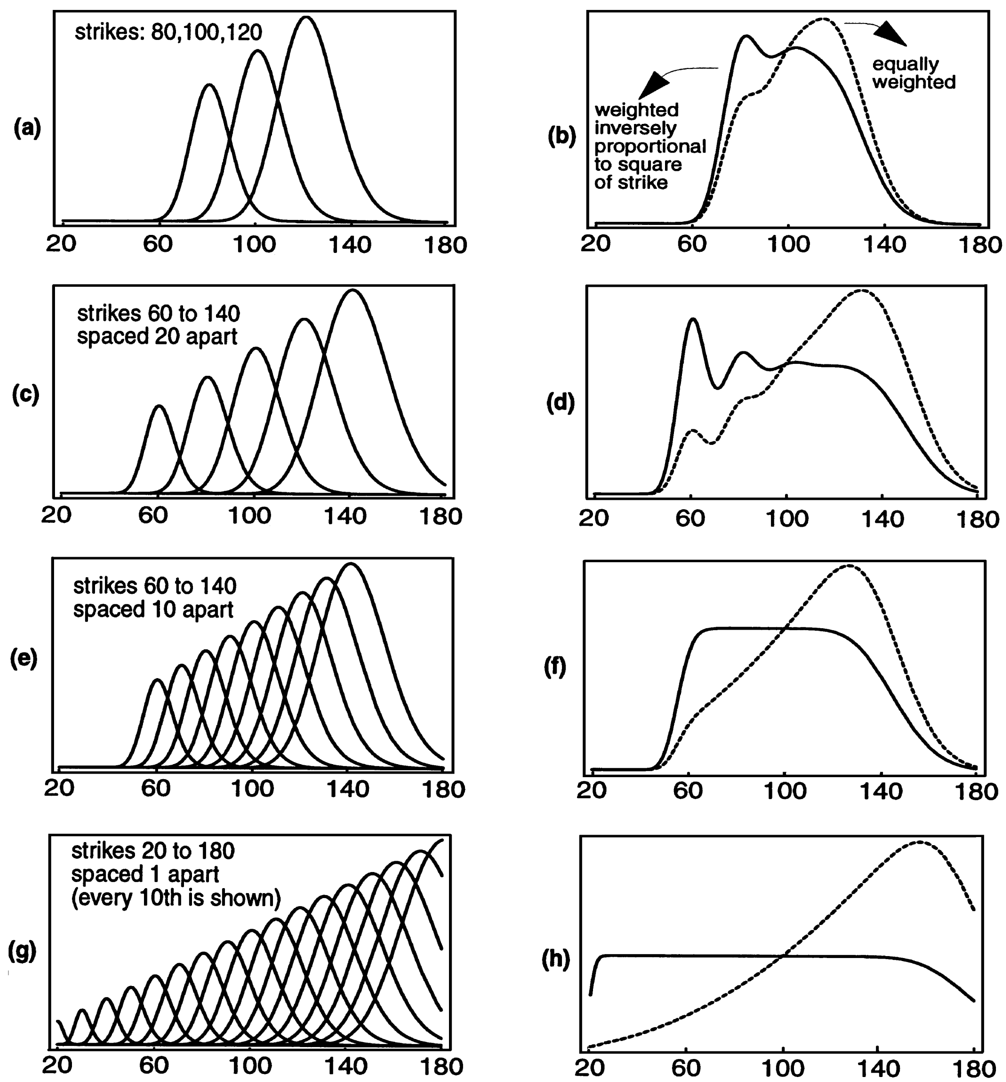
Similarly, at time t you can sum all the Black-Scholes options values to show that the total portfolio value is

$$\Pi(S, S_*, \sigma\sqrt{\tau}) = \frac{S - S_*}{S_*} - \log\left(\frac{S}{S_*}\right) + \frac{\sigma^2\tau}{2} \tag{7}$$

where S is the stock price at time t. Note how little the

EXHIBIT 1

Variance Exposure, $\mathcal{V}$, of Portfolios of Call Options of Different Strikes as a Function of Stock Price S



Graphs on the left show the individual $\mathcal{V}_i$ contributions for each option of strike K_i. Graphs on the right show the sum of the contributions, weighted two different ways; the dotted line corresponds to an equally weighted sum of options; the solid line corresponds to weights inversely proportional to K^2, and becomes totally independent of stock price S inside the strike range.

value of the portfolio before expiration differs from its value at expiration at the same stock price. The only difference is the additional value due to half the total variance $\sigma^2\tau$.

Clearly, the variance exposure of Π is

$$\mathcal{V} = (\tau/2) \tag{8}$$

To obtain an initial exposure of \$1 per volatility point squared, you need to hold (2/T) units of the portfolio Π. From now on, we will use Π to refer to the value of this new portfolio, namely:

$$\Pi(S, S_*, \sigma\sqrt{\tau}) = \frac{2}{T}\left[\frac{S - S_*}{S_*} - \log\left(\frac{S}{S_*}\right)\right] + \frac{\tau}{T}\sigma^2 \tag{9}$$

The first term in the payoff in Equation (9), $(S - S_*)/S_*$, describes $1/S_*$ forward contracts on the stock with delivery price S_*. It is not really an option; its value represents a long position in the stock (value S) and a short position in a bond (face value S_*), which can be statically replicated, once and for all, without any knowledge of the stock's volatility.

The second term, $-\log(S/S_*)$, describes a short position in a *log contract* with reference value S_*, a so-called exotic option whose payoff is proportional to the log of the stock at expiration, and whose correct hedging depends on the volatility of the stock. All the volatility sensitivity of the weighted portfolio of options we have created is incorporated in the log contract.[2]

Exhibit 2 illustrates the equivalence between 1) the summed, weighted payoffs of puts and calls, and 2) a long position in a forward contract and a short position in a log contract.

Trading Realized Volatility with a Log Contract

Assume that we are in a Black-Scholes world where the implied volatility σ_I is the estimate of future realized volatility. If you take a position in the portfolio Π, the fair value you should pay at time $t = 0$ when the stock price is S_0 is

$$\Pi_0 = \frac{2}{T}\left[\frac{S_0 - S_*}{S_*} - \log\left(\frac{S_0}{S_*}\right)\right] + \sigma_I^2$$

At expiration, if the realized volatility turns out to have been σ_R, the initial fair value of the position would have been

$$\Pi_0 = \frac{2}{T}\left[\frac{S_0 - S_*}{S_*} - \log\left(\frac{S_0}{S_*}\right)\right] + \sigma_R^2$$

The net payoff on the position, hedged to expiration, will be

$$\text{Payoff} = (\sigma_R^2 - \sigma_I^2) \tag{10}$$

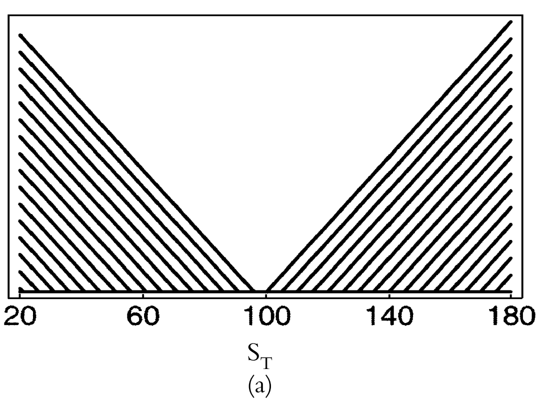
Looking back at Equation (2), you will see that by rehedging the position in log contracts, you have, in effect, been the owner of a position in a variance swap with fair strike $K_{var} = \sigma_I^2$ and face value \$1. You will have profited (or lost) if realized volatility has exceeded (or been exceeded by) implied volatility.

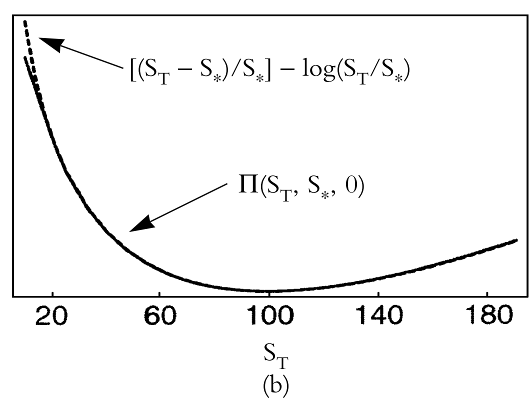
Vega, Gamma, and Theta of a Log Contract

In a Black-Scholes world with zero interest rates

EXHIBIT 2

Equivalence at Expiration of Weighted Payoffs of Puts and Calls and Payoff of Long Position in Forward Contract and Short Position in Log Contract





(a) Individual contributions to the payoff from put options with all integer strikes from 20 to 99, and call options with all integer strikes from 100 to 180. Only every fifth option is shown for clarity. (b) The payoff of 1/100 of a long position in a forward contract with delivery price 100 and one short position in a log contract with reference value 100.

and zero dividend yield, the portfolio of options whose variance vega is independent of the stock price S is given by Equation (9). Only the log( ) term in Equation (9) needs continual dynamic rehedging. Therefore, let us concentrate on the log contract term alone, whose value at time t for a logarithmic payoff at time T is

$$L(S, S_*, \sigma, t, T) = -\frac{2}{T}\log\left(\frac{S}{S_*}\right) + \frac{(T-t)}{T}\sigma^2 \tag{11}$$

The sensitivities of the value of this portfolio are precisely appropriate for trading variance. The variance vega of the portfolio in Equation (11) is

$$\mathcal{V} = \left(\frac{T-t}{T}\right) \tag{12}$$

The exposure to variance is equal to 1 at $t = 0$, and decreases linearly to zero as the contract approaches expiration.

The time decay of the log contract, the rate at which its value changes if the stock price remains unchanged as time passes, is

$$\theta = -\frac{1}{T}\sigma^2 \tag{13}$$

The contract loses time value at a constant rate proportional to its variance, so that at expiration, all the initial variance has been lost. The log contract's exposure to stock price is

$$\Delta = -\frac{2}{T}\frac{1}{S}$$

shares of stock. That is, since each share of stock is worth S, you need a constant long position in \$(2/T) worth of stock to be hedged at any time.

The gamma of the portfolio, the rate at which the exposure changes as the stock price moves, is

$$\Gamma = \frac{2}{T}\frac{1}{S^2} \tag{14}$$

Gamma is a measure of the risk of hedging an option. The log contract's gamma, being the sum of the gammas of a portfolio of puts and calls, is a smoother function of S than the sharply peaked gamma of a single option.

Equations (13) and (14) can be combined to show that

$$\theta + \frac{1}{2}\Gamma S^2\sigma^2 = 0 \tag{15}$$

Equation (15) is the essence of the Black-Scholes option pricing theory. It states that the disadvantage of negative theta (the decrease in value as time to expiration elapses) is offset by the benefit of positive gamma (the curvature of the payoff).

Imperfect Hedges

It takes an infinite number of strikes, appropriately weighted, to replicate a variance swap. In practice, this isn't possible. Even when the stock and options markets satisfy all the Black-Scholes assumptions, there are only a finite number of options available at any maturity.

Exhibit 1 illustrates that a finite number of strikes fail to produce a uniform $\mathcal{V}$ as the stock price moves outside the range of the available strikes. As long as the stock price remains within the strike range, trading the imperfectly replicated log contract will allow variance to accrue at the correct rate. Whenever the stock price moves outside, the reduced vega of the imperfectly replicated log contract will make it less responsive than a true variance swap.

Exhibit 3 shows how the variance vega of a three-month variance swap is affected by imperfect replication. Exhibit 3A shows the ideal variance vega that results from a portfolio of puts and calls of all strikes from zero to infinity, weighted in inverse proportion to the strike squared. Here the variance vega is independent of stock price level, and decreases linearly with time to expiration, as expected for a swap whose value is proportional to the remaining variance $\sigma^2\tau$ at any time.

Exhibit 3B shows strikes from \$75 to \$125, uniformly spaced \$1 apart. Here, deviation from constant-variance vega develops at the tail of the strike range, and the deviation is greater at earlier times.

Finally, Exhibit 3C shows the vega for strikes from \$20 to \$200, spaced \$10 apart. Now, although the range of strikes is wider, the coarser spacing causes the vega surface to develop corrugations between strike values that grow more pronounced closer to expiration.

Limitations of the Intuitive Approach

A variance swap has a payoff proportional to realized variance. Assuming the Black-Scholes world for stock and options markets, we have shown that the dynamic, continuous hedging of a log contract produces a payoff whose value is proportional to future realized variance. We have also shown that you can use a portfolio of appropriately weighted puts and calls to approximate a log contract.

Our somewhat intuitive derivations assume that interest rates and dividend yields are zero, but it is not hard to generalize them. We also assume that all the Black-Scholes assumptions hold. In practice, in the presence of an implied volatility skew, it is difficult to extend these arguments clearly.

We need a more general and rigorous derivation of the value of variance swaps based on replication. Many of the results are similar, but the conditions under which they hold, and the correct answers when they do not hold, are more easily understandable.

IV. REPLICATING VARIANCE SWAPS: GENERAL RESULTS

We now show that the dynamic hedging of a log contract captures realized volatility under more general conditions. The only assumption we need make about the evolution of the future underlying is that it is diffusive, or continuous — this means that no jumps are allowed. Therefore, we assume that the stock price evolution is given by

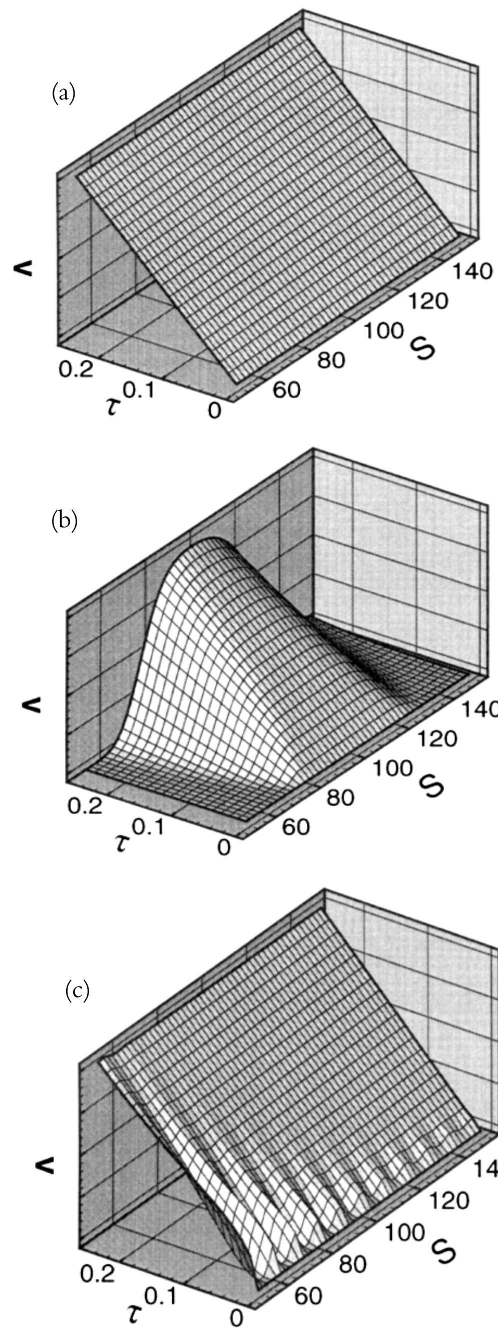
$$\frac{dS_t}{S_t} = \mu(t,\ldots,)dt + \sigma(t,\ldots,)dZ_t \qquad (16)$$

Here, the drift μ and the continuously sampled volatility σ are arbitrary functions of time and other parameters. The assumptions include, but are not restricted to, implied tree models in which the volatility is a function $\sigma(S, t)$ of stock price and time only. For simplicity of presentation, we assume the stock pays no dividends; allowing for dividends does not significantly alter the derivation.

The theoretical definition of realized variance

EXHIBIT 3

Variance Vega of Portfolio of Puts and Calls, Weighted Inversely Proportional to the Square of the Strike Level, and Chosen to Replicate a Three-Month Variance Swap



(a) An infinite number of strikes. (b) Strikes from \$75 to \$125, uniformly spaced \$1 apart. (c) Strikes from \$20 to \$200, uniformly spaced \$10 apart.

for a given price history is the continuous integral

$$V = \frac{1}{T}\int_0^T \sigma^2(t,\ldots)dt \tag{17}$$

This is a good approximation of the variance of daily returns used in the contract terms of most variance swaps.

Conceptually, valuing a variance forward contract or "swap" is no different from valuing any other derivative security. The value of a forward contract F on future realized variance with strike K is the expected present value of the future payoff in the risk-neutral world:

$$F = E[e^{-rT}(V - K)] \tag{18}$$

where r is the risk-free discount rate corresponding to the expiration date T, and E[ ] denotes the expectation.

The fair delivery value of future realized variance is the strike K_{var} for which the contract has zero present value:

$$K_{var} = E[V] \tag{19}$$

If the future volatility in Equation (16) is specified, then one approach for calculating the fair price of variance is to directly calculate the risk-neutral expectation

$$K_{var} = \frac{1}{T}E\left[\int_0^T \sigma^2(t,\ldots)dt\right] \tag{20}$$

No one knows with certainty the value of future volatility. In implied tree models, the so-called local volatility $\sigma(S, t)$ consistent with all current options prices is extracted from the market prices of traded stock options.[3] You can then use simulation to calculate the fair variance K_{var} as the average of the experienced variance along each simulated path consistent with the risk-neutral stock price evolution given in Equation (16), where the drift μ is set equal to the riskless rate.

This approach is good for valuing the contract, but it does not provide insight into how to replicate it. The essence of the replication strategy is to devise a position that, over the next instant of time, generates a payoff proportional to the incremental variance of the stock during that time.[4]

By applying Itô's lemma to log S_t, we find

$$d(\log S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dZ_t \tag{21}$$

Subtracting Equation (21) from Equation (16), we obtain

$$\frac{dS_t}{S_t} - d(\log S_t) = \frac{1}{2}\sigma^2 dt \tag{22}$$

in which all dependence on the drift μ has canceled. Integrating Equation (22) over all times from 0 to T, we obtain the continuously sampled variance

$$\begin{aligned} V &\equiv \frac{1}{T}\int_0^T \sigma^2 dt \\ &= \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}\right] \end{aligned} \tag{23}$$

This mathematical identity dictates the replication strategy for variance. The first term in the brackets can be thought of as the net outcome of continuous rebalancing of a stock position so that it is always instantaneously long $1/S_t$ shares of stock worth \$1. The second term represents a static short position in a contract that, at expiration, pays the logarithm of the total return. Following this continuous rebalancing strategy captures the realized variance of the stock from inception to expiration at time T.

Note that no expectations or averages have been taken. Equation (23) guarantees that variance can be captured no matter which path the stock price takes, as long as it moves continuously.

Equation (23) provides another method for calculating the fair variance. Instead of averaging over future variances, as in Equation (20), one can take the expected risk-neutral value of the right-hand side of Equation (23) to obtain the cost of replication directly, so that

$$K_{var} = \frac{2}{T}E\left[\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}\right] \tag{24}$$

The expected value of the first term in Equation

(24) accounts for the cost of rebalancing. In a risk-neutral world with a constant risk-free rate r, the underlying evolves according to:

$$\frac{dS_t}{S_t} = r dt + \sigma(t, \ldots) dZ \tag{25}$$

so that the risk-neutral price of the rebalancing component of the hedging strategy is given by

$$E\left[\int_0^T \frac{dS_t}{S_t}\right] = rT \tag{26}$$

This equation represents the fact that a share position, continuously rebalanced to be worth \$1, has a financing cost that grows at the riskless rate.

As there are no actively traded log contracts for the second term in Equation (24), one must duplicate the log payoff, at all stock price levels at expiration, by decomposing its shape into linear and curved components, and then duplicating each of these separately. The linear component can be duplicated with a forward contract on the stock with delivery time T; the remaining curved component, representing the quadratic and higher-order contributions, can be duplicated using standard options with all possible strike levels and the same expiration time T.

For practical reasons, we want to duplicate the log payoff with liquid options — that is, with a combination of out-of-the-money calls for high stock values and out-of-the-money puts for low stock values. We introduce a new arbitrary parameter S_* to define the boundary between calls and puts. The log payoff can then be rewritten as

$$\log \frac{S_T}{S_0} = \log \frac{S_T}{S_*} + \log \frac{S_*}{S_0} \tag{27}$$

The second term, $\log(S_*/S_0)$, is constant, independent of the final stock price S_T, so only the first term has to be replicated.

A mathematical identity, which holds for all future values of S_T, suggests the decomposition of the log payoff:

$$-\log \frac{S_T}{S_*} = -\frac{S_T - S_*}{S_*} + \qquad \text{(forward contract)}$$

$$\int_0^{S_*} \frac{1}{K^2} \mathrm{Max}(K - S_T, 0) dK + \qquad \text{(put options)}$$

$$\int_{S_*}^{\infty} \frac{1}{K^2} \mathrm{Max}(S_T - K, 0) dK \qquad \text{(call options)}$$

$$\tag{28}$$

Equation (28) represents the decomposition of a log payoff into a portfolio consisting of:

- A short position in $(1/S_*)$ forward contracts struck at S_*.
- A long position in $(1/K^2)$ put options struck at K, for a continuum of strikes from 0 to S_*.
- A long position in $(1/K^2)$ call options struck at K, for a continuum of strikes from S_* to °.

All contracts expire at time T. Exhibit 4 shows this decomposition schematically.

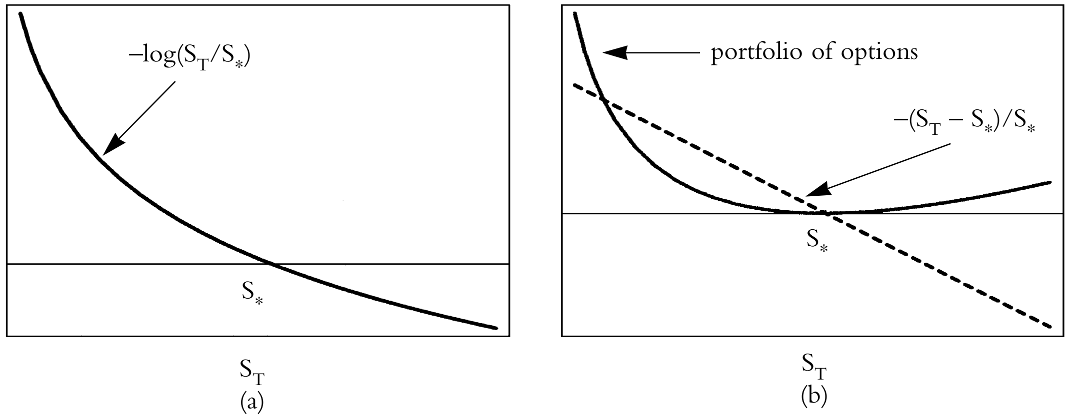
The fair value of future variance can be related to the initial fair value of each term on the right-hand side of Equation (24). By using the identities in Equations (26) and (28), we obtain

$$K_{var} = \frac{2}{T}\Bigg(rT - \left(\frac{S_0}{S_*} e^{rT} - 1\right) - \log \frac{S_*}{S_0} + e^{rT} \int_0^{S_*} \frac{1}{K^2} P(K) dK + e^{rT} \int_{S_*}^{\infty} \frac{1}{K^2} C(K) dK \Bigg) \tag{29}$$

where P(K) and C(K), respectively, denote the current fair value of a put and a call option of strike K. If you use the market prices of these options, you obtain an estimate of the current market price of future variance.

This approach to the fair value of future variance is the most rigorous from a theoretical point of view, and makes fewer assumptions than our initial intuitive treatment. Equation (29) makes precise the intuitive notion that implied volatilities can be regarded as the

EXHIBIT 4
Replication of Log Payoff as Function of S_T



(a) Payoff of a short position in a log contract at expiration. (b) Dashed line: linear payoff at expiration of a forward contract with delivery price S_*; Solid line: curved payoff of calls struck above S_* and puts struck below S_*. Each option is weighted by the inverse square of its strike. The sum of the payoffs for the dashed and solid lines provide the same payoff as the log contract.

market's expectation of future realized volatilities. It provides a direct connection between the market cost of options and the strategy for capturing future realized volatility, *even when there is an implied volatility skew*, and the simple Black-Scholes formula is invalid.

V. EXAMPLE OF A VARIANCE SWAP

Suppose you want to price a swap on the realized variance of the daily returns of some hypothetical equity index. The fair delivery variance is determined by the cost of the replicating strategy.

If you could buy options of all strikes between zero and infinity, the fair variance would be given by Equation (29) with some choice of S_*, say, $S_* = S_0$. In practice, however, only a small set of discrete option strikes are available, and using Equation (29) with only a few strikes leads to appreciable errors. Here we suggest a better approximation.

We start with the definition of fair variance given by Equation (24), which can be written as

$$K_{var} \equiv \frac{2}{T} E\left[\int_0^T \frac{dS_t}{S_t} - \frac{S_T - S_*}{S_*} - \log\frac{S_*}{S_0} + \right.$$

$$\left. \frac{S_T - S_*}{S_*} - \log\frac{S_T}{S_*}\right]$$

Taking expectations, this becomes

$$K_{var} = \frac{2}{T}\left[rT - \left(\frac{S_0}{S_*}e^{rT} - 1\right) - \log\frac{S_*}{S_0}\right] + e^{rT}\Pi_{CP} \qquad (30)$$

where Π_{CP} is the present value of the portfolio of options with payoff at expiration given by

$$f(S_T) = \frac{2}{T}\left(\frac{S_T - S_*}{S_*} - \log\frac{S_T}{S_*}\right) \qquad (31)$$

Suppose that you can trade call options with strikes K_{ic} such that $K_0 = S_* < K_{1c} < K_{2c} < K_{3c} < \ldots <$ and put options with strikes K_{ip} such that $K_0 = S_* > K_{1p} > K_{2p} > K_{3p} > \ldots >$. In Appendix A we derive the formula that determines how many options of each strike you need in order to approximate the payoff $f(S_T)$ by piecewise linear options payoffs. The procedure in Appendix A guarantees that these payoffs will always exceed or match the value of the log contract, but never be worth less.

Once these weights are calculated, Π_{CP} is obtained from

$$\Pi_{CP} = \sum_i w(K_{ip})P(S, K_{ip}) + \sum_i w(K_{ic})C(S, K_{ic}) \qquad (32)$$

We can illustrate this procedure using a concrete numerical example. Assume that the index level S_0 is 100, the continuously compounded annual riskless interest rate r is 5%, the dividend yield is zero, and the maturity of the variance swap is three months (T = 0.25). Suppose that you can buy options with strikes in the range from 50 to 150, uniformly spaced five points apart. We assume that at-the-money implied volatility is 20%, with a skew such that the implied volatility increases by one volatility point for every five-point decrease in the strike level.

In Exhibit 5 we provide the list of strikes and their corresponding implied volatilities. We then show the weights, the value of each individual option, and the contribution of each strike level to the total cost of the portfolio. The total cost of the options portfolio, Π_{CP}, is 419.8671.

It is clear from Exhibit 5 that most of the cost comes from options with strikes near the spot value. Although the number of options that are far out of the money is large, their value is small and contributes little to the total cost.

The cost of capturing variance is now simply calculated using Equation (30) with the result $K_{var} = (20.467)^2$. This is not strictly the fair variance; because the procedure of approximating the log contract in Appendix A always overestimates the value of the log contract, this value is higher than the true theoretical value for the fair variance obtained by approximating the log contract with a continuum of strikes.

In Exhibit 6 we illustrate the cost of variance as a function of the spacing between strikes, for two cases, with and without a volatility skew. You can see that as the spacing between strikes approaches zero, the cost of capturing variance approaches the theoretically fair variance.

VI. EFFECTS OF THE VOLATILITY SKEW

The general strategy can be used to determine the fair variance and the hedging portfolio from the set of available options and their implied volatilities. To explore the effects of a volatility skew on the fair variance, we assume that there is no term structure (implied volatility is not affected by option maturity), and that the skew can be parameterized in a simple, yet realistic form. We compare the numerically correct value of fair variance, computed from Equation (29), with an approximate analytic formula that we derive. This formula provides a good rule of thumb for a quick estimate of the impact of the volatility skew on the fair variance.

Let us consider a skew that varies linearly with the Black-Scholes delta of the option, so that:

$$\Sigma(\Delta_p) = \Sigma_0 + b\left(\Delta_p + \frac{1}{2}\right) \tag{33}$$

Here Δ_p is the Black-Scholes exposure of a put option, given by $\Delta_p = -N(-d_1)$, where d_1 is defined in Equation (4), Σ_0 is the implied volatility of a "50-delta" put option, and b is the slope of the skew — that is, the change in the skew per unit delta. A value of b = 0.2 means that the difference between implied volatilities for 25-delta put and 25-delta call is ten volatility points.

Note that there is an implicit dependence on time

EXHIBIT 5

Portfolio of European-Style Put and Call Options to Capture Realized Variance When There is a Volatility Skew

	Strike	Volatility	Weight	Value per Option	Contribution to Cost
Puts	50	30	163.04	0.000002	0.0004
	55	29	134.63	0.00003	0.0035
	60	28	113.05	0.0002	0.0241
	65	27	96.27	0.0013	0.1289
	70	26	82.98	0.0067	0.5560
	75	25	72.26	0.0276	1.9939
	80	24	63.49	0.0958	6.0829
	85	23	56.23	0.2854	16.0459
	90	22	50.15	0.7384	37.0260
	95	21	45.00	1.6747	75.3616
	100	20	20.98	3.3537	70.3615
Calls	100	20	19.63	4.5790	89.8691
	105	19	36.83	2.2581	83.1580
	110	18	33.55	0.8874	29.7752
	115	17	30.69	0.2578	7.9130
	120	16	28.19	0.0501	1.4119
	125	15	25.98	0.0057	0.1476
	130	14	24.02	0.0003	0.0075
	135	13	22.27	0.000006	0.0001
Total Cost					419.8671

to expiration in Equation (33), because of the Δ_p term. Since Δ_p is bounded, the implied volatility is always positive, provided $b < 2\Sigma_0$. For unrealistically large skews, the parameterization in Equation (33) leads to arbitrage violation, anyway.

It is useful to see how this parameterization of the skew looks as a function of strike. The skew by delta in Equation (33) and the same skew as a function of strike are displayed in Exhibit 7.

Appendix B presents a detailed derivation of an approximate formula for the fair variance of the contract with time to expiration T:

$$K_{var} \approx \Sigma_0^2\left(1 + \frac{1}{\sqrt{\pi}} b\sqrt{T} + \frac{1}{12}\frac{b^2}{\Sigma_0^2} + \ldots +\right) \quad (34)$$

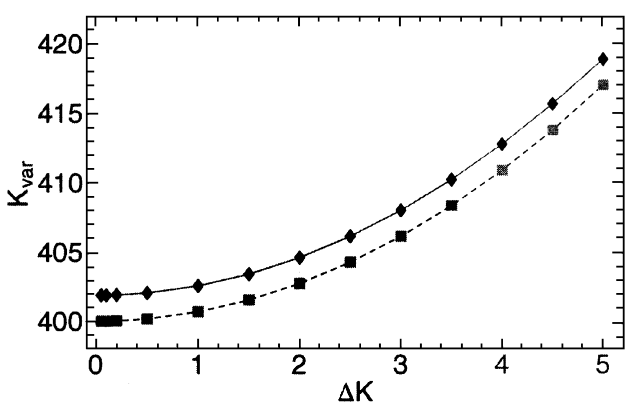
Note that the first-order correction is of magnitude $b\sqrt{T}$, and the second-order correction is of the order b^2/Σ_0^2.

Exhibit 8 compares the results for fair variance, computed numerically, with the approximate values given by the analytic formula in Equation (34). The analytic formula works very well for the three-month variance swap, and truly impressively for the one-year swap, as displayed in Exhibit 9.

The method described in Appendix B generalizes easily to other parameterizations of the skew that are polynomials in delta or in strike. Demeterfi et al. [1999] explicitly derive a formula analogous to Equation (34) for the skew linear in strike.

EXHIBIT 6

Convergence of Cost of Capturing Variance with a Discrete Set of Strikes Toward Fair Value of Variance as a Function of Spacing Between Strikes



The line with square symbols shows the convergence for no skew, with all implied volatilities at the same value of 20%. The theoretical fair variance for $\Delta K = 0$ is then $(20)^2 = 400$. The line with diamond symbols shows similar convergence to a higher fair variance of about 402, the extra contribution coming from the effect of the skew.

VII. PRACTICAL PROBLEMS WITH REPLICATION

Equation (23) shows that a variance swap is theoretically equivalent to a dynamically adjusted, constant-dollar exposure to the stock, together with a static long position in a portfolio of options and a forward that together replicate the payoff of a log contract. This portfolio strategy captures variance exactly, provided the portfolio of options incorporates all strikes between zero and infinity in the appropriate weight to match the log payoff, and provided the stock price evolves continuously.

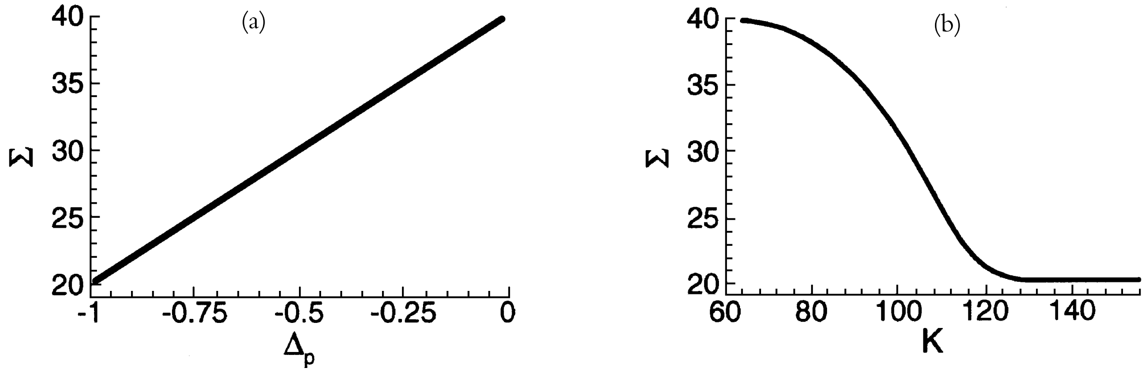
Two obvious things can go wrong. First, you may be able to trade only a limited range of option strikes, insufficient to accurately replicate the log payoff. Second, the stock price may jump. Both effects cause the strategy to capture a quantity that is not the true realized variance. We will focus on the effects of these two limitations, although other practical issues, like liquidity, may also corrupt the ideal strategy.

Imperfect Replication Due to Limited Strike Range

Variance replication requires a log contract. Since log contracts are not traded in practice, we replicate the payoff with traded standard options in a limited strike range. Because these strikes fail to duplicate the log contract exactly, they will capture less than the true realized variance. Therefore, they have lower value than that of a true log contract, and so produce an inaccurate, lower estimate of the fair variance.

Exhibit 10 shows how the estimated value of fair variance is affected by the range of strikes that make up the replicating portfolio. The fair variances are estimated from 1) a replicating portfolio with a narrow range of strikes, ranging from 75% to 125% of the initial spot level, and 2) a portfolio with a wide range of strikes, from 50% to 200% of the initial spot level. In both cases the strikes are uniformly spaced, one point apart. (The

EXHIBIT 7
Skews by Delta and as a Function of Strike

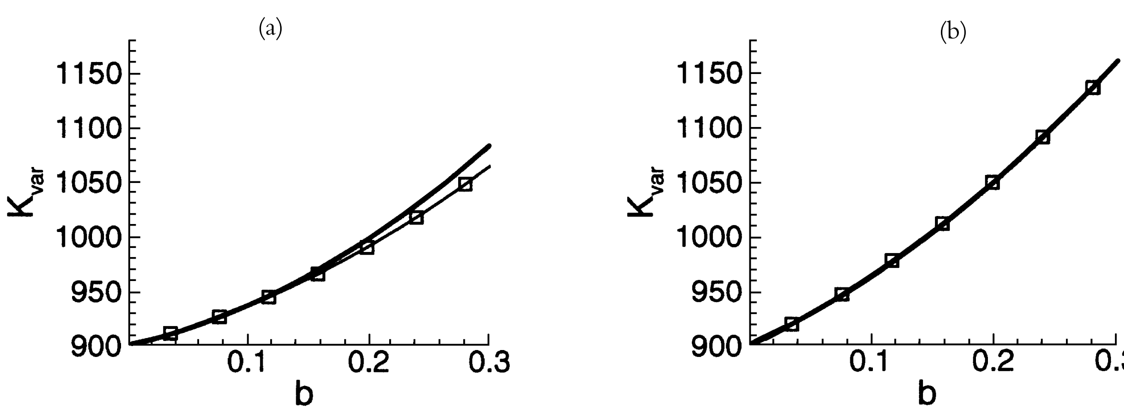


(a) A volatility skew that varies linearly in delta. (b) The corresponding skew plotted as a function of strike.

EXHIBIT 8
Comparison of Fair Variance Computed Numerically with Approximate Analytic Formula

Skew Slope	T = 3 months		T = 1 year	
b	Exact Value	Analytic Approximation	Exact Value	Analytic Approximation
0.0	$(30.01)^2$	$(30.00)^2$	$(29.97)^2$	$(30.00)^2$
0.1	$(30.61)^2$	$(30.62)^2$	$(31.06)^2$	$(31.03)^2$
0.2	$(31.49)^2$	$(31.60)^2$	$(32.42)^2$	$(32.40)^2$
0.3	$(32.64)^2$	$(32.93)^2$	$(34.06)^2$	$(34.06)^2$

EXHIBIT 9
Comparison of Exact Value of Fair Variance with Approximate Analytic Formula as Function of Skew Slope b



The thin line with squares shows the exact values obtained by replicating the log payoff. The thick line depicts the approximate value given by Equation (34). (a) Three-month variance swap. (b) One-year variance swap.

fair variance is calculated according to Equation (29), except that the integrals are replaced by sums over the available option strikes whose weights are chosen according to the procedure of Appendix A.)

We assume here that implied volatility is 25% per year for all strikes, with no volatility skew, so that all options are valued at the same implied volatility. We also assume a continuously compounded annual interest rate of 5%.

For both expirations, the wide strike range accurately approximates the actual square of the implied volatility. The narrow strike range, however, underestimates the fair variance, more dramatically so for longer expirations. This is expected because the probability of falling outside a fixed range is greater for longer expirations.

As shown in Exhibit 3, the vega and gamma of a limited strike range both fall to zero when the index moves outside the strike range, and the strategy then fails to accrue realized variance as the stock price moves. Consequently, the estimated variance is lower than the true fair value for both three-month and one-year expirations, and the reduction in value is greater for the one-year case. Over a longer time period, it is more likely that the stock price will evolve outside the strike range.

In essence, capturing variance requires owning the full log contract, whose duplication demands an infinite range of strikes. If you own a limited number of strikes, still appropriately weighted, you pay less than the full value, and, when the stock price evolves into regions where the curvature of the portfolio is insufficiently large, you capture less than the full realized variance, even if no jumps occur and the stock always moves continuously. In order to keep capturing variance, you need to maintain the curvature of the log contract at the current stock price, whatever value it takes.

A simpler way of understanding why a narrow strike range leads to a lower fair variance is to compare the payoff of the narrow-strike replicating portfolio at expiration to the terminal payoff that the portfolio is attempting to replicate, that is, the non-linear part of the log payoff:

$$\frac{S_T - S_0}{S_0} - \log\frac{S_T}{S_0} \tag{35}$$

Exhibit 11 displays the mismatch between the two payoffs. The narrow-strike option portfolio matches the curved part of the log payoff well at stock price levels between the range of strikes, that is, from 75 to 125. Beyond this range, the option portfolio payoff remains linear, always growing less rapidly than the non-linear part of the log contract. The lack of curvature (or gamma, or vega) in the options portfolio outside the narrow strike range is responsible for the inability to capture variance.

The Effect of Jumps on a Perfectly Replicated Log Contract

When the stock price jumps, the log contract may no longer capture realized volatility, for two reasons. First, if the log contract has been approximately replicated by only a finite range of strikes, a large jump may take the stock price into a region in which variance does not accrue at the right rate. Second, even with perfect replication, a discontinuous stock price jump causes Equation (23) to capture an amount not equal to the true realized variance.

In reality, both these effects contribute to the replication error. We focus only on the second effect, however, and examine the effects of jumps assuming that the log payoff can be replicated perfectly with options.

For the sake of discussion, from now on we assume that we are short the variance swap, which we will hedge by following a discrete version of the variance capture strategy:

$$V = \frac{2}{T}\left[\sum_{i=1}^{N}\frac{\Delta S_i}{S_{i-1}} - \log\frac{S_T}{S_0}\right] \tag{36}$$

where $\Delta S_i = S_i - S_{i-1}$ is the change in stock price between successive observations. Rather than rebalance

EXHIBIT 10
Effect of Strike Range on Estimated Fair Variance

Expiration	Wide Strike Range (50%-200%)	Narrow Strike Range (75%-125%)
Three-Month	$(25.0)^2$	$(24.9)^2$
One-Year	$(25.0)^2$	$(23.0)^2$

continuously as the stock price moves, we instead adjust the exposure to (2/T) dollars worth of stock only when a new stock price is recorded for updating the realized variance.

Because of the additive properties of the logarithm function, the terminal log payoff is equivalent to a daily accumulation of log payoffs:

$$V = \frac{2}{T}\sum_{i=1}^{N}\left[\frac{\Delta S_i}{S_{i-1}} - \log\frac{S_i}{S_{i-1}}\right] \tag{37}$$

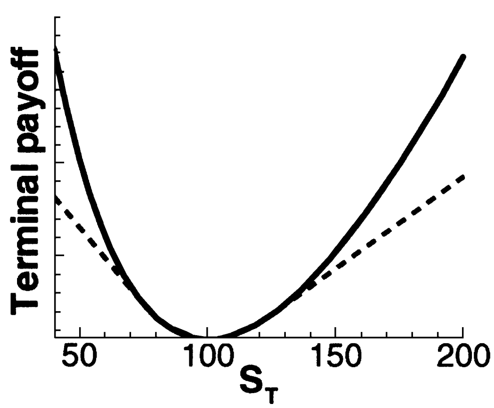
Suppose that all but one of the daily price changes are well-behaved — that is, all changes are diffusive, except for a single jump event. We characterize the jump by the parameter J, the percentage jump *downward*, from $S \to S(1-J)$; a jump downward of 10% corresponds to $J = 0.1$. A jump up corresponds to a value $J < 0$.

The contribution of this one jump to the variance is easy to isolate, because variance is additive; the total (unannualized) realized variance for a zero-mean contract is the sum

$$V = \frac{1}{T}\Sigma\left(\frac{\Delta S_i}{S_{i-1}}\right)^2$$

$$= \frac{1}{T}\sum_{\text{no jumps}}\left(\frac{\Delta S_i}{S_{i-1}}\right)^2 + \frac{1}{T}\left(\frac{\Delta S}{S}\right)^2_{\text{jump}} \tag{38}$$

EXHIBIT 11
Comparison of Terminal Payoff of Narrow-Strike Replicating Portfolio (dashed line) and Non-Linear Part of Log Payoff (solid line)



The contribution of the jump to the realized total variance is given by:[5]

$$\frac{1}{T}\left(\frac{\Delta S}{S}\right)^2_{\text{jump}} = \frac{J^2}{T} \tag{39}$$

The impact of the jump on the quantity captured by our variance replication strategy in Equation (37) is

$$\frac{2}{T}\left(\frac{\Delta S_i}{S_{i-1}} - \log\frac{S_i}{S_{i-1}}\right)_{\text{jump}} = \frac{2}{T}[-J - \log(1-J)] \tag{40}$$

In the limit that the jump size J is small enough to be regarded as part of a continuous stock evolution process, the right-hand side of Equation (40) does reduce to the contribution of this (now small) move to the true realized variance. It is only because J is not small that the variance capture strategy is inaccurate. Therefore, the replication error, or the P&L (profit/loss) due to the jump for a short position in a variance swap hedged by a long position in a variance capture strategy is

$$\text{P \& L due to jump} = \frac{2}{T}[-J - \log(1-J)] - \frac{J^2}{T} \tag{41}$$

To understand this result better, it is helpful to expand the log function as a series in J:

$$-\log(1-J) = J + \frac{J^2}{2} + \frac{J^3}{3} + \ldots + \tag{42}$$

The leading contribution to the replication error is then

$$\text{P \& L due to jump} = \frac{2}{3}\frac{J^3}{T} + \ldots + \tag{43}$$

The quadratic contribution of the jump is the same for the variance swap as it is for the variance capture strategy, and has no impact on the hedging mismatch. The leading correction is cubic in the jump size J and has a different sign for upward or downward jumps. A large move downward ($J > 0$) leads to a profit for the (short variance swap) – (long variance capture strategy), while a large move upward ($J < 0$) leads to a loss. Furthermore, a large move one day followed by a large move in the opposite direction the next day would tend to offset.

Exhibit 12 shows the impact of the jump on the strategy for a range of jump values. Note that the cubic approximation of Equation (43) correctly predicts the sign of the P&L for all values of the jump size.

There is an analogy between the cancellation of the quadratic jump term in variance replication and the linear jump term in options replication. When you are long an option you are long linear, quadratic, and higher-order dependence on the stock price. If you are also short the option's delta-hedge, then the linear dependence of the net position cancels, leaving only the quadratic and higher-order dependencies. Because the leading-order term is quadratic, large moves in either direction benefit the position; this is precisely why hedged long options positions capture variance.

In the case of variance replication considered here, the variance replication strategy is long quadratic, cubic, and higher-order terms in the stock price, while the position in the variance swap is short only the quadratic dependence. Now the quadratic term in the net position cancels, leaving only cubic and higher-order dependencies on the jump size. Since the leading term is cubic, the direction of the jump determines whether there is a net profit or loss.

Exhibit 13 displays the profit or loss due to jumps of varying sizes for three-month and one-year variance swaps with a notional value of \$1 per squared variance point, which is hedged with the variance replication strategy of Equation (37) for $T = 1$ year.

The Effect of Jumps When Replicating with a Finite Strike Range

In practice, both the effects of jumps and the risks of log replication with only a limited strike range cause the strategy to capture a quantity different from the true realized variance of the stock price. The combined effect of both these risks is harder to characterize, because the risks interact with one another in a complicated manner.

Consider again a short position in a variance contract that is being hedged by the variance capture strategy. Suppose that a downward jump occurs, large enough to move the stock price outside the range of option strikes.

If the log payoff were replicated perfectly, the constant-dollar exposure would cancel the linear part of the stock price change, and lead to a convexity gain. Although the log payoff is not being replicated perfectly, there is still a convexity gain from the jump, but it is smaller in size. After the jump, however, with the stock price now outside the strike range, the vega and gamma of the replicating portfolio are now too low to accrue sufficient variance, even if no further jumps occur.

In this scenario, the gain from the jump has to be balanced against the subsequent failure of the hedge to capture the smooth variance. The net results will depend on the details of the scenario.

A large move upward will be doubly damaging; there will be convexity loss due to the jump, and the hedge will not capture variance if the jump takes the index outside the strike range.

VIII. FROM VARIANCE TO VOLATILITY CONTRACTS

There is no simple replication strategy for synthesizing a volatility swap; it is variance that emerges naturally from hedged options trading. The replication strategy for the variance swap makes no assumptions about the level of future volatility, other than assuming that the stock price evolves continuously (without jumps). Changes in volatility have no effect on the strat-

EXHIBIT 12

Impact of Single Jump on Profit or Loss of a Short Position in a Variance Swap and a Long Position in the Variance Replication Strategy

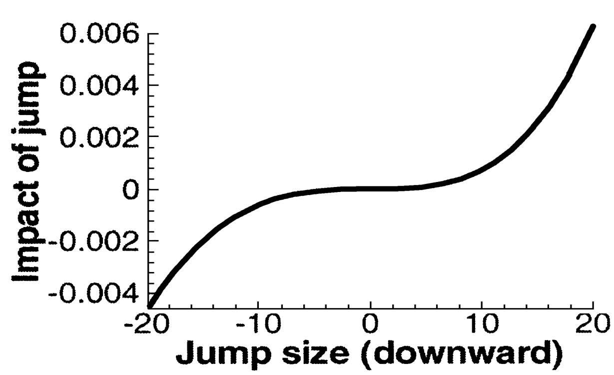


EXHIBIT 13
Profit/Loss Due to a Single Jump

Jump Size and Direction	Three-Month	One-Year
J = 15% (down)	101.5	25.4
J = 10% (down)	28.8	7.2
J = 5% (down)	3.5	0.9
J = –5% (up)	–3.2	–0.8
J = –10% (up)	–24.8	–6.2
J = –20% (up)	–80.9	–20.2

egy, which still captures the total variance over the life of the log contract.

The replication strategy for a volatility swap is fundamentally different; it is affected by changes in volatility, and its value depends on the volatility of future realized volatility. In essence, from a contingent claims or derivatives point of view, variance is the primary underlier, and all other volatility payoffs, such as volatility swaps, are best regarded as derivative securities on the variance as underlier.

From this perspective, volatility itself is a nonlinear function (the square root) of variance and is therefore more difficult, both theoretically and practically, to value and hedge.

To illustrate the issues involved, let's consider a naive strategy. Approximate a volatility swap by statically holding a suitably chosen variance contract. In order to approximate a volatility swap struck at K_{vol}, which has payoff $\sigma_R - K_{vol}$, we can use the approximation

$$\sigma_R - K_{vol} \approx \frac{1}{2K_{vol}}(\sigma_R^2 - K_{vol}^2) \tag{44}$$

This means that $1/(2K_{vol})$ variance contracts with strike K_{vol}^2 can approximate a volatility swap with a notional \$1/(vol point), for realized volatilities near K_{vol}. With this choice, the variance and volatility payoffs agree in value and volatility sensitivity (the first derivative with respect to σ_R) when $\sigma_R = K_{vol}$.

Naively, this would also imply that the fair price of future volatility (the strike for which the volatility swap has zero value) is simply the square root of fair variance K_{var}:

$$K_{vol} = \sqrt{K_{var}} \qquad \text{(naive estimate)} \tag{45}$$

In Exhibit 14 we compare the two sides of Equation (44) for K_{vol} = 30% for different values of the realized volatility. The actual volatility swap and the approximating variance swap differ appreciably only when the future realized volatility moves away from K_{vol}; you cannot fit a line everywhere with a parabola.

The naive estimate of Equations (44) and (45) is not quite correct. With this choice, the variance swap payoff is always greater than the volatility swap payoff. The mismatch between the variance and volatility swap payoffs in Equation (44) is the

$$\text{Convexity bias} = \frac{1}{2K_{vol}}(\sigma_R - K_{vol})^2$$

This square is always positive, so that with this choice of the fair delivery price for volatility, the variance swap always outperforms the volatility swap. To avoid this arbitrage, we should correct our naive estimate to make the fair strike for the volatility contract lower than the square root of the fair strike for a variance contract, so that $K_{vol} < \sqrt{K_{var}}$. In this way, the straight line in Exhibit 14 will shift to the left and will not always lie below the parabola.

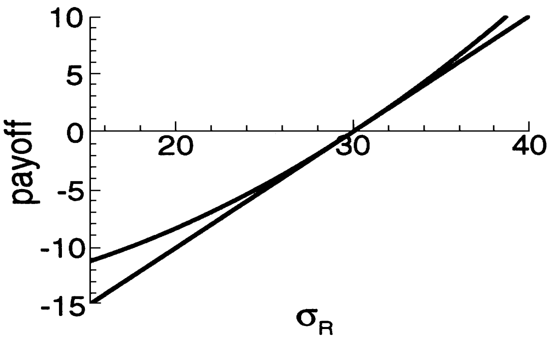
In order to estimate the size of the convexity bias, and therefore the fair strike for the volatility swap, it is necessary to make an assumption about both the level and the volatility of future realized volatility. In Appendix C we estimate the expected hedging mismatch and static hedging parameters under the assumption that future realized volatility is normally distributed.

In principle, some of the risks inherent in the static approximation of a volatility swap by a variance swap could be reduced by dynamically trading new variance contracts throughout the life of the volatility swap. This dynamic replication of a volatility swap by means of variance swaps would (in principle) produce the payoff of a volatility swap independent of the moves in future volatility.

This is closely analogous to replicating a curved stock option payoff by means of delta-hedging using the linear underlying stock price. In practice, of course, there is no market in variance swaps liquid enough to provide a usable underlier.

In the same way that the appropriate option hedge

EXHIBIT 14
Payoff of a Volatility Swap (straight line) and Variance Swap (curved line) as a Function of Realized Volatility



ratio depends on the assumed future volatility of the stock, the dynamic replication of a volatility swap requires a model for the volatility of volatility. Taking the analogy further, one could imagine that the strategy would call for holding at every instant a "variance-delta" equivalent of variance contracts to hedge a volatility derivative.

Practical implementation of these ideas requires an arbitrage-free model for the stochastic evolution of the volatility surface. Because of the complexity of the mathematics involved, it is only very recently that such models have been developed (see, for example, Derman and Kani [1998] and Ledoit and Santa-Clara [1998]). When there is a liquid market in variance swaps, these models may be useful in hedging volatility swaps and other variance derivatives.

IX. CONCLUSIONS AND FUTURE INNOVATIONS

We have tried to present a comprehensive and didactic account of both the principles and the methods used to value and hedge variance swaps. We have explained both the intuitive and the rigorous approach to replication. In markets with a volatility skew (the real world for most swaps of interest), the intuitive approach loses its footing. In this case, using the rigorous approach, one can still value variance swaps by replication.

We have derived analytic approximations that work well for the swap value under commonly used skew parameterizations. These formulas enable traders to update price quotes quickly as the market skew changes.

There are at least two areas where further development is warranted. First, our ability to effectively price and hedge volatility swaps is still limited. To fully implement a replication strategy for volatility swaps, we need a consistent stochastic volatility model for options. Much work remains to be done in this area.

Second, some market participants prefer to enter a capped variance swap or volatility swap that limits the possible loss on the position. The capped variance swap has embedded in it an option on realized variance. The development of a truly liquid market in volatility swaps, forwards, or futures would lead to the possibility of trading and hedging volatility options. Once again, this requires a consistent model for stochastic volatility.

APPENDIX A
Replicating Logarithmic Payoffs

Constant Vega Requires Options Weighted Inversely Proportional to the Square of the Strike

Consider a portfolio of standard options

$$\Pi(S) = \int_0^\infty \rho(K)O(S,K,v)dK \tag{A-1}$$

where O(S, K, v) represents a standard Black-Scholes option of strike K and total variance $v = \sigma^2\tau$ when the stock price is S.

Vega, the sensitivity to the total variance of an individual option O in this portfolio, is given by

$$V_O = \tau\frac{\partial}{\partial v}(O) = \tau S f\left(\frac{K}{S}, v\right)$$

where

$$f(S,K,v) = \frac{1}{2\sqrt{v}}\frac{\exp(-d_1^2/2)}{\sqrt{2\pi}}$$

and

$$d_1 = \frac{\ln(S/K) + v/2}{\sqrt{v}}$$

The variance sensitivity of the whole portfolio is therefore

$$V_\Pi(S) = \tau \int_0^\infty \rho(K) S f\left(\frac{K}{S}, v\right) dK \qquad \text{(A-2)}$$

The sensitivity of vega to S is

$$\frac{\partial V_\Pi}{\partial S} = \tau \int_0^\infty \frac{\partial}{\partial S}[S^2 \rho(xS)] f(x, v) dx$$

$$= \tau \int_0^\infty S[2\rho(xS) + xS\rho'(xS)] f(x, v) dx$$

where, in the second line, we change the integration variable to $x = K/S$.

We want vega to be independent of S, that is, $(\partial V_\Pi/\partial S) = 0$, which implies that

$$2\rho + K\frac{\partial \rho}{\partial K} = 0$$

The solution to this equation is

$$\rho = \frac{\text{const}}{K^2} \qquad \text{(A-3)}$$

Log Payoff Replication with a Discrete Set of Options

It is shown in the main text that the realized variance is related to trading a log contract. Since there is no log contract traded, we want to represent it in terms of standard options. It is useful to subtract the linear part (corresponding to the forward contract) and look at the function

$$f(S_T) = \frac{2}{T}\left[\frac{S_T - S_*}{S_*} - \log\frac{S_T}{S_*}\right] \qquad \text{(A-4)}$$

where S_* is some reference price. In practice, only a discrete set of option strikes are available for replicating $f(S_T)$, and we need to determine the number of options for each strike.

Assume that you can trade call options with strikes

$$K_0 = S_* < K_{1c} < K_{2c} < K_{3c} < \ldots <$$

and put options with strikes

$$K_0 = S_* > K_{1p} > K_{2p} > K_{3p} > \ldots >$$

We can approximate $f(S_T)$ with a piecewise linear function as in Exhibit 15. The first segment to the right of S_* is equivalent to the payoff of a call option with strike K_0. The number of options is determined by the slope of this segment:

$$w_c(K_0) = \frac{f(K_{1c}) - f(K_0)}{K_{1c} - K_0} \qquad \text{(A-5)}$$

Similarly, the second segment looks like a combination of calls with strikes K_0 and K_{1c}. Given that we already hold $w_c(K_0)$ options with strike K_0, we need to hold $w_c(K_1)$ calls with strike K_1 where

$$w_c(K_1) = \frac{f(K_{2c}) - f(K_{1c})}{K_{2c} - K_{1c}} - w_c(K_0) \qquad \text{(A-6)}$$

Continuing in this way, we can build the entire payoff curve one step at a time. In general, the number of call options of strike $K_{n,c}$ is given by

$$w_c(K_{n,c}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n+1,c} - K_{n,c}} - \sum_{i=0}^{n-1} w_c(K_{i,c}) \qquad \text{(A-7)}$$

The other side of the curve can be built using put options:

$$w_p(K_{n,p}) = \frac{f(K_{n+1,p}) - f(K_{n,p})}{K_{n,p} - K_{n+1,p}} - \sum_{i=0}^{n-1} w_p(K_{i,p}) \qquad \text{(A-8)}$$

APPENDIX B

Skew Linear in Delta

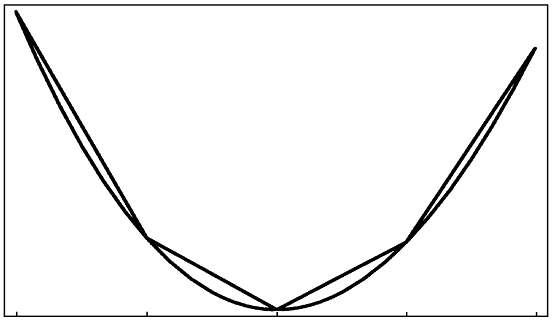
Here, we derive a formula that gives the approximate value of the variance swap when implied volatility varies linearly with delta. Such a skew can be parameterized in terms of Δ_p, the delta of a European-style put, as:

$$\Sigma(\Delta_p) = \Sigma_0 + b\left(\Delta_p + \frac{1}{2}\right) \qquad \text{(B-1)}$$

where Σ_0 is the implied volatility of options with $\Delta_p = -1/2$ (the "50-delta volatility"). (We could also parameterize the skew in terms of the call delta as $\Sigma(\Delta_c) = \Sigma_0 + b[\Delta_c - 1/2]$.)

We start with the general expression for the fair variance discussed in the text:

EXHIBIT 15
Log Payoff and Options Portfolio at Maturity



$$K_{var} = \frac{2}{T}\left(rT - \left(\frac{S_0}{S_*}e^{rT} - 1\right) - \log\frac{S_*}{S_0} + \right.$$

$$\left. e^{rT}\int_0^{S_*}\frac{1}{K^2}P[K, \Sigma(b)]dK + \right. \qquad \text{(B-2)}$$

We then expand option prices as a power series in b around a flat implied volatility (b = 0):

$$C[K,\Sigma(b)] = C(K,\Sigma_0) + b\left.\frac{\partial C}{\partial b}\right|_{b=0} + \frac{1}{2}b^2\left.\frac{\partial^2 C}{\partial b^2}\right|_{b=0} + \ldots +$$

$$P[K,\Sigma(b)] = P(K,\Sigma_0) + b\left.\frac{\partial P}{\partial b}\right|_{b=0} + \frac{1}{2}b^2\left.\frac{\partial^2 P}{\partial b^2}\right|_{b=0} + \ldots + \qquad \text{(B-3)}$$

Using this expansion, we can formally write an expansion of fair variance in powers of b as follows:

$$K_{var} = \Sigma_0^2 + b\left(\frac{2}{T}e^{rT}\right)\left\{\int_0^{S_*}\frac{1}{K^2}\left.\frac{\partial P}{\partial b}\right|_{b=0}dK + \int_{S_*}^{\infty}\frac{1}{K^2}\left.\frac{\partial C}{\partial b}\right|_{b=0}dK\right\} +$$

$$\frac{1}{2}b^2\left(\frac{2}{T}e^{rT}\right)\left\{\int_0^{S_*}\frac{1}{K^2}\left.\frac{\partial^2 P}{\partial b^2}\right|_{b=0}dK + \int_{S_*}^{\infty}\frac{1}{K^2}\left.\frac{\partial^2 C}{\partial b^2}\right|_{b=0}dK\right\} + \ldots + \qquad \text{(B-4)}$$

Here Σ_0^2 is the fair variance in the "flat world" where volatility is constant and is given by Equation (B-2) with $\Sigma(b)$ replaced by Σ_0.

The derivatives that enter Equation (B-4) are given by:

$$\left.\frac{\partial P}{\partial b}\right|_{b=0} = \left.\frac{\partial P}{\partial \Sigma}\right|_{\Sigma_0}\left.\frac{\partial \Sigma}{\partial b}\right|_{b=0}$$

$$\left.\frac{\partial C}{\partial b}\right|_{b=0} = \left.\frac{\partial C}{\partial \Sigma}\right|_{\Sigma_0}\left.\frac{\partial \Sigma}{\partial b}\right|_{b=0}$$

$$\left.\frac{\partial^2 P}{\partial b^2}\right|_{b=0} = \left.\frac{\partial^2 P}{\partial \Sigma^2}\right|_{\Sigma_0}\left.\left(\frac{\partial \Sigma}{\partial b}\right)^2\right|_{b=0} + \left.\frac{\partial P}{\partial \Sigma}\right|_{\Sigma_0}\left.\frac{\partial^2 \Sigma}{\partial b^2}\right|_{b=0}$$

$$\left.\frac{\partial^2 C}{\partial b^2}\right|_{b=0} = \left.\frac{\partial^2 C}{\partial \Sigma^2}\right|_{\Sigma_0}\left.\left(\frac{\partial \Sigma}{\partial b}\right)^2\right|_{b=0} + \left.\frac{\partial C}{\partial \Sigma}\right|_{\Sigma_0}\left.\frac{\partial^2 \Sigma}{\partial b^2}\right|_{b=0} \qquad \text{(B-5)}$$

The derivatives with respect to volatility are easily calculated using the Black-Scholes formula:

$$\left.\frac{\partial P}{\partial \Sigma}\right|_{\Sigma_0} = \left.\frac{\partial C}{\partial \Sigma}\right|_{\Sigma_0} = \frac{S\sqrt{T}}{\sqrt{2\pi}}e^{-d_1^2/2}$$

$$\left.\frac{\partial^2 P}{\partial \Sigma^2}\right|_{\Sigma_0} = \left.\frac{\partial^2 C}{\partial \Sigma^2}\right|_{\Sigma_0} = -\frac{S\sqrt{T}}{\sqrt{2\pi}}d_1\frac{\partial d_1}{\partial \Sigma_0}e^{-d_1^2/2}$$

$$d_1 = \frac{\log\left(\frac{S_F}{K}\right) + \frac{1}{2}\Sigma_0^2 T}{\Sigma_0\sqrt{T}} \qquad \text{(B-6)}$$

Other derivatives we need are given by:

$$\left.\frac{\partial \Sigma}{\partial b}\right|_{b=0} = \Delta_p + \frac{1}{2}$$

$$\left.\frac{\partial^2 \Sigma}{\partial b^2}\right|_{b=0} = 2\left(\Delta_p + \frac{1}{2}\right)\frac{\partial \Delta_p}{\partial \Sigma_0} \qquad \text{(B-7)}$$

where

$$\Delta_p = -N(-d_1)$$

Combining these relations, the fair variance can be written as

$$K_{var} = \Sigma_0^2 - b\left(\frac{2}{T}e^{rT}\right)\frac{S\sqrt{T}}{\sqrt{2\pi}}\int_0^\infty \frac{1}{K^2}\left(\Delta_p + \frac{1}{2}\right)e^{-d_1^2/2}dK -$$

$$\frac{1}{2}b^2\left(\frac{2}{T}e^{rT}\right)\frac{S\sqrt{T}}{\sqrt{2\pi}}\left(\int_0^\infty \frac{1}{K^2}\left(\Delta_p + \frac{1}{2}\right)^2 \times\right.$$

$$\left. d_1\frac{\partial d_1}{\partial \Sigma_0}e^{-d_1^2/2}dK - 2\int_0^\infty \frac{1}{K^2}\left(\Delta_p + \frac{1}{2}\right)\frac{\partial \Delta_p}{\partial \Sigma_0}e^{-d_1^2/2}dK\right) \tag{B-8}$$

Integrals can be evaluated by changing the integration variable to

$$z = \left(\log\frac{S_F}{K} + \frac{1}{2}v_0\right)/\sqrt{v_0} \equiv d_1$$

where

$$v_0 = \Sigma_0^2 T$$

so that

$$K_{var} = \Sigma_0^2 - b\left\{2\Sigma_0\int_{-\infty}^{\infty}\left[N(z) - \frac{1}{2}\right]e^{\sqrt{v_0}z - v_0/2}e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}\right\} +$$

$$b^2\left(\int_{-\infty}^{\infty}\left[N(z) - \frac{1}{2}\right]^2(z^2 - \sqrt{v_0}z)e^{\sqrt{v_0}z - v_0/2}e^{-z^2/2}\frac{dz}{\sqrt{2\pi}} -\right.$$

$$\left.2\int_{-\infty}^{\infty}\left[N(z) - \frac{1}{2}\right](z - \sqrt{v_0})e^{\sqrt{v_0}z - v_0/2}e^{-z^2}\frac{dz}{\sqrt{2\pi}}\right)$$

All these integrals can be evaluated exactly. Since we are eventually interested in expanding the result in powers of $v_0 = \Sigma_0^2 T$, one can first expand $e^{\sqrt{v_0}z - v_0/2}$ in powers of z, and integrate term by term. It is also useful to note that $N(z) - 1/2$ is antisymmetric in z to simplify calculations.

The following results are also useful:

$$\int_0^\infty z^{2n}e^{-az^2/2}\left[N(bz) - \frac{1}{2}\right]\frac{dz}{\sqrt{2\pi}} = \frac{(-2)^n}{2\pi}\frac{\partial^n}{\partial a^n}\left[\frac{1}{\sqrt{a}}\arctan\frac{b}{\sqrt{a}}\right]$$

$$\int_0^\infty z^{2n+1}e^{-az^2/2}\left[N(bz) - \frac{1}{2}\right]\frac{dz}{\sqrt{2\pi}} = \frac{(-2)^n b}{2\sqrt{2\pi}}\frac{\partial^n}{\partial a^n}\left[\frac{1}{a\sqrt{a + b^2}}\right]$$

$$\int_0^\infty z^{2n}e^{-az^2/2}\left[N(bz) - \frac{1}{2}\right]^2\frac{dz}{\sqrt{2\pi}} =$$

$$\frac{(-2)^n}{2\pi}\frac{\partial^n}{\partial a^n}\left[\frac{1}{\sqrt{a}}\left(\arctan\frac{\sqrt{a + 2b^2}}{\sqrt{a}} - \frac{\pi}{4}\right)\right]$$

$$\int_0^\infty z^{2n+1}e^{-az^2/2}\left[N(bz) - \frac{1}{2}\right]^2\frac{dz}{\sqrt{2\pi}} =$$

$$\frac{(-2)^n b}{(2\pi)^{3/2}}\frac{\partial^n}{\partial a^n}\left[\frac{1}{a\sqrt{a + b^2}}\left(\arctan\frac{b}{\sqrt{a + b^2}}\right)\right] \tag{B-9}$$

After evaluating all integrals, we find the final answer to be

$$K_{var} = \Sigma_0^2 + b\Sigma_0^2\sqrt{\frac{T}{\pi}} + \frac{1}{12}b^2 + \ldots + \tag{B-10}$$

Our calculations can easily be generalized to the model where the slope of the skew is different for put and call options; that is:

$$\Sigma_p(\Delta_p) = \Sigma_0 + b_p\left(\Delta_p + \frac{1}{2}\right) \quad \text{for} \quad -\frac{1}{2} \le \Delta_p \le 0$$

$$\Sigma_c(\Delta_c) = \Sigma_0 + b_c\left(\Delta_c - \frac{1}{2}\right) \quad \text{for} \quad 0 \le \Delta_c \le \frac{1}{2} \tag{B-11}$$

We sketch the derivation emphasizing only the differences with the previous detailed calculations. We start with the same fundamental expression as in Equation (B-2):

$$K_{var} = \frac{2}{T}\left(rT - \left(\frac{S_0}{S_*}e^{rT} - 1\right) - \log\frac{S_*}{S_0} + \right.$$

$$\left. e^{rT}\int_0^{S_*}\frac{1}{K^2}P(K, \Sigma_p(b_p))dK + e^{rT}\int_{S_*}^{\infty}\frac{1}{K^2}C(K, \Sigma_c(b_c))dK\right)$$

Then we use different implied volatility parameterizations for put and call options, as given by Equation (B-11). Note that we should choose S_* so that

$$S_* = S_F e^{-\Sigma_0^2 T/2}$$

This ensures that we use the put (call) parameterization in Equation (B-11) for strikes below (above) S_*. We expand put option prices in powers of b_p and call option prices in powers of b_c.

Evaluating all integrals as above, we find

$$K_{var} = \Sigma_0^2 + \left[\frac{1}{4}\Sigma_0(b_p - b_c) + \Sigma_0 \frac{b_p + b_c}{2}\left(\Sigma_0\sqrt{\frac{T}{\pi}}\right)\right] +$$

$$\frac{1}{12}\frac{b_p^2 + b_c^2}{2} + \ldots + \tag{B-12}$$

Obviously, for $b_p = b_c$, this reduces to the result for single slope given in Equation (B-10). Note that by changing the sign of b_c we turn the implied skew into a smile.

APPENDIX C

Static and Dynamic Replication of a Volatility Swap

We have argued that volatility swaps are fundamentally different from variance swaps and that, unlike the variance swap, there is no simple replicating strategy to create a volatility swap synthetically. We show that attempting to create a volatility swap from a variance swap by means of a buy-and-hold strategy invariably leads to misreplication, since this amounts to trying to fit a linear payoff (the volatility payoff) with a quadratic payoff (the variance swap).

Given a view on both the direction and the volatility of future volatility, we will show that it is possible to pick the strike and notional size of a variance contract to match the payoff of a volatility contract, on average, as closely as possible. The extent of the replication mismatch will depend on how close the realized volatility is to its expected value.

The hedging instrument is the realized variance (Σ_T^2), while the target of the replication is the realized volatility (Σ_T). We want to approximate the volatility as a function of the variance by writing:

$$\Sigma_T \approx a\Sigma_T^2 + b \tag{C-1}$$

and choose a and b to minimize the expected squared deviation of the two sides of Equation (C-1):

$$\min\ E[(\Sigma_T - a\Sigma_T^2 - b)^2] \tag{C-2}$$

Differentiation leads to two equations for the coefficients a and b:

$$E[\Sigma_T] = aE[\Sigma_T^2] + b$$

$$E[\Sigma_T^3] = aE[\Sigma_T^4] + bE[\Sigma_T^2] \tag{C-3}$$

The distribution of future volatility could be assumed to be normal, with mean $\bar{\Sigma}$ and standard deviation σ_Σ:

$$\Sigma_T \sim N(\bar{\Sigma}, \sigma_\Sigma) \tag{C-4}$$

This model makes sense only if the probability of negative volatilities is negligible. This strategy will replicate only on average; the expected squared replication error is given by:

$$\min\ E[(\Sigma_T - a\Sigma_T^2 - b)^2] = Var(\Sigma_T)[1 - [corr(\Sigma_T, \Sigma_T^2)]^2] \tag{C-5}$$

For realized volatilities distributed normally as in Equation (C-4), the hedging coefficients are

$$a = \frac{1}{2\bar{\Sigma} + \dfrac{\sigma_\Sigma^2}{\bar{\Sigma}}}$$

$$b = \frac{\bar{\Sigma}}{2 + \dfrac{\sigma_\Sigma^2}{\bar{\Sigma}^2}} \tag{C-6}$$

and the expected squared replication error is:

$$\min\ E[(\Sigma_T - a\Sigma_T^2 - b)^2] = \frac{\sigma_\Sigma^2}{1 + \dfrac{2\bar{\Sigma}^2}{\sigma_\Sigma^2}} \tag{C-7}$$

ENDNOTES

The authors thank Emmanuel Boussard, Llewellyn Connolly, Rustom Khandalavala, Cyrus Pirasteh, David Rogers, Emmanuel Roman, Peter Selman, Richard Sussman, and Nicholas Warren, and several clients, for many discussions and insightful questions about volatility swaps; and Barbara Dunn for editorial assistance.

[1]For a discussion of volatility as an asset class, see Derman et al. [1996].

[2]The log contract is first discussed in Neuberger [1994]. See also Neuberger [1996].

[3]See, for example, Derman and Kani [1994], Dupire

[1994], and Derman, Kani, and Zou [1996].

[4]This approach was first outlined in Derman et al. [1996]. For an alternative discussion, see Carr and Madan [1998].

[5]Note that this separation is possible because we have chosen to define the variance payoff as the zero-mean variance. If instead we had calculated the variance using the sample mean, then

$$V_{mean} = \frac{1}{T}\sum_i \left[\frac{\Delta S_i}{S_{i-1}} - \frac{1}{N}\sum_j \frac{\Delta S_j}{S_{j-1}}\right]^2$$

With this definition, the total variance involves cross-products of price changes from different times

$$\left(\frac{\Delta S_i}{S_{i-1}}\right)\left(\frac{\Delta S_j}{S_{j-1}}\right)$$

which would make the variance contract much more difficult to hedge. This is why zero-mean variance contracts are generally preferred.

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