Stochastic Finance (FIN 519)
Midterm Exam

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BM stands for Brownian motion. Assume that B_t is a standard **BM**. **RN** and **RV** stand for random number and random variable, respectively. The PDF and CDF of the standard normal distribution are denoted by $n(z)$ and $N(z)$ respectively. You can use $n(z)$ and $N(z)$ in your answers without further evaluation.

1. (5 points) **[Student's *t*-distribution]** Student's t-distribution has the probability density function (PDF) given by

$$f_\nu(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}},$$

where $\nu > 0$ is the degrees of freedom and $\Gamma(\cdot)$ is the gamma function. Gamma function has a property, $\Gamma(x) = (x-1)\Gamma(x-1)$. Obtain the mean and variance of the distribution. Clearly specify the range of ν where the mean and variance are defined.

(**Hint**: For any $\nu > 0$,

$$1 = \int_{-\infty}^{\infty} f_\nu(t)\,dt = \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+u^2\right)^{-\frac{\nu+1}{2}}\,du, \tag{1}$$

after the change of variable, $t = \sqrt{\nu}\,u$.)

Solution:

Mean: Because $f(t)$ is symmetric,

$$E(T) = \int_{-\infty}^{\infty} t f_\nu(t) dt = 0.$$

The integrand in $E(T)$ is

$$t\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \approx O(t^{-\nu}),$$

and the integral has finite value when $\nu > 1$

$$\int_0^\infty t^{-\nu} < \infty \quad \text{if} \quad \nu > 1.$$

Therefore $E(T) = 0$ is valid when $\nu > 1$.

Variance: With the change of variable, $t = \sqrt{\nu}\, u$,

$$\mathrm{Var}(T) = \int_{-\infty}^{\infty} t^2 f_\nu(t) dt = \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \nu u^2 \left(1+u^2\right)^{-\frac{\nu+1}{2}} du.$$

Since $\nu u^2 \left(1+u^2\right)^{-\frac{\nu+1}{2}} = \nu \left(1+u^2\right)^{-\frac{\nu-1}{2}} - \nu \left(1+u^2\right)^{-\frac{\nu+1}{2}}$,

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \nu \left(1+u^2\right)^{-\frac{\nu-1}{2}} = \frac{\frac{\nu-1}{2}}{\frac{\nu-2}{2}} \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu-2}{2}\right)} \nu \left(1+u^2\right)^{-\frac{\nu-1}{2}}.$$

The variance is simplified as

$$\begin{aligned}\mathrm{Var}(T) &= \frac{\nu(\nu-1)}{\nu-2} \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu-2}{2}\right)} \left(1+u^2\right)^{-\frac{\nu-1}{2}} du - \nu \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1+u^2\right)^{-\frac{\nu+1}{2}} du \\ &= \frac{\nu(\nu-1)}{\nu-2} \cdot 1 - \nu \cdot 1 = \frac{\nu}{\nu-2}.\end{aligned}$$

The first integral is obtained from Eq. (1) by $\nu \to \nu - 2$ for $\nu - 2 > 0$, and the second integral is from Eq. (1) for $\nu > 0$. Therefore, the variance is valid when $\nu > 2$.

2. (3×3 points) **[Martingale related to BM]** If B_t is a standard BM, determine whether the following is a martingale or not. Give a brief reason.
   (a) $Y_t = B_{\lambda t}^2 - (\lambda t)^2$
   (b) $Y_t = \exp(2\sigma B_t - 2\sigma^2 t)$
   (c) $S_t = S_0 + \sigma B_{t\wedge\tau}$ where $\tau = \min\{t : S_t = A\}$ for some $A > 0$.

Solution:

(a) **No**. $Y_t = B_{\lambda t}^2 - \lambda t$ is a martingale.

(b) **Yes**. $Y_t = \exp\left(2\sigma B_t - \frac{1}{2}(2\sigma)^2 t\right)$ is a martingale.

(c) **Yes**. τ is a proper stopping time because it is based on information up to time t.

3. (2×3 points) Consider the following process:

$$M_n = A_0 X_1 + A_1 X_2 + \cdots + A_{n-1} X_n \quad \text{where} \quad X_k = \pm 1 \ (p = q = 0.5).$$

A_{k-1} is understood as the bet size for the n-th game (X_n), and M_n is your wealth after n games. The bet size A_{k-1} follows the rule:

- The bet on the first game is one: $A_0 = 1$
- If you win a game ($X_n = +1$), stop gambling: $A_n = A_{n+1} = \cdots = 0$.
- If you lose a game ($X_n = -1$), double the bet for the next game: $A_n = 2A_{n-1}$.

Note that the stopping time for the gambling is given by:

$$\tau = \min\{k : X_k = 1\}.$$

(a) What is your balance M_τ when you just stop the game ($n = \tau$)? Consider $\tau = 1, 2, 3$ and generalize your answer.

(b) Calculate $E(M_\tau)$. Is M_n a martingale?

(c) M_n is the martingale transformation of $S_n = X_1 + \cdots + X_n$ by A_n. A_n is non-anticipating (i.e., is determined by information up to $t = n$.) Therefore, should M_n be a martingale? Give your answer and explain.

Solution: This betting strategy was popular in 18th-century France and it was originally called as *martingale* (WIKIPEDIA). The martingale in stochastic process was named after this betting strategy. Also read the history of martingale (WIKIPEDIA) and St. Petersburg paradox (WIKIPEDIA).

(a)
- If $\tau = 1$, $M_1 = 1$.
- If $\tau = 2$, $M_2 = -1 + 2 = 1$.
- If $\tau = 3$, $M_3 = -1 - 2 + 4 = 1$.
- If $\tau = n$, $M_n = -(1 + 2 + \cdots + 2^{n-2}) + 2^{n-1} = 1$.

(b)

$$E(M_\tau) = \sum_{n=1}^{\infty} P(\tau = n)\, M_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot 1 = 1.$$

M_n is a martingale because (i) $|M_n|$ is bounded because $1 - 2^n \le M_n \le 1$, and (ii)

$$E(M_{n+1} \,|\, \mathcal{F}_n) = M_n + A_n\, E(X_{n+1} \,|\, \mathcal{F}_n) = M_n.$$

Two properties, $E(M_\tau) = 1$ and $0 = M_0 = E(M_n)$ for any n can coexist.

$$0 = M_0 = E(M_{n\wedge\tau}) \ \text{ for any } n \ne \lim_{n\to\infty} E(M_{n\wedge\tau}) = E(M_\tau) = 1,$$

The equality does not hold in the middle because M_n does not satisfy

$$|M_{n\wedge\tau}| < C \quad \text{for all } n \text{ and some } C \quad .$$

In class we discussed $B_{t\wedge\tau}$ as a similar example, where τ is the first hitting time of level $a > 0$:

$$0 = B_0 = E(B_{t\wedge\tau}) \ \text{ for any } t \ne \lim_{t\to\infty} E(B_{t\wedge\tau}) = E(B_\tau) = a.$$

because $|B_{t\wedge\tau}|$ is not bounded for all t.

(c) Yes, the martingale Transform Theorem (Theorem 2.1) holds in this case because A_n is not infinite for any n. This is another way to prove that M_n is a martingale.

4. (5 points) [**Geometric BM**] Let τ be the first hitting time of $B_t + \gamma t$ at the level δ:

$$\tau = \min\{t : B_t + \gamma t = \delta\} \quad (\gamma \ge 0,\ \delta > 0).$$

Then, τ follows an inverse Gaussian distribution. The probability density of τ is given by

$$f_\tau(t) = \frac{\delta}{\sqrt{2\pi t^3}} \exp\left(-\frac{(\gamma t - \delta)^2}{2t}\right) = \frac{\delta}{\sqrt{t^3}}\, n\left(\frac{\delta - \gamma t}{\sqrt{t}}\right), \tag{2}$$

and the CDF is given by

$$F(t) = N\left(\gamma\sqrt{t} - \frac{\delta}{\sqrt{t}}\right) + e^{2\gamma\delta} N\left(-\gamma\sqrt{t} - \frac{\delta}{\sqrt{t}}\right).$$

Assume that a stock price S_t follows a geometric BM with volatility σ and initial price S_0,

$$S_t = S_0 \exp\left(\sigma B_t - \frac{\sigma^2 t}{2}\right).$$

Calculate the probability that S_t never falls below L ($< S_0$) until time $t = T$.

Solution: For $0 \le t \le T$,

$$\begin{aligned}
\text{Prob}\,(S_t \ge L) &= \text{Prob}\left(\sigma B_t - \frac{\sigma^2 t}{2} \ge \log(L/S_0)\right) \\
&= \text{Prob}\left((-B)_t + \frac{\sigma t}{2} \le \frac{\log(S_0/L)}{\sigma}\right) \\
&= \text{Prob}\,(\tau > T) \quad \text{with} \quad \gamma = \sigma/2,\ \delta = \log(S_0/L)/\sigma \\
&= 1 - F(T) \\
&= N\left(-\frac{\sigma\sqrt{T}}{2} + \frac{\log(S_0/L)}{\sigma\sqrt{T}}\right) - \frac{S_0}{L} N\left(-\frac{\sigma\sqrt{T}}{2} - \frac{\log(S_0/L)}{\sigma\sqrt{T}}\right).
\end{aligned}$$

From the second to the third line, we use that $-B_t$ is also a standard BM.

5. (5 points) **[Look-back option]** Assume that a stock price follows a BM, $S_t = S_0 + \sigma B_t$. Calculate the put option price whose payout at expiry $t = T$ is

$$P_T = (S_T^* - S_T)^+,$$

where S_T^* is the maximum stock price until $t = T$, $S_T^* = \max\{S_t : 0 \le t \le T\}$. In other words, the strike price of the option is S_T^* instead of a constant value K. Compared to the regular at-the-money put option (i.e., $K = S_0$), how much is this look-back put option more expensive or cheaper?

Solution: From class, we know the joint PDF of (B_t^*, B_t):

$$f_{(B_t^*, B_t)}(v, u) = \frac{2(2v - u)}{t^{3/2}}\, n\left(\frac{2v - u}{\sqrt{t}}\right) \quad \text{for} \quad v \ge u \quad \text{and} \quad v \ge 0.$$

The payout is

$$P_T = \max(\sigma B_T^* - \sigma B_T) = \sigma \max(B_T^* - B_T),$$

and the put option price is expressed by the double integral:

$$P = \sigma E\left((B_T^* - B_T)^+\right) = \int_{v=0}^{\infty} \int_{u=-\infty}^{v} (v-u) f_{(B_T^*, B_T)}(v,u) du dv.$$

With change of variable, $x = v$ and $y = v - u$ ($dudv = -dxdy$),

$$\begin{aligned} P &= \sigma \int_{x=0}^{\infty} \int_{y=0}^{\infty} y \frac{2(x+y)}{T^{3/2}}\, n\left(\frac{x+y}{\sqrt{T}}\right) dydx \\ &= \sigma \int_{y=0}^{\infty} \frac{2y}{\sqrt{T}} \int_{x=0}^{\infty} \frac{(x+y)}{T}\, n\left(\frac{x+y}{\sqrt{T}}\right) dxdy \\ &= \sigma \int_{y=0}^{\infty} \frac{2y}{\sqrt{T}} \left[-n\left(\frac{x+y}{\sqrt{T}}\right)\right]_{x=0}^{\infty} dy = \sigma \int_{y=0}^{\infty} \frac{2y}{\sqrt{T}} n\left(\frac{y}{\sqrt{T}}\right) dy \\ &= 2\sigma\sqrt{T} \int_{y=0}^{\infty} \frac{y}{T} n\left(\frac{y}{\sqrt{T}}\right) dy = 2\sigma\sqrt{T}\left[-n\left(\frac{y}{\sqrt{T}}\right)\right]_{y=0}^{\infty} \\ &= 2\sigma\sqrt{T} n(0) = \sigma\sqrt{2T/\pi} \approx 0.8\sigma\sqrt{T}. \end{aligned}$$