Stochastic Finance (FIN 519)
Problems and Solutions

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Contents

- **SCFA** stands for the exercise problems of Stochastic Calculus and Financial Applications, **HW** homework, **ME** midterm exam, and **FE** final exam.
- **BM** stands for Brownian motion. Assume that B_t is a standard **BM**.
- **RN** and **RV** stand for random number and random variable, respectively.
- **PDF** and **CDF** stand for probability density function and cumulative distribution function, respectively.
- $P(\cdot)$ and $E(\cdot)$ are probability and expectation, respectively.
- The PDF and CDF of the standard normal variable are denoted by $n(z)$ and $N(z)$, respectively. (In **SCFA**, they are denoted by $\phi(z)$ and $\Phi(z)$.) You can use $n(z)$ and $N(z)$ in your answers without further evaluation.
- Assume interest rate (r) and dividend rate (q) are zero in option pricing unless stated otherwise.
- $(x)^+ = \max(x, 0)$
- $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.

Probability and Statistics Review

1. **[2018HW 2-1]** In class, we derived the moments of the standard normal distribution:

$$E(Z^{2n}) = (2n-1)(2n-3)\cdots 3\cdot 1 \quad \text{for} \quad Z \sim N(0,1).$$

We can derive the same result using the moment generating function. First, derive the moment generating function,

$$M_X(t) = E(\exp(tX)) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \quad \text{for} \quad X \sim N(\mu, \sigma^2).$$

Then, using the Taylor expansion of $M_X(t)$, derive the moment of Z. (After this problem, you can understand SCFA 3.4 better.)

Solution: The PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Therefore,

$$\begin{aligned} M_X(t) = E(\exp(tX)) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\ &= \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \int_{-\infty}^{\infty} f_X(x-\sigma^2 t)dx = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right). \end{aligned}$$

For $Z \sim N(0,1)$, the moment generating function is $M_Z(t) = \exp(t^2/2)$. Expanding $M_Z(t)$,

$$M_Z(t) = 1 + \frac{t^2}{2} + \cdots + \frac{1}{2^n \cdot n!} t^{2n} + \cdots$$

The $2n$-th moment is given by

$$E(Z^{2n}) = \frac{(2n)!}{2^n \cdot n!} = \frac{(2n)!}{(2n)(2n-2)\cdots 2} = (2n-1)(2n-3)\cdots 3\cdot 1$$

2. **[2019HW 1-1]** Find the moment generating function (MGF) of uniform distribution on $[0,1]$. From the MGF, find the mean and variance of the distribution.

Solution: The MGF is

$$M_U(t) = \int_0^1 e^{tu} du = \frac{1}{t}\left(e^t - 1\right) = 1 + \frac{1}{2}t + \frac{1}{6}t^2 + \cdots.$$

Therefore,

$$\begin{aligned} E(U) &= \frac{1}{2} \\ \mathrm{Var}(U) &= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12} \end{aligned}$$

3. **[2019HW 1-2]** Find the moment generating function (MGF) of the exponential distribution with intensity λ. From the MGF, find the mean and variance of the distribution.

Solution: If we let X be the exponential random variable, the MGF is

$$M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - t} = 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \cdots \quad (t < \lambda)$$

Therefore,

$$E(X) = \frac{1}{\lambda}$$
$$\mathrm{Var}(U) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

4. **[2019HW 1-3]** Assume that the random variable X follows the uniform distributions on $[0, 1]$.
   (a) If Y is another random variable given by $Y = X^2$, what is $\rho(X, Y)$, the correlation coefficient between X and Y?
   (b) If Y is another random variable given by $Y = 4X(1 - X)$, what is $\rho(X, Y)$, the correlation coefficient between X and Y?
   (c) Regarding (b), are X and Y independent? We know that the independence between X and Y implies $\rho(X, Y) = 0$. What can you say about the opposite?

Solution:

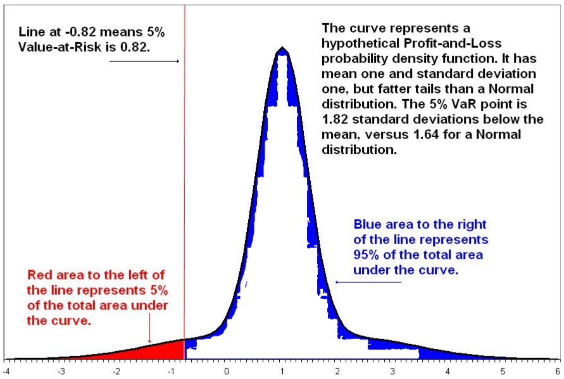
(a)

$$E(Y) = \int_0^1 x^2 dx = \frac{1}{3}, \quad E(Y^2) = \int_0^1 x^4 dx = \frac{1}{5}, \quad E(XY) = \int_0^1 x^3 dx = \frac{1}{4},$$
$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{1/4 - 1/6}{\sqrt{1/12}\sqrt{1/5 - 1/9}} = 96.8\%$$

(b) Because Y is symmetric at $X = 1/2$, the correlation should be zero.

(c) The opposite does not hold. Y is completely dependent on X.

5. **[2018ME, Probability]** Value-at-risk (VaR) (WIKIPEDIA) is a measure of the risk of loss for investments. It estimates how much a set of investments might lose (with a given probability p). VaR is typically used by firms and regulators in the financial industry to gauge the amount of assets needed to cover possible losses. For a given portfolio, time horizon, and probability p, the p-VaR is defined such that the probability of a loss greater than VaR is (at most) p while the probability of a loss less than VaR is (at least) $1 - p$. In other words, p-VaR is the loss at the worst p percentile.



(From Wikipedia. The graph is illustration only. **Ignore** the numbers in the graph.)

Assume that you invest in one share of stock today and that your profit & loss is distributed as $S_T - S_0 = X$ for some random variable X with the CDF, $F_X(x)$, and the PDF, $f_X(x)$.

(a) If $X \sim N(0, 10^2)$ (i.e., $\sigma = 10$), what is your 5%-VaR, $\text{VaR}(p = 0.05)$? You may use $N(-1.64) \approx 0.05$.

(b) Express the put option price with strike price K, $P(K)$, in terms of $f_X(x)$. (You may use integral in the answer.)

(c) Conditional VaR (CVaR or expected shortfall) is another risk measure to improve VaR. It is defined as the expected loss conditional on that the loss is within the worst p percentile. Find the expression for $\text{CVaR}(p)$. You can simplify the expression using $\text{VaR}(p)$ and the put option price, $P(K)$. Between $\text{VaR}(p)$ and $\text{CVaR}(p)$, which one assumes bigger loss?

Solution:

(a) $X \sim 10Z$ where $Z \sim N(0, 1)$.

$$\text{VaR}(0.05) = 10\, N^{-1}(0.05) = 10 \cdot (-1.64) = -16.4$$

(b)

$$\begin{aligned} P(K) &= \int_{x=-\infty}^{K-S_0} (K - S_0 - x) f_X(x) dx \\ \text{or} \quad &= \int_{x=-\infty}^{K} (K - x) f_X(x - S_0) dx \end{aligned}$$

(c) Let $S_T = K$ be the price which give the p-percentile loss. Then, $K - S_0 = \text{VaR}(p)$. We express $\text{CVar}(p)$ using K, and substitute using $K = S_0 + \text{VaR}(p)$ in the end.

$$\begin{aligned} \text{CVar}(p) &= \frac{1}{p} \int_{x=-\infty}^{K-S_0} x f_X(x) dx \\ &= \frac{K - S_0}{p} \int_{x=-\infty}^{K-S_0} f_X(x) dx + \frac{1}{p} \int_{x=-\infty}^{K-S_0} (x - K + S_0) f_X(x) dx \\ &= \frac{\text{VaR}(p)}{p} p - \frac{P(K)}{p} = \text{VaR}(p) - \frac{P(S_0 + \text{VaR}(p))}{p} \end{aligned}$$

CVaR assumes more severe loss.

6. **[2020ME, Lognormal distribution]** A lognormal random variables with parameters (μ, σ) is given by

$$Y = \mu \exp(\sigma Z - \sigma^2/2) \quad \text{for a standard normal variable} \quad Z.$$

(a) Obtain the mean and variance of Y.

(b) Suppose that two lognormal random variables are give by

$$Y_1 = \mu_1 \exp(\sigma_1 Z_1 - \sigma_1^2/2) \quad \text{and} \qquad Y_2 = \mu_2 \exp(\sigma_2 Z_2 - \sigma_2^2/2),$$

and that the two standard normals, Z_1 and Z_2, are correlated by ρ (i.e., $E(Z_1 Z_2) = \rho$). Obtain the covariance and correlation between Y_1 and Y_2.

Solution:

(a) The mean and variance are given by μ and $\mu^2(e^{\sigma^2} - 1)$ respectively.

$$E(Y) = \mu E(\exp(\sigma Z - \sigma^2/2)) = \mu.$$
$$E(Y^2) = \mu^2 E(\exp(2\sigma Z - \sigma^2)) = \mu^2 \exp(\sigma^2) E(\exp(2\sigma Z - (2\sigma)^2/2)) = \mu^2 \exp(\sigma^2).$$
$$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = \mu^2 \left(\exp(\sigma^2) - 1\right).$$

(b) Using $\sigma_1 Z_1 + \sigma_2 Z_2 \sim N(0, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$,

$$\begin{aligned}
E(Y_1 Y_2) &= \mu_1 \mu_2 E(\exp(\sigma_1 Z_1 + \sigma_2 Z_2 - (\sigma_1^2 + \sigma_2^2)/2)) \\
&= \mu_1 \mu_2 \exp(\rho\sigma_1\sigma_2) E(\exp(\sigma_1 Z_1 + \sigma_2 Z_2 - (\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)/2)) \\
&= \mu_1 \mu_2 \exp(\rho\sigma_1\sigma_2) \\
\mathrm{Cov}(Y_1, Y_2) &= E(Y_1 Y_2) - E(Y_1)E(Y_2) = \mu_1 \mu_2 \left(\exp(\rho\sigma_1\sigma_2) - 1\right) \\
\mathrm{Corr}(Y_1, Y_2) &= \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)\mathrm{Var}(Y_2)}} = \frac{\exp(\rho\sigma_1\sigma_2) - 1}{\sqrt{(\exp(\sigma_1^2) - 1)(\exp(\sigma_2^2) - 1)}}.
\end{aligned}$$

7. **[2021HW 1-1]** An RN X follows the gamma distribution with parameters (a, b). The PDF is given by

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx},$$

where the gamma function $\Gamma(a)$ is defined by

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx \quad (a > 0).$$

(a) If a is a positive integer, show that $\Gamma(a) = (a-1)! = (a-1)(a-2) \cdots 1$.

(b) Find the MGF of X, that is, $E(e^{tX})$.

(c) From (b), show that

$$E(X) = \frac{a}{b} \quad \text{and} \quad \mathrm{Var}(X) = \frac{a}{b^2}.$$

(d) Find the skewness and ex-kurtosis of X.

Solution:

(a) For $a = 1$,

$$\Gamma(0) = \int_0^\infty e^{-x} dx = 1.$$

For $a \geq 2$, we use the integral by part:

$$\begin{aligned}\Gamma(a) &= \int_0^\infty x^{a-1} e^{-x} dx = \left[-x^{a-1}e^{-x}\right]_0^\infty + (a-1)\int_0^\infty x^{a-2}e^{-x}dx \\ &= (a-1)\Gamma(a-1).\end{aligned}$$

Therefore, $\Gamma(a) = (a-1)\Gamma(a-1) = (a-1)(a-2)\cdots 1 \cdot \Gamma(1) = (a-1)!$

(b) The MGF can be derived as

$$\begin{aligned}E(e^{tX}) &= \int_0^\infty e^{tx} f(x) dx = \int_0^\infty \frac{b^a}{\Gamma(a)} x^{a-1} e^{-(b-t)x} dx \\ &= \left(\frac{b}{b-t}\right)^a \int_0^\infty \frac{(b-t)^a}{\Gamma(a)} x^{a-1} e^{-(b-t)x} dx = \left(1 - \frac{t}{b}\right)^{-a}.\end{aligned}$$

The MGF is well-defined for $t < b$.

(c) Taylor's expansion of the MGF is

$$\begin{aligned}E(e^{tX}) &= 1 + a\left(\frac{t}{b}\right) + \frac{a(a+1)}{2}\left(\frac{t}{b}\right)^2 + \frac{a(a+1)(a+2)}{3!}\left(\frac{t}{b}\right)^3 \\ &\quad + \frac{a(a+1)(a+2)(a+3)}{4!}\left(\frac{t}{b}\right)^4 + \cdots.\end{aligned}$$

Therefore,

$$E(X) = \bar{X} = \frac{a}{b} \quad \text{and} \quad E(X^2) = \frac{a(a+1)}{b^2}.$$

Finally,

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{a(a+1)}{b^2} - \frac{a^2}{b^2} = \frac{a}{b^2}.$$

(d) From the MGF, we know

$$E(X^3) = \frac{a(a+1)(a+2)}{b^3} \quad \text{and} \quad E(X^4) = \frac{a(a+1)(a+2)(a+3)}{b^4}$$

We need to obtain the 3rd and 4th central moments.

$$\begin{aligned}E((X-\bar{X})^3) &= E(X^3) - 3\bar{X}E(X^2) + 2\bar{X}^3 \\ &= \frac{a(a+1)(a+2)}{b^3} - 3\frac{a^2(a+1)}{b^3} + 2\frac{a^3}{b^3} = \frac{2a}{b^3}. \\ E((X-\bar{X})^4) &= E(X^4) - 4\bar{X}E(X^3) + 6\bar{X}^2E(X^2) - 3\bar{X}^4 \\ &= \frac{a(a+1)(a+2)(a+3)}{b^4} - 4\frac{a^2(a+1)(a+2)}{b^4} + 6\frac{a^3(a+1)}{b^4} - 3\frac{a^4}{b^4} \\ &= \frac{3a(a+2)}{b^4}\end{aligned}$$

Therefore,

$$\text{Skewness} = \frac{(X-\bar{X})^3}{\mathrm{Var}(X)^{3/2}} = \frac{2}{\sqrt{a}}$$
$$\text{Ex-kurtosis} = \frac{(X-\bar{X})^4}{\mathrm{Var}(X)^2} - 3 = \frac{3a(a+2)}{a^2} - 3 = \frac{6}{a}$$

8. **[2021ME, Cumulants]** The cumulants (and cumulant generating function) of an RV X provide interesting alternatives to the moments (and moment generating function) of X. The cumulant generating function $K_X(t)$ is defined as the log of MGF, and the cumulants κ_n are defined as the coefficients of Taylor's expansion of $K_X(t)$:

$$\log M_X(t) = K_X(t) = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!} = \kappa_1\, t + \kappa_2 \frac{t^2}{2} + \kappa_3 \frac{t^3}{6} + \kappa_4 \frac{t^4}{24} + \cdots .$$

The MGF, $M_X(t)$, and the moments, μ_n, are defined as usual:

$$M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} \mu_n \frac{t^n}{n!} \quad \text{where} \quad \mu_n = E(X^n).$$

As you will see, the cumulants are particularly useful when $M_X(t)$ is in an exponential form. In the following questions, clearly show your derivation.

(a) Express κ_1 and κ_2 in terms of μ_n. What are the statistical meanings of κ_1 and κ_2?

(b) Express κ_3 and κ_4 in terms of μ_n. How are κ_3 and κ_4 related to the skewness and ex-kurtosis of X?

(c) The MGF of a normal RV, $X \sim N(\mu, \sigma^2)$, is $\exp(\mu t + \sigma^2 t^2/2)$ (see **2018HW 2-1**). What are the cumulants of X?

(d) In **HW 1-1**, we derived the MGF of the gamma RV with parameter (a, b). But the calculation for variance, skewness, and ex-kurtosis was very tedious. Derive them again by using $K_X(t)$.

Solution:

(a) We will rely on the expansion of $\log(1+x)$ for small x:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots .$$

Comparing the coefficients,

$$\kappa_1\, t + \kappa_2 \frac{t^2}{2} + \cdots = \log\left(1 + \mu_1\, t + \mu_2 \frac{t^2}{2} + \cdots\right)$$
$$= \mu_1\, t + (\mu_2 - \mu_1^2)\frac{t^2}{2} + \cdots ,$$

we obtain

$$\kappa_1 = \mu_1 \quad \text{and} \quad \kappa_2 = \mu_2 - \mu_1^2.$$

Therefore, κ_1 and κ_2 are the mean and variance, respectively, of the distribution.

(b) Equation the t^3 and t^4 terms, we find

$$\kappa_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$$
$$\kappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4.$$

In turn, they can be expressed as

$$\kappa_3 = E((X-\mu_1)^3) = \text{Skewness} \times \kappa_2^{3/2}$$
$$\kappa_4 = E((X-\mu_1)^4) - 3\kappa_2^2 = \text{Ex-kurtosis} \times \kappa_2^2.$$

(c) Because $K_X(t) = \mu t + \sigma^2 t^2/2$,

$$\kappa_1 = \mu, \quad \kappa_2 = \sigma^2, \quad \text{and} \quad \kappa_3 = \kappa_4 = \cdots = 0.$$

This is consistent with what we already know!

(d) The cumulant generating function of the gamma distribution is

$$\log E(e^{tX}) = -a\log\left(1 - \frac{t}{b}\right) = a\left(\frac{t}{b} + \frac{t^2}{2b^2} + \frac{t^3}{3b^3} + \frac{t^4}{4b^4} + \cdots\right)$$

The mean and variance are

$$\mu_1 = \kappa_1 = \frac{a}{b} \quad \text{and} \quad \kappa_2 = \frac{a}{b^2}.$$

From $\kappa_3 = 2a/b^3$ and $\kappa_4 = 6a/b^4$, the skewness and ex-kurtosis are

$$\text{Skewness} = \frac{2a/b^3}{(a/b^2)^{3/2}} = \frac{2}{\sqrt{a}} \quad \text{and} \quad \text{Ex-kurtosis} = \frac{6a/b^4}{(a/b^2)^2} = \frac{6}{a}.$$

9. **[2022ME, Poisson Distribution]** The RV, N, follows a Poisson distribution with rate λ. The Poisson distribution is a discrete probability distribution (i.e., $N = 0, 1, 2, \cdots$) of the number of the events occurring in a unit time interval $T = 1$. The probability function is given by

$$P(N = k) = f_\lambda(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

(a) Find the moment generating function (MGF) of N:

$$M_N(t) = E(e^{tN}) = \sum_{k=0}^{\infty} e^{tk} f_\lambda(k).$$

(Hint: use that $\sum_{k=0}^{\infty} f_\lambda(k) = 1$ for any $\lambda > 0$.)

(b) Prove that $E(N) = \lambda$ and $\mathrm{Var}(N) = \lambda$. (If you obtained the MGF from (a), use it. You may still be able to prove them without MGF.)

(c) Find the skewness and ex-kurtosis of N.

Solution:

(a) Although you don't have to show this, $\sum_{k=0}^{\infty} f_\lambda(k)$ because of the Taylor's expansion of e^λ:

$$e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \,.$$

The MGF of N is

$$M_N(t) = E(e^{tN}) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{\eta-\lambda} \sum_{k=0}^{\infty} \frac{\eta^k}{k!} e^{-\eta} = e^{\lambda(e^t-1)},$$

where $\eta = \lambda e^t$.

(b) From the expansions of the MGF in (a),

$$M_N(t) = e^{\lambda(e^t-1)} = \exp\big(\lambda(t + t^2/2 + \cdots)\big) = \lambda t + \lambda \frac{t^2}{2} + \cdots + \lambda^2 \frac{t^2}{2} + \cdots \,,$$

we prove that

$$E(N) = \lambda, \quad \mathrm{Var}(N) = \lambda + \lambda^2 - E(N)^2 = \lambda.$$

(c) The skewness and ex-kurtosis (the mean and variance as well) can be easily obtained from the 3rd and 4th terms of the cumulant generating function:

$$K_N(t) = \log M_N(t) = \lambda(e^t - 1) = \lambda t + \lambda \frac{t^2}{2} + \lambda \frac{t^3}{6} + \lambda \frac{t^4}{4!} + \cdots \,.$$

Therefore, skewness and ex-kurtosis are

$$s = \frac{E((N-\lambda)^3)}{\mathrm{Var}(N)^{1.5}} = \frac{\lambda}{\lambda^{1.5}} = \frac{1}{\sqrt{\lambda}} \quad \text{and} \quad \kappa = \frac{E((N-\lambda)^4)}{\mathrm{Var}(N)^2} - 3 = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}.$$

1 Random Walks and First Step Analysis

1. **[SCFA 1.1]**

Solution: Let $T_{i,j}$ denote the expected time to go from level i to j. We are going to compute the answer as

$$T_{25,18} = T_{25,20} + T_{20,19} + T_{19,18}.$$

First, $T_{25,20} = 15$ from Eq. (1.15):

$$E(\tau \mid S_0 = 0) = \frac{B}{q-p} - \frac{A+B}{q-p}\frac{1-(q/p)^B}{1-(q/p)^{A+B}}$$

with $p = 1/3$, $q = 2/3$, $A = \infty$, $B = 5$. We also know $T_{21,20} = 3$ from $B = 1$. Next, $T_{20,19} = 37$ is calculated from

$$T_{20,19} = \frac{1}{10} \cdot 1 + \frac{9}{10}(1 + T_{21,20} + T_{20,19}).$$

Finally $T_{19,18} = 77$ is obtained from

$$T_{19,18} = \frac{1}{3} \cdot 1 + \frac{2}{3}(1 + T_{20,19} + T_{19,18}).$$

Therefore, $T_{25,18} = 15 + 37 + 77 = 129$.

2. **[SCFA 1.3]**

Solution: Let N_k be the number of visits to the level $k \neq 0$ before returning to 0 for the first time. First, we prove that $P(N_k \geq 1) = 1/(2k)$ for $k \geq 1$. In order for the event, $N_k > 0$, to happen, the first step should be $+1$, ($X_1 = +1$). If the first step is -1, the random walk has to hit 0 before it reaches $k \geq 1$. Given that the first step is $+1$, the probability to hit k ($A = k-1$ more steps up) before hitting 0 ($B = 1$ step down) is given by $1/k$ from Eq. (1.2). Combining the two results together, we have $P(N_k > 0) = 1/(2k)$ for $k \geq 1$.

Next, we prove that

$$P(N_k \geq j+1 \mid N_k \geq j) = \frac{1}{2} + \frac{k-1}{2k}.$$

Imagine that the random walk just hit the level k for the j-th time before hitting 0. If the next step is up, it is guaranteed that it will hit k at least one more time before returning to 0. If the next step is down, we know that the probability to hit k one more time ($A = 1$) before hitting 0 ($B = k-1$) is $(k-1)/k$. Adding the two probabilities together, we obtain the result.

Therefore, we can say that, for $j \geq 0$,

$$\begin{aligned} P(N_k > j) &= P(N_k > 0)P(N_k > 1|N_k > 0)\cdots P(N_k > j|N_k > j-1) \\ &= \frac{1}{2k}\left(\frac{1}{2} + \frac{k-1}{2k}\right)^j = \frac{1}{2k}\left(\frac{2k-1}{2k}\right)^j. \end{aligned}$$

We can prove the final statement:

$$\begin{aligned}
E(N_k) =& \sum_{j=1}^{\infty} j \cdot P(N_k = j) = P(N_k = 1) + 2P(N_k = 2) + 3P(N_k = 3) + \cdots \\
=& (P(N_k = 1) + P(N_k = 2) + P(N_k = 3) + \cdots) \\
& + (P(N_k = 2) + P(N_k = 3) + \cdots) \\
& + (P(N_k = 3) + \cdots) + \cdots \\
=& \sum_{j=1}^{\infty} P(N_k \geq j) = \sum_{j=0}^{\infty} P(N_k > j) = \frac{1}{2k} \sum_{j=0}^{\infty} \left(\frac{2k-1}{2k} \right)^j = 1.
\end{aligned}$$

3. **[2016HW 1, A popular interview quiz, Recurrence relation]**
Imagine that you keep tossing a fair coin (50% for head and 50% for tail) until you get two heads in a row. On average, how many times do you need to toss the coin?

Solution: Let X be the answer (expected number of tosses) and branch on the following three cases based on the outcomes in the beginning. The head is denoted by **H** and tail by **T**.

1. **T** (Prob $= 1/2$): You start from the scratch with 1 toss wasted. So the expected number of tosses in the branch is 1+X.
2. **HT** (Prob $= 1/4$): You start from the scratch with 1 toss wasted. So the expected number of tosses in the branch is 2+X.
3. **HH** (Prob $= 1/4$): You get two heads in a row in 2 tosses.

Therefore, we obtain the following equation on X

$$X = \frac{1}{2}(1 + X) + \frac{1}{4}(2 + X) + \frac{1}{4} \cdot 2$$

and conclude that $X = 6$.

4. **[2021ME, Coin toss for two heads in a row (HH)]** From 2016HW 1, A popular interview quiz, we know it takes on average 6 times to get 2 heads (**H**) in a row. We are going to solve this again, but using a martingale similar to the gambler's ruin.

Suppose you bet A_{n-1} ($A_0 = 1$) for head in the n-th toss. Therefore, your wealth after the n-th toss is

$$M_n = A_0 X_1 + \cdots + A_{n-1} X_n + \cdots \quad (M_0 = 0) \quad \text{where} \quad X_n = +1(\mathbf{H}) \text{ or } -1(\mathbf{T}).$$

You stop this gamble when you get **HH**. That is, the stopping time τ is defined as the first n such that $X_n = X_{n-1} = +1$ (this is obviously a stopping time).

(a) You want to determine the bet A_{n-1} such that $M_n = -n$ if $X_n = -1(\mathbf{T})$. For example, you want $M_1 = -1$ if $X_n = -1$. Therefore $A_0 = 1$ is correct. You can also satisfy $M_n = -n$ by betting the same amount $A_{n-1} = 1$ until you get $X_n = 1(\mathbf{H})$ for the first time. What is M_n when you get $X_n = 1(\mathbf{H})$ for the first time? What should be your next bet A_n to make sure $M_{n+1} = -(n+1)$ if $X_{n+1} = -1$?

(b) What is M_n when you get two **H**'s in a row? In other words, what is M_τ?

(c) What is $E(\tau)$? Hint: The process M_n is martingale and so is $M_{n\wedge\tau}$. Use $M_0 = E(M_\tau)$, assuming that all mathematical conditions are satisfied.

(d) What is the expected number of coin toss until you get 3 **H**'s in a row? Use martingale property. You can get the same answer using the recurrence relation, but you will get only 3 points.

Solution:

(a) Because $M_{n-1} = -(n-1)$, $M_n = 2 - n$ when you get **H** for the first time. For the $(n+1)$-th toss, you should bet $A_n = 3$ because, when you get **T** next time, $M_{n+1} = 2 - n - 3 = -(n+1)$.

(b) You get two **H**'s (for the first time) at $t = n$ means that you had **T** at $t = n-2$, so $M_{n-2} = 2 - n$. Now you had **H** at $t = n-1$, so $M_{n-1} = 3 - n$. From (a), you bet $A_{n-1} = 3$ and won it (because you get the second **H**), so $M_n = 6 - n$.

(c) We arrive at the same answer, $E(\tau) = 6$, because

$$0 = M_0 = E(6 - \tau) = 6 - E(\tau).$$

(d) From (b), $M_n = 6 - n$ when you get two **H**'s in a row for the first time. To have $M_{n+1} = -(n+1)$ in case you get **T**, you need to bet $A_n = 7$ ($M_{n+1} = 6 - n - 7 = -(n+1)$). Having 3 **H**'s in a row at $t = n$ means that from $M_{n-3} = -(n-3)$, you won 1, 3, and 7 to arrive at $M_n = 14 - n$. From the martingale property,

$$0 = M_0 = E(14 - \tau) = 14 - E(\tau),$$

it takes 14 coin toss until you get 3 **H**'s in a row.

5. **[2016HW 2-1]** Related to the last statement of Chapter 1 of **SCFA**, prove that the expected time for the gambler to become first positive is infinite: $E(\tau) = \infty$. [Hint: consider the derivative of $\phi(z)$ with respect to z]

Solution:

$$\begin{aligned} E(\tau) &= \sum_{k=1}^{\infty} k\,P(\tau = k) = \sum_{k=1}^{\infty} k\,P(\tau = k)\,z^{k-1}\Big|_{z=1} \\ &= \frac{d}{dz}\phi(z)\Big|_{z=1} = -\frac{1 - \sqrt{1 - z^2}}{z^2} + \frac{1}{\sqrt{1 - z^2}}\Big|_{z=1} = \infty \end{aligned}$$

6. **[2019HW 1-4]** Let

$$S_n = S_0 + X_1 + X_2 + \cdots + X_n, \quad \text{where} \quad X_k = 1\ (p = 0.5) \text{ or } -1\ (q = 1/2)$$

and τ be the first time n when S_n hits either A or $-B$. From lectures, We know that

$$E(\tau | S_0 = k) = (A - k)(B + k)$$

and that it can be understood as the price of the accumulator derivative which pays $\$\,\tau$ when the underlying stock price S_n hits either A or $-B$ for the first time. Imagine that you are the bank who sold the derivative to a client and you want to hedge the derivative by buying or selling some amount of the underlying stock.

(a) On the n-th day with $S_n = k$ $(n < \tau)$, how many shares (positive or negative) of the underlying stock do you need to hold to hedge the derivative? (Hint: consider the derivative with respect to k.)

(b) You can verify your answer in (a) under specific scenarios. Assume $S_0 = 0$, $A = 2$, and $B = 1$, and consider the following two scenarios:

1. $S_1 = -1$ $(\tau = 1)$. Therefore, you pay \$1 to your client on the day $n = 1$.
2. $S_n = 0, 1, 0, 1, 2$ on $n = 0, \cdots, 4$ $(\tau = 4)$. Therefore, you pay \$4 to your client on the day $n = 4$.

On each scenario, track your the profit & loss (P&L) from your hedge position and record the accumulated value (of cash and stock) on each day. Do not forget that you received $\$AB$ from your client as the premium, so your initial value is AB. On the day τ, you have to pay $\$\,\tau$ to your client. What is the accumulated cash you hold (after selling the stocks for the hedge)? Is that cash amount same as τ?

Solution:

(a)

$$E(\tau | S_n = k) = n + (A - k)(B + k), \qquad \frac{d}{d\,k} E(\tau | S_n = k) = A - B - 2k$$

Therefore, you have to hold $A - B - 2k$ shares of the underlying stock.

(b) In both scenarios, the accumulated cash is same as the payout, τ.

Scenario 1:

Day (n)	0	1
S_n	0	-1
Hedge (Share)	1	x
P&L from hedge	x	-1
Accumulated value	2	1

Scenario 2:

Day (n)	0	1	2	3	4
S_n	0	1	0	1	2
Hedge (Share)	1	-1	1	-1	x
P&L from hedge	x	1	1	1	-1
Accumulated value	2	3	4	5	4

Note: P&L on day k is the hedge position at $t = k - 1$ times $(S_k - S_{k-1})$. Accumulated value is the sum the cash and the value of the hedge.

7. **[2017ME, Recurrence relation]** A startup company in Shenzhen either fails or succeeds every year with probability of 25% and 75% respectively. If a company is successful, employees spin off a new startup at the end of the year (and it will face with the same fail/success probability afterwards). There is no correlation between companies. Assume that Shenzhen sets off with one startup company when the city was established as a special economic zone in 1980. What is the probability that all startups in Shenzhen eventually fail.

> **Solution:** Let p be the probability and branch on the whether the first company fails or succeeds. If the first company succeeds and there are two companies in Shenzhen next year, the probability that both fail is p^2 due to the independence assumption. Therefore,
>
> $$p = \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot p^2 \quad \Rightarrow \quad 3p^2 - 4p + 1 = (3p-1)(p-1) = 0$$
>
> Among the two solutions, $p = 1/3$ is the right solution.

8. **[2018ME]** In the gambler's ruin problem,

$$S_n = X_1 + \cdots + X_n, \quad X_k = \pm 1 \quad \text{with probability} \quad p : q \ (p+q=1),$$

what is the probability that S_n ever hits a level $A > 0$? How does this probability changes when p changes? (Hint: consider $B \to \infty$ from the results we know from class.)

> **Solution:** The probability of hitting A before hitting $-B$ is given by
>
> $$P(S_\tau = A) = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1} \quad (p \neq q) \quad \text{or} \quad \frac{B}{A+B} \quad (p = q = 1/2)$$
>
> If we let $B \to \infty$, the probability of S_n ever hitting A is
>
> $$1 \quad \text{if} \quad p \geq 0.5 \quad \text{or} \quad (p/q)^A \quad \text{if} \quad p < 0.5$$

2 First Martingale Steps

1. **[SCFA 2.1]**

Solution: The roots of the equation qualify for the x

$$E(x_1^X) = \frac{0.52}{x} + 0.45x + 0.03x^2 = 1.$$

Form

$$\frac{0.52}{x} + 0.45x + 0.03x^2 - 1 = \frac{x-1}{x}\big(0.03x^2 + 0.48x - 0.52\big)$$

we have the following three values.

$$x = 1,\ \frac{-0.24 \pm \sqrt{0.24^2 + 0.03 \times 0.24}}{0.03} = 1,\ 1.01850,\ -17.0185.$$

We pick $x = 1.01850$ since the change under high powers are reasonable. If we let τ be the first time Gambler's wealth is either 100, 101 or -100, we have the equation from the Martingale property,

$$1 = E(M_\tau) = x^{100}P(S_\tau = 100) + x^{101}P(S_\tau = 101) + x^{-100}P(S_\tau = -100).$$

Letting $p = P(S_\tau = 100) + P(S_\tau = 101)$ and using the fact that $x > 1$,

$$x^{100}p + x^{-100}(1-p) < 1 < x^{101}p + x^{-100}(1-p)$$
$$\frac{1-x^{-100}}{x^{101}-x^{-100}} < p < \frac{1-x^{-100}}{x^{100}-x^{-100}} \quad \Rightarrow \quad 0.13531 < x < 0.13788.$$

2. **[SCFA 2.4]**

Solution: From the hint, let us define

$$A_{n+1} = A_n + E[(M_{n+1} - M_n)^2 | \mathcal{F}_n]$$

and prove the three required properties:

(i) N_n is a martingale.

$$\begin{aligned} E[N_{n+1} \,|\, \mathcal{F}_n] &= E[M_{n+1}^2 - A_{n+1} \,|\, \mathcal{F}_n] \\ &= E[2M_{n+1}M_n - M_n^2 - A_n \,|\, \mathcal{F}_n] \\ &= 2E[M_{n+1} \,|\, \mathcal{F}_n]\, M_n - M_n^2 - A_n \\ &= M_n^2 - A_n = N_n. \end{aligned}$$

(ii) $A_{n+1} \geq A_n$ is trivial.

(iii) A_n is non-anticipating because it is defined via the expectation under $\mathcal{F}_n$.

3. **[2017HW 1-1]** Using martingale property, re-drive that

$$E(\tau) = AB \quad \text{for} \quad \tau = \min\{n : S_n = A \text{ or } S_n = -B\}.$$

Solution: You can find the answer in **SCFA** Section 2.3.

4. **[2018HW 2-2]** Martingale page (WIKIPEDIA) gives the following example of Martingale. Prove (or disprove) the statement.

> Suppose each amoeba either splits into two amoebas, with probability p, or eventually dies, with probability $q = 1-p$. Let X_n be the number of amoebas surviving in the n-th generation (in particular $X_n = 0$ if the population has become extinct by that time). Let r be the probability of eventual extinction. (Finding r as a function of p is an instructive exercise. Hint: The probability that the descendants of an amoeba eventually die out is equal to the probability that either of its immediate offspring dies out, given that the original amoeba has split.) Then
>
> $$\{\, r^{X_n} : n = 1, 2, 3, \dots \,\}$$
>
> is a martingale with respect to $\{X_n : n = 1, 2, 3, \dots\}$.

See 2017ME Recurrence Relation to derive the extinction probability r as a function of p (and q).

Solution: First, we find r in terms of p and q. Since the survival of amoeba's are independent, the probability of n amoebas' extinction is r^n. Branching on the 2nd generation (i.e., multiplication vs death of the first amoeba), we get the following recurrence relation,

$$r = p \cdot r^2 + q,$$

and we can solve $r = q/p$. In fact, the relation above indicate the first step proof of the Martingale:

$$E(r^{X_2}|X_1 = 1) = E(r^{X_1}) = r.$$

For the rest, we do not need $r = q/p$, but just the relation, $1 = pr + q/r$.

At each step, each of X_n amoebas either becomes 2 or 0 with probability of p and q respectively. We can write

$$X_{n+1} = X_n + \sum_{k=1}^{X_n} I_k, \quad r^{X_{n+1}} = r^{X_n} \cdot r^{I_1 + \cdots + I_{X_n}}$$

where I_k takes value of $+1$ or -1 with probability p and q respectively and $\{I_k\}$ are independent events. The probability of j amoebas multiplying by 2 ($X_n - j$ amoebas die) is given by the binomial distribution, $\binom{n}{j}p^j\ q^{X_n-j}$. Therefore,

$$E(r^{I_1+\cdots+I_{X_n}}) = \sum_{j=0}^{X_n}\binom{X_n}{j}p^j\ q^{X_n-j}\cdot\frac{r^j}{r^{X_n-j}} = \sum_{j=0}^{X_n}\binom{X_n}{j}(pr)^j\ \left(\frac{q}{r}\right)^{X_n-j}$$
$$= \left(pr+\frac{q}{r}\right)^{X_n} = 1.$$

Therefore,

$$E(r^{X_{n+1}}|X_n) = r^{X_n}\cdot E(r^{I_1+\cdots+I_{X_n}}) = r^{X_n}.$$

5. **[2020HW 1-1]** Consider the gambler's fortune with an unfair coin:

$$S_n = X_1+X_2+\cdots X_n \quad \text{where} \quad X_n = \begin{cases} 1 & \text{(probability } p) \\ -1 & \text{(probability } q) \end{cases}.$$

(a) Prove that $M_n = (q/p)^{S_n}$ is a martingale.

(b) If τ is the first time n that S_n hits A or $-B$, find $\text{Prob}(S_\tau = A)$ using the martingale property,

$$1 = M_0 = E(M_{n\wedge\tau}) \text{ for all } n = E(M_\tau).$$

Solution:

(a)

$$E(M_{n+1}|\mathcal{F}_n) = M_n E\left((q/p)^{X_{n+1}}\right) = \left(\frac{q}{p}p+\frac{p}{q}q\right)M_n = M_n$$

(b)

$$1 = E(M_\tau) = \text{Prob}(S_\tau = A)(q/p)^A + (1-\text{Prob}(S_\tau = A))(q/p)^{-B}$$
$$\text{Prob}(S_\tau = A) = \frac{1-(q/p)^{-B}}{(q/p)^A-(q/p)^{-B}} = \frac{(q/p)^B-1}{(q/p)^{A+B}-1}$$

6. **[2016ME(ASP), Martingale, Polya's urn]**
A box has 1 red ball and 9 blue balls. Pick up one ball randomly. If it is red, put it back and add one more red ball into the box. If it is blue, put it back and add one more blue ball into the box. If Y_n is the proportion of the red balls in the box after the process is repeated n times ($Y_0 = 0.1$), show that $\{Y_n\}$ is a martingale. So what is the expected number of the red balls after you repeat the process 100 times?

Solution:

$$Y_{n+1} = \begin{cases} \frac{(n+10)Y_n+1}{n+11} & \text{if a red ball is picked with probability } Y_n \\ \frac{(n+10)Y_n}{n+11} & \text{if a blue ball is picked with probability } 1-Y_n \end{cases}$$

Therefore,

$$E(Y_{n+1}|Y_n) = \frac{(n+10)Y_n + 1}{n+11} Y_n + \frac{(n+10)Y_n}{n+11}(1 - Y_n) = Y_n,$$

so Y_n is a martigale. The expected number of the red balls at $n = 100$ is

$$E\big((10 + 100)Y_{100}\big) = 110\, Y_0 = 110 \times 0.1 = 11.$$

7. **[2016ME(ASP), Wald's equation]**
When $\{X_k\}$ are independent identically distributed random variable and N is a random variable taking positive integer values, Wald's equation says

$$E(X_1 + X_2 + \cdots X_N) = E(N)\, E(X_1)$$

if either (i) N is independent from $\{X_k\}$ or (ii) N is a stopping time with respect to $\{X_k\}$. Consider an example where $X_k = 0$ or 1 with 50% and 50% probability and N is given by

$$N = X_1 + X_2 + 1.$$

Obviously, $E(X_k) = 1/2$ and $E(N) = 2/2+1 = 2$. Find $E(X_1+X_2+\cdots+X_N)$ and explain why Wald's equation does not hold in this example.

Solution: We can branch on the first scenarios (with probability 1/4) depending on the outcome of X_1 and X_2: 0–0, 0–1, 1–0 and 1–1.

$$\begin{aligned} E(X_1 + X_2 + \cdots + X_N) &= \frac{1}{4}E(0) + \frac{1}{4}E(0+1) + \frac{1}{4}E(1+0) + \frac{1}{4}E(1+1+X_3) \\ &= \frac{1}{4}(0 + 1 + 1 + 2 + \frac{1}{2}) = \frac{9}{8} \neq E(X_k)E(N) = 1. \end{aligned}$$

Wald's equation does not hold because $N = X_1 + X_2 + 1$ is neither
(i) independent from $\{X_k\}$: N is defined via X_1 and X_2
nor (ii) a stopping time w.r.t. $\{X_k\}$: it looks into the future, i.e., X_2 at $k = 1$.

8. **[2016ME, Wald's equation]**
When $\{X_k\}$ are independent identically distributed random variable and N is a random variable taking positive integer values, Wald's equation says

$$E(X_1 + X_2 + \cdots X_N) = E(N)\, E(X_1)$$

if either (i) N is independent from $\{X_k\}$ or (ii) N is a stopping time with respect to $\{X_k\}$. Consider an example where $X_k = 0$ or 1 with 50% and 50% probability and N is given by

$$N = X_2 + 1.$$

Obviously, $E(X_k) = 1/2$ and $E(N) = 1/2 + 1 = 3/2$. Find $E(X_1 + X_2 + \cdots + X_N)$ and explain why Wald's equation does not hold in this example. If N is given instead as

$$N = X_1 + 1,$$

does Wald's equation hold? Is N a stopping time?

Solution: Branching on the value of X_2:

$$N = \begin{cases} 1 & (X_2 = 0, \quad \text{Prob} = 1/2) \\ 2 & (X_2 = 1, \quad \text{Prob} = 1/2), \end{cases}.$$

we compute

$$\begin{aligned} E(X_1 + X_2 + \cdots + X_N) &= \frac{1}{2}E(X_1) + \frac{1}{2}E(X_1 + 1) \\ &= \frac{1}{2}(\frac{1}{2} + \frac{1}{2} + 1) = 1 \neq E(X_k)E(N) = \frac{3}{4}. \end{aligned}$$

Wald's equation does not hold because $N = X_1 + X_2 + 1$ is
(i) not independent from $\{X_k\}$ as N is defined via X_2 and (ii) not a stopping time w.r.t. $\{X_k\}$ because we need future information X_2 in order to determine N (X_1 is not enough to tell $N = 1$ or not.)

If $N = X_1 + 1$, N is a stopping time. Based on X_1 we can tell whether $N = 1$ or not: $N = 1$ if $X_1 = 0$ and $N \neq 1$ (in fact, $N = 2$) if $X_1 = 1$. Therefore Wald's inequality should hold. We can directly verify that

$$E(X_1 + X_2 + \cdots + X_N) = \frac{1}{2}E(X_1) + \frac{1}{2}E(X_1 + X_2) = \frac{1}{2}(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) = \frac{3}{4}.$$

3 Brownian Motion

1. **[SCFA 3.1, Brownian Bridge]**

Solution:

(a) This problem is based on a series representation of BM (WIKIPEDIA). See p.286 of **SCFA** also.

$$B_t = tZ_0 + \sqrt{2}\sum_{k=1}^{\infty} Z_k \frac{\sin \pi k t}{\pi k}$$

for independent standard normal random variables, $\{Z_{k\geq 0}\}$. (But I think the author somehow dropped this in the current version of **SCFA**.) So, $\Delta_0(t) = t$ and $\lambda_0 = 1$. Because $B_1 = Z_0$ (from $\Delta_{k\geq 1}(1) = 0$), the first term can be expressed as $\lambda_0 Z_0 \Delta_0(t) = t\,B_1$. Therefore,

$$U_t = B_t - tB_1$$

(b)

$$\begin{aligned}\mathrm{Cov}(U_s, U_t) &= E\Big((B_s - sB_1)(B_t - tB_1)\Big) = E\Big(B_sB_t - sB_1B_t - tB_sB_1 + stB_1^2\Big) \\ &= \min(s,t) - s\min(1,t) - t\min(s,t) + st = s(1-t)\end{aligned}$$

(c) We need to find any set of functions, $g(\cdot)$ and $h(\cdot)$, such that

$$\mathrm{Cov}(X_s, X_t) = g(s)g(t)\min(h(s), h(t)) = s(1-t) \quad \text{for} \quad s \leq t.$$

If we narrow down the search by assuming $h(\cdot)$ is monotonically increasing,

$$\mathrm{Cov}(X_s, X_t) = g(s)g(t)h(s) = s(1-t),$$

so we get

$$g(t) = 1-t, \quad h(s) = \frac{s}{1-s},$$

where $h(s)$ is indeed an increasing function. Therefore we obtained a representation of Brownian bridge,

$$X_t = (1-t)B_{\frac{t}{1-t}}$$

Since U_{1-t} is also a Brownian bridge due to the symmetry,

$$X_{1-t} = tB_{\frac{1-t}{t}}, \quad \Big(g(t) = t,\ h(t) = \frac{1-t}{t}\Big)$$

is also a valid solution.

(d) We use the inequality $s/(1+s) \leq t/(1+t)$ if $0 \leq s \leq t$.

$$\begin{aligned}\mathrm{Cov}(Y_s, Y_t) &= \mathrm{Cov}\Big((1+s)U_{\frac{s}{1+s}}(1+t)U_{\frac{t}{1+t}}\Big) = (1+s)(1+t)\mathrm{Cov}(U_{\frac{s}{1+s}}, U_{\frac{t}{1+t}}) \\ &= (1+s)(1+t)\ \frac{s}{1+s}\Big(1 - \frac{t}{1+t}\Big) = s = \min(s,t).\end{aligned}$$

2. **[SCFA 3.2, Cautionary Tale]**

Solution: Suppose X is a standard normal, consider an independent U such that $P(U = 1) = 1/2 = P(U = -1)$, and set $Y = UX$. Then, Y is also a standard normal as X and $-X$ are also standard normal.

In order to show X and Y are not independent, we need to show

$$P(I_X \ \& \ J_Y) \neq P(I_X)P(J_Y)$$

for some event I_X and I_Y regarding X and Y respectively.

For any $h > 0$,

$$\begin{aligned} P(X > h \ \& \ Y > h) &= \frac{1}{2}P(X > h \ \& \ X > h \mid U = 1) \\ &\quad + \frac{1}{2}P(X > h \ \& \ -X > h \mid U = -1) \\ &= \frac{1}{2}(1 - N(h)) + 0. \end{aligned}$$

However,

$$P(X > h)P(Y > h) = (1 - N(h))(1 - N(h))$$

is not same as the previous value. Therefore

3. **[SCFA 3.3, Multivariate Gaussians]**

Solution:

(a) Let us work on each components of the vectors and matrices; $V = (v_i)$, $\mu = (\mu_i)$, $A = (a_{ij})$ and $\Sigma = (\sigma_{ij})$.

$$E\Big((AV)_i\Big) = E\Big(\sum_j a_{ij}V_j\Big) = \sum_j a_{ij}E\Big(V_j\Big) = \sum_j a_{ij}\mu_j = (A\mu)_i$$

$$E\big(AV\big) = A\mu$$

$$\begin{aligned} \mathrm{Cov}\Big((AV)_i, (AV)_j\Big) &= \mathrm{Cov}\Big(\sum_l a_{il}V_l, \sum_m a_{jm}V_m\Big) = \sum_{l,m} a_{il}\mathrm{Cov}(V_l, V_m)a_{jm} \\ &= \sum_{l,m} a_{il}\sigma_{lm}a_{jm} = (A\Sigma A^T)_{ij} \end{aligned}$$

Therefore,

$$\mathrm{Cov}\Big(AV, AV\Big) = A\Sigma A^T.$$

(b)

$$\begin{aligned} E(X \pm Y) &= E(X) \pm E(Y) = 0 \pm 0 = 0 \\ \mathrm{Var}(X \pm Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) \pm 2\,\mathrm{Cov}(X, Y) = 1 + 1 + 0 = 2 \\ \mathrm{Cov}(X + Y, X - Y) &= \mathrm{Var}(X) - \mathrm{Var}(Y) = 1 - 1 = 0 \end{aligned}$$

(c) When $\mathrm{Cov}(X,Y)=0$, the covariance matrix Σ is given by

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix}, \quad \Sigma^{-1} = \begin{pmatrix} 1/\sigma_{11} & 0 \\ 0 & 1/\sigma_{22} \end{pmatrix}, \quad \det\Sigma = \sigma_{11}\,\sigma_{22}$$

The joint density function can be factored to the product of the single variable density function,

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sqrt{\sigma_{11}\,\sigma_{22}}} \exp\Big(-\frac{(x-\mu_X)^2}{2\sigma_{11}} - \frac{(y-\mu_Y)^2}{2\sigma_{22}}\Big) \\ &= \frac{1}{2\pi\sqrt{\sigma_{11}}} \exp\Big(-\frac{(x-\mu_X)^2}{2\sigma_{11}}\Big) \frac{1}{2\pi\sqrt{\sigma_{22}}} \exp\Big(-\frac{(y-\mu_Y)^2}{2\sigma_{22}}\Big) = f(x)f(y). \end{aligned}$$

Therefore X and Y are independent.

(d) We first find the matrix A such that, for the independent standard normal variables W and Z,

$$\begin{pmatrix} X-\mu_X \\ Y-\mu_Y \end{pmatrix} = A \begin{pmatrix} W \\ Z \end{pmatrix}$$

has the given covariance matrix

$$\begin{pmatrix} \sigma_{XX} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{YY} \end{pmatrix} = A\,I\,A^T = A\,A^T.$$

One of the solution from Cholesky decomposition is

$$A = \begin{pmatrix} \sqrt{\sigma_{XX}} & 0 \\ \sigma_{XY}/\sqrt{\sigma_{XX}} & \sqrt{\sigma_{YY} - \sigma_{XY}^2/\sigma_{XX}} \end{pmatrix}.$$

Conditional on that $X=x$,

$$Y = \mu_Y + \frac{\sigma_{XY}}{\sqrt{\sigma_{XX}}}\frac{x-\mu_X}{\sqrt{\sigma_{XX}}} + Z\sqrt{\sigma_{YY} - \sigma_{XY}^2/\sigma_{XX}}.$$

Therefore

$$\begin{aligned} E(Y|X=x) &= \frac{\sigma_{XY}}{\sigma_{XX}}(x-\mu_X) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}(x-\mu_X) \\ \mathrm{Var}(Y|X=x) &= \sigma_{YY} - \frac{\sigma_{XY}^2}{\sigma_{XX}} = \mathrm{Var}(Y) - \frac{\mathrm{Cov}^2(X,Y)}{\mathrm{Var}(X)} \end{aligned}$$

4. **[SCFA 3.4, Auxiliary Functions and Moments]**

Solution:

$$E(e^{tz}) = \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = \int_{-\infty}^{\infty} e^{t^2/2}\frac{1}{\sqrt{2\pi}}e^{-(z-t)^2/2}dz = e^{t^2/2}$$

If M_n is the $n-$th moment,

$$E(e^{tz}) = \sum_{k=0}^{\infty} M_k \frac{t^k}{k!} = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \cdots$$

By matching the coefficients, we get

$$\begin{aligned} &M_0 = 1 \\ &M_1 = M_3 \; (= M_{2k-1}) = 0 \\ &M_2 = 1 \\ &M_4 = 4!/(2!\;2^2) = 3 \\ &M_6 = 6!/(3!\;2^3) = 15. \end{aligned}$$

For $t > 0$,

$$E(e^{tz^4}) = \int_{-\infty}^{\infty} e^{tz^4} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \to \infty,$$

so the MGF of Z^4 does not exist.

5. **[2019HW 2-1]** For a standard BM, B_t, find the probability that

$$B_1 + B_2 + B_3 \geq 3.$$

Solution: Let $x = B_1$, $y = B_2 - B_1$, and $z = B_3 - B_2$, then x, y, and z follow $\mathcal{N}(0,1)$ and they are independent. Since,

$$B_1 + B_2 + B_3 = x + (x+y) + (x+y+z) = 3x + 2y + z \sim \mathcal{N}(0, 14),$$

the probability that $B_1 + B_2 + B_3 \geq 3$ is $1 - N(3/\sqrt{14}) = 0.2113$.

6. **[2022ME, BM]**

   (a) For a standard BM, B_t, find the probability that

   $$B_1 + B_2 + B_3 - 3B_4 \geq 2.$$

   (b) When a stock price follows $S_t = S_0 + \sigma B_t$, consider the following option payout at the expiry $T = 4$:

   $$\max\left(S_4 - \frac{S_1 + S_2 + S_3}{3}, 0\right).$$

   The payout is similar to that of the regular call option, except that the strike price K is determined by the average stock price at $t = 1$, 2, and 3. What is the price of this option (i.e., the expectation of the payout)?

Solution: Let $x = B_1$, $y = B_2 - B_1$, $z = B_3 - B_2$, and $w = B_4 - B_3$, then x, y, z and w follow independent standard normal variables. It follows that

$$B_1 + B_2 + B_3 - 3B_4 = x + (x+y) + (x+y+z) - 3(x+y+z+w) = -(y+2z+3w) \sim \mathcal{N}(0, 14).$$

(a) The probability is

$$P(B_1 + B_2 + B_3 - 3B_4 \geq 2) = 1 - N(2/\sqrt{14}) \approx 0.2965.$$

(b) The payout is normally distributed as

$$S_4 - \frac{S_1 + S_2 + S_3}{3} = \frac{\sigma}{3}(y + 2z + 3w) \sim \mathcal{N}\left(0, \frac{14}{9}\sigma^2\right).$$

From the extended Bachelier model, the call option price is given by

$$C \approx 0.4\frac{\sqrt{14}}{3}\sigma.$$

7. **[2020HW 1-2]** Prove that, if B_t is a standard BM, the inverted process,

$$Y_0 = 0 \quad \text{and} \quad Y_t = t\, B_{1/t} \quad \text{for} \quad t > 0,$$

is also a standard BM.

Solution: Y_t satisfy the following requirements to be a standard BM:

(i) $Y_0 = 0$ by definition. (In fact, you can prove this too!)

(ii) Let $s < t < u$ so that $1/u < 1/t < 1/s$. We will show that the increments of Y_t are independent by showing $\mathrm{Cov}(Y_t - Y_s, Y_u - Y_s) = 0$. We can decompose $Y_t - Y_s$ and $Y_u - Y_s$ into

$$\begin{aligned} Y_t - Y_s &= tB_{1/t} - sB_{1/s} = (t-s)B_{1/u} + (t-s)(B_{1/t} - B_{1/u}) - s(B_{1/s} - B_{1/t}) \\ Y_u - Y_t &= uB_{1/u} - tB_{1/t} = (u-t)B_{1/u} - t(B_{1/t} - B_{1/u}). \end{aligned}$$

Because $B_{1/u}$, $B_{1/t} - B_{1/u}$, and $B_{1/s} - B_{1/t}$ are mutually independent, the covariance is calculated as

$$\begin{aligned} \mathrm{Cov}(Y_t - Y_s, Y_u - Y_s) &= E\left((t-s)(u-t)B_{1/u}^2\right) - E\left((t-s)t(B_{1/t} - B_{1/u})^2\right) \\ &= \frac{(t-s)(u-t)}{u} - (t-s)t\left(\frac{1}{t} - \frac{1}{u}\right) = 0. \end{aligned}$$

(iii) For $s \leq t$,

$$Y_t - Y_s = tB_{1/t} - sB_{1/s} = (t-s)B_{1/t} + s(B_{1/t} - B_{1/s}).$$

Since

$$(t-s)B_{1/t} \sim N(0,(t-s)^2/t) \quad \text{and} \quad s(B_{1/t}-B_{1/s}) \sim N(0, s-s^2/t),$$

and $B_{1/t}$ and $B_{1/t}-B_{1/s}$ are independent from (iii), $Y_t - Y_s$ are normally distribute with variance

$$\text{Var}(Y_t - Y_s) = (t-s)^2/t + s - s^2/t = t - s.$$

(iv) Y_t is continuous for $t > 0$. Y_t is also continuous at $t = 0$. $\lim_{t\to 0} Y_t \to Y_0 = 0$ because $E(Y_t) \to 0$ and $\text{Var}(Y_t) = t^2/t = t \to 0$ as $t \to 0$.

Alternatively, from **Lemma 3.1** of **SCFA**, you can prove (ii) by showing that Y_t is a Gaussian process with $E(Y_t) = 0$ and $\text{Cov}(Y_s, Y_t) = \min(s,t)$. Y_t is naturally a Gaussian process. For $s \le t$ $(1/t \le 1/s)$,

$$\text{Cov}(Y_s, Y_t) = \text{Cov}(sB_{1/s}, tB_{1/t}) = \text{Cov}(sB_{1/t} + s(B_{1/s} - B_{1/t}), tB_{1/t}) = \frac{s\,t}{t} = s.$$

8. **[2016ME, Standard BM]**
If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Provide a brief reason for your answer.
   (a) $4B_{t/2}$
   (b) $tB_{1/t}$ with $B_0 = 1$.
   (c) $2(B_{1+t/4} - B_1)$
   (d) $\sqrt{t}Z$ for a standard normal RV Z

Solution:
(a) No. $\text{Var}(4B_{t/2}) = 16 \times t/2 = 8t \neq t$.

(b) No. B_0 should be 0.

(c) Yes. $B_{1+t/4} - B_1$ is equivalent to $B_{t/4}$ and $2B_{t/4}$ is equivalent to B_t.

(d) No. For any value of Z, $\sqrt{t}Z$ is not a stochastic process. For example, $\sqrt{s}Z$ and $(\sqrt{t} - \sqrt{s})Z$ for $s < t$ are correlated.

9. **[2017ME, Standard BM]** If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Provide a brief reason for your answer.
   (a) $\frac{1}{2}B_{4t}$
   (b) $\frac{1}{2}(B_{1+2t} - B_1)$
   (c) $\frac{3}{5}B_t + \frac{4}{5}W_t$, where W_t is another BM independent from B_t.

Solution:

(a) **Yes**. $\mathrm{Var}(\frac{1}{2}B_{4t}) = \frac{1}{4} \cdot 4t = t$.

(b) **No**. It is equivalent to $\frac{1}{2}B_{2t}$, however $\frac{1}{2}B_{2t}$ is not a standard BM.

(c) **Yes**. $aB_t + bW_t$ is a standard BM when $a^2 + b^2 = 1$.

10. **[2018ME, Standard BM]** If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Explain briefly why it is a standard BM or not.

(a) $(1/\sqrt{2})B_{2t}$

(b) $\begin{cases} B_t & \text{if} \quad t \le \tau_a \\ 2a - B_t & \text{if} \quad t > \tau_a \end{cases}$, where τ_a is the first time B_t hitting the level a.

(c) $\frac{1}{13}(5B_t + 12W_t)$ where W_t is another standard BM independent from B_t.

(d) $B_{2t} - B_t$

Solution:

(a) Yes. The scaling property.

(b) Yes. The reflection principle.

(c) Yes.

$$\frac{5^2 + 12^2}{13^2} = 1$$

(d) No. Let $X_t = B_{2t} - B_t$.

$$\begin{aligned} \mathrm{Cov}(X_s, X_t) &= E((B_{2s} - B_s)(B_{2t} - B_t) = E(B_{2s}B_{2t} - B_{2s}B_t - B_sB_{2t} + B_sB_t) \\ &= 3\min(s,t) - \min(2s,t) - \min(s,2t) \end{aligned}$$

If X_t is a standard BM, $\mathrm{Cov}(X_1, X_2) = \min(1,2) = 1$. However, $\mathrm{Cov}(X_1, X_2) = 3 - 2 - 1 = 0$.

11. **[2021ME, Standard BM]** If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Explain briefly why.

(a) $\frac{1}{2}B_{4t}$

(b) $\frac{3}{5}B_t + \frac{4}{5}X_t$ where X_t is another BM independent from B_t.

(c) $B_t^2 - t$

(d) $X_t = B_t - B_{t/2}$

Solution:

(a) Yes from the scaling property: $\mathrm{Var}(\frac{1}{2}B_{4t}) = \frac{1}{4}4t = t$.

(b) Yes. $\mathrm{Var}(\frac{3}{5}B_t + \frac{4}{5}X_t) = \frac{9t}{25} + \frac{16t}{25} = t$.

(c) No. At $t = 1$, $B_1^2 - 1 \sim Z^2 - 1$ for $Z \sim N(0,1)$ is not a Gaussian distribution.

(d) No. At $t = 1$, $\mathrm{Var}(X_1) = 1/2 \neq 1$.

12. **[2016ME, Average of a BM path]**
If B_t for $0 \leq t \leq 1$ is a standard BM, what is the distribution of the average of the BM values observed at three different times, $T = 1/3, 2/3$ and 1,

$$A = \frac{1}{3}\Big(B_{\frac{1}{3}} + B_{\frac{2}{3}} + B_1\Big)?$$

Please make sure to provide the mean and the standard deviation of the distribution.

Solution:

$$\begin{aligned}
\mathrm{Var}\Big(\frac{1}{3}(B_{1/3} + B_{2/3} + B_1)\Big) &= \frac{1}{9}E\Big((B_{1/3} + B_{2/3} + B_1)^2\Big) \\
&= \frac{1}{9}E\Big(B_{1/3}^2 + B_{2/3}^2 + B_1^2 + 2B_{1/3}(B_{2/3} + B_1) + 2B_{2/3}B_1\Big) \\
&= \frac{1}{9}\Big(\frac{1}{3} + \frac{2}{3} + 1 + 2 \cdot \frac{1}{3} \cdot 2 + 2 \cdot \frac{2}{3}\Big) = \frac{1}{9}\frac{14}{3} = \frac{14}{27}.
\end{aligned}$$

4 Martingales: The next steps

1. **[SCFA 4.6]**

Solution: The stopped process is a martingale. By the symmetry, $\mathrm{P}(B_\tau = A) = \mathrm{P}(B_\tau = -A) = 0.5$.

$$1 = E(X_\tau) = \frac{1}{2}e^{\alpha A}E(e^{-\alpha^2\tau/2}) + \frac{1}{2}e^{-\alpha A}E(e^{-\alpha^2\tau/2}).$$

Therefore, we have

$$E\left(e^{-\alpha^2\tau/2}\right) = \frac{1}{\cosh(\alpha A)}$$

or

$$\phi(\lambda) = E(e^{-\lambda\tau}) = \frac{1}{\cosh(A\sqrt{2\lambda})}.$$

In order to calculate $E(\tau^2)$, we need to obtain the x^4 term in the expansion of $1/\cosh(x)$ given that $\sqrt{\lambda}$ appears in the expression. From the expansion, $\cosh x \sim 1 + x^2/2! + x^4/4! + \cdots$,

$$\begin{aligned}\frac{1}{\cosh x} &\sim \frac{1}{1 + (x^2/2! + x^4/4! + \cdots)} = 1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} + \cdots\right)^2 \\ &= 1 - \frac{x^2}{2} + \frac{5}{24}x^4 + \cdots .\end{aligned}$$

Finally we get

$$E(\tau^2) = 2\frac{5}{24}(A\sqrt{2\lambda})^4|_{\lambda=1} = \frac{5}{3}A^4$$

For the non-symmetric case ($A \neq B$), we can not use $\mathrm{P}(B_\tau = A) = \mathrm{P}(B_\tau = -B) = 0.5$ anymore.

2. **[2016ME, Martingale related to BM]**
If B_t is a standard BM, find the value of the coefficient λ in order for each of the following expressions to be a martingale.
   (a) $B_{at}^2 - \lambda t$
   (b) $\exp(-B_{at} + \lambda t)$

Solution:
(a) $\lambda t = E(B_{at}^2) = at$. Therefore, $\lambda = a$.
(b) $\sqrt{a}B_t$ is a BM equivalent to $-B_{at}$. Therefore, $\lambda = -a/2$.

3. **[2020ME, Martingale related to BM]** If B_t is a standard BM, determine whether the following is a martingale or not. Give a brief reason.
   (a) $Y_t = B_{\lambda t}^2 - \lambda^2 t$

(b) $Y_t = \exp(-B_{at} - a^2t/2)$

(c) $Y_t = \begin{cases} 2A - B_t & \text{if} \quad t \le \tau \\ B_t & \text{if} \quad t > \tau \end{cases}$, where τ is the first time B_t hits A ($\tau = \min\{t : B_t = A\}$)

(d) $S_t = S_0 + \sigma B_{t\wedge\tau}$ where $\tau = \min\{t : S_{t+1} - S_t < -A\}$ for some $A > 0$.

Solution:

(a) No. $Y_t = B_{\lambda t}^2 - \lambda t$ is a martingale.

(b) No. $Y_t = \exp(-B_{at} - at/2)$ is a martingale.

(c) Yes. Y_t is a BM starting with $2A$ from the reflection principle. Although Y_t is not a standard BM, it is still a martingale.

(d) No. τ is not a proper stopping time because it is based on forward-looking information (i.e., S_{t+1}).

4. **[2022ME, Martingale related to BM]** If B_t is a standard BM, determine whether the following is a martingale or not. Give a brief reason.

(a) $Y_t = B_{\lambda t}^2 - \lambda t^2$

(b) $Y_t = \exp(2\sigma B_t - \sigma^2 t)$

(c) $S_t = S_0 \exp(\sigma B_{t\wedge\tau} - \frac{\sigma^2 (t\wedge\tau)}{2})$ where $\tau = \min\{t : |B_t - B_{t-1}| > A\}$ for some $A > 0$.

Solution:

(a) **No**. $Y_t = B_{\lambda t}^2 - \lambda t$ is a martingale.

(b) **No**. $Y_t = \exp(2\sigma B_t - 2\sigma^2 t)$ is a martingale.

(c) **Yes**. τ is a proper stopping time and $M_t = S_0 \exp(\sigma B_t - \sigma^2 t/2)$ is a martingale. Therefore, $M_{t\wedge\tau}$ is a martingale.

5. **[2022ME, Gambler's ruin with BM]** Assume that a stock price follows a BM with volatility σ and initial price S_0:

$$S_t = S_0 + \sigma B_t,$$

and that the stopping time τ is the first time S_t hits $S_0 + A$ or $S_0 - B$ for $A, B > 0$. Find $E(\tau)$.

Solution: From class, we know

$$E(\tau) = AB \quad \text{where} \quad \tau = \min\{t : B_t = A \quad \text{or} \quad B_t = -B\}.$$

The stopping time in the problem is

$$\begin{aligned} \tau &= \min\{t : S_t = S_0 + A \quad \text{or} \quad S_t = S_0 - B\} \\ &= \min\{t : \sigma B_t = A \quad \text{or} \quad \sigma B_t = -B\} \\ &= \min\{t : B_t = A/\sigma \quad \text{or} \quad B_t = -B/\sigma\}. \end{aligned}$$

Therefore,

$$E(\tau) = \frac{AB}{\sigma^2}.$$

6. **[2020ME, Geometric BM]** Assume that a stock price follows a geometric BM with volatility σ and initial price S_0,

$$S_t = S_0 \exp\left(\sigma B_t - \frac{\sigma^2 t}{2}\right)$$

When you observe the stock price ever year ($t = 1, 2, \cdots$), find the probability that the annual stock return is positive for the following 3 years,

$$P(S_1 > S_0 \text{ and } S_2 > S_1 \text{ and } S_3 > S_2).$$

Express the answer with $n(\cdot)$ or $N(\cdot)$.

Solution: For $t = n$,

$$\begin{aligned} P(S_{n+1} > S_n) &= P(S_{n+1}/S_n > 1) = P(\sigma(B_{n+1} - B_n) - \sigma^2/2 > 0) \\ &= P(\sigma Z - \sigma^2/2 > 0) = P(Z > \sigma/2) \quad \text{for a standard normal} \quad Z \\ &= 1 - N(\sigma/2) \quad \text{or} \quad N(-\sigma/2) \end{aligned}$$

Thanks to the independent increments of BM, $S_1 > S_0$, $S_2 > S_1$, and $S_3 > S_2$ are independent events. Therefore,

$$P(S_1 > S_0 \text{ and } S_2 > S_1 \text{ and } S_3 > S_2) = (1 - N(\sigma/2))^3$$

For example, if $\sigma = 20\%$ $(= 0.2)$, the probability is $(1 - N(0.1))^3 \approx (0.460)^3 = 9.74\%$.

Bachelier (Normal) Model

1. **[2016HW 3-1, Digital option]** Derive the (forward) price of the digital(binary) call/put option struck at K at maturity T. The digital(binary) call/put option pays \$1 if S_T is above/below the strike K, i.e. $1_{S_T \ge K}/1_{S_T \le K}$.

Solution: Similarly following the derivation of the call price, the digital call option price is

$$C_D(K) = E(1_{S_T \ge K}) = P(S_T \ge K) = \int_{-d_\text{N}}^{\infty} n(z)dz = 1 - N(-d_\text{N}) = N(d_\text{N}).$$

and the digital put option price is

$$P_D(K) = E(1_{S_T \le K}) = P(S_T \le K) = \int_{-\infty}^{-d} n(z)dz = N(-d_\text{N}) = 1 - N(d_\text{N}),$$

where d_N is given by

$$d_\text{N} = \frac{S_0 - K}{\sigma\sqrt{T}}.$$

Notice that $N(d_\text{N})$ has another meaning as the probability of the stock price ends up in-the-money in addition the delta of the call option.

2. **[2016HW 3-2, Asset-or-nothing option]** The payoff of the call option, $\max(S_T - K, 0)$ can be decomposed into two parts,

$$S_T \cdot 1_{S_T \ge K} \quad - \quad K \cdot 1_{S_T \ge K}.$$

The first payout is the payout of the **asset-or-nothing** call option and the second payout if the binary call option multiplied with $-K$. Under the Bachelier model, what is the price of the asset-or-nothing call option?

Solution: From the binary call option price above and the (regular) call option price from the class,

$$C(K) = (F - K)N(d_\text{N}) + \sigma\sqrt{T}\, n(d_\text{N}),$$

we conclude that

$$C_\text{A-or-N} = F\, N(d_\text{N}) + \sigma\sqrt{T}\, n(d_\text{N}).$$

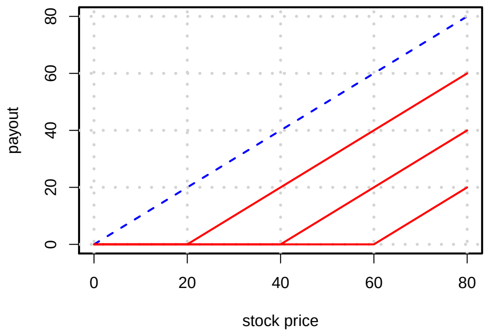
3. **[2016HW 4-4, Maximum call option value]**

   (a) Pleased with the outstanding performance of the Stochastic Finance class, Professor is going to give each student a gift of **EITHER** one share of Tencent stock **OR** one unit of the call option on Tencent stock struck just at 1 yuan (so the call option is deep in-the-money) with maturity at the end of this module. Which gift has more financial value? (Assume that Tencent pays no dividend. No calculation required. Use your common sense.)

(b) What is the upper limit of the call option value under the Bachelier model? In the Bachelier model, under which circumstance the option is more valuable than the underling stock itself? How does it affect your choice of the gift in the previous question?

Solution:

(a) Given that a stock price can not go below zero, the payoff of a stock is always greater than that of a call option with any strike K. Therefore, the underlying stock is more valuable. See the plot below for the payoff of a stock (dashed blue) versus that of the options with $K = 20, 40$ and 60.



(b) The price of a call option (and put option as well) is unbounded (can go to infinity) under the Bachelier model. This happens when volatility is very high. As $\sigma \to \infty$, $d \to 0$, $N(d) \to 0$ and $n(d) \to 1/\sqrt{2\pi}$. Therefore $C \approx 0.4\sigma\sqrt{T} \to \infty$. Intuitively, it is because a call option gives you a protection against the negative underlying price (underlying asset becoming liability), which is possible under the Bachelier model. On the other hand, under Black-Scholes-Merton model the call option price is always bounded by the price of the underlying asset as mentioned in the class. Since a stock value can not be negative, you still better of by choosing a stock rather than a call option.

4. **[2019HW 2-2, Option delta and vega]** Under the Bachelier (normal) model, the undiscounted price of call option is given by

$$C_{\mathrm{N}}(K) = (S_0 - K)N(d_{\mathrm{N}}) + \sigma\sqrt{T}\,n(d_{\mathrm{N}}) \quad \text{for} \quad d_{\mathrm{N}} = \frac{S_0 - K}{\sigma_{\mathrm{N}}\sqrt{T}}$$

(a) Show that the delta, the sensitivity on S_0, is

$$\frac{\partial C_{\mathrm{N}}}{\partial S_0} = N(d_{\mathrm{N}})$$

(b) Show that the vega, the sensitivity on σ, is

$$\frac{\partial C_{\mathrm{N}}}{\partial \sigma_{\mathrm{N}}} = \sqrt{T}\,n(d_{\mathrm{N}})$$

In the questions above, make sure to apply the derivative to d_{N} since d_{N} depends on S_0 and σ_{N}.

Solution:

(a)

$$\begin{aligned}\frac{\partial C_{\text{N}}}{\partial S_0} &= N(d_{\text{N}}) + (S_0 - K)n(d_{\text{N}})\frac{\partial d_{\text{N}}}{\partial S_0} + \sigma\sqrt{T}(-d_{\text{N}})n(d_{\text{N}})\frac{\partial d_{\text{N}}}{\partial S_0} \\ &= N(d) + \Big((S_0 - K)n(d_{\text{N}}) - \sigma\sqrt{T}\, d_{\text{N}}\, n(d_{\text{N}})\Big)\, \frac{\partial d_{\text{N}}}{\partial S_0} \\ &= N(d_{\text{N}}) + 0\cdot\frac{\partial d_{\text{N}}}{\partial S_0} = N(d_{\text{N}})\end{aligned}$$

(b)

$$\begin{aligned}\frac{\partial C_{\text{N}}}{\partial \sigma_{\text{N}}} &= (S_0 - K)n(d_{\text{N}})\frac{\partial d_{\text{N}}}{\partial \sigma_{\text{N}}} + \sqrt{T}\, n(d_{\text{N}}) + \sigma_{\text{N}}\sqrt{T}\,(-d_{\text{N}})n(d_{\text{N}})\frac{\partial d_{\text{N}}}{\partial \sigma_{\text{N}}} \\ &= \sqrt{T}\, n(d_{\text{N}}) + \Big((S_0 - K) - \sigma\sqrt{T}\, d_{\text{N}}\Big)\, n(d)\frac{\partial d_{\text{N}}}{\partial \sigma_{\text{N}}} \\ &= \sqrt{T}\, n(d_{\text{N}}) + 0\cdot n(d_{\text{N}})\frac{\partial d_{\text{N}}}{\partial \sigma_{\text{N}}} = \sqrt{T}\, n(d_{\text{N}})\end{aligned}$$

5. **[2019HW 2-3, A simple credit default swap]** Assume that the stock price of a firm follows an arithmetic BM,

$$S_t = S_0 + \sigma_{\text{N}} B_t \quad \text{where } B_t \text{ is a standard BM}$$

and that the firm is considered default if $S_t = 0$. You want to buy a credit default swap (CDS) from an investment bank. You continuously pay at the rate of λ to the bank as premium (i.e., pay λdt during the time period between t and $t + dt$). The bank will pay you \$$M$ if the firm defaults. We want to determine the fair value of λ. Assume that the continuously compounded interest rate is $r > 0$.

(a) What is the expected value of the payout (\$$M$) at the default event.

(b) What is the expected value of the premium payment.

(c) Determine the fair rate λ by equating the two values above.

(Hint: use the result from **Hitting time of a level** in SCFA Section 4.5)

Solution: The Laplace transform of the first hitting time τ is given by

$$E(e^{-r\tau}) = e^{-(S_0/\sigma_{\text{N}})\sqrt{2r}}.$$

(a) The expectation of the CDS payout is

$$M \cdot E(e^{-r\tau}) = Me^{-(S_0/\sigma_{\text{N}})\sqrt{2r}}.$$

(b) The present value of the premium paid until τ is

$$\int_0^\tau e^{-rt}\lambda dt = \frac{\lambda}{r}(1-e^{-r\tau}).$$

Taking the expectation,

$$E\left(\frac{\lambda}{r}(1-e^{-r\tau})\right) = \frac{\lambda}{r}\left(1-E\left(e^{-r\tau}\right)\right) = \frac{\lambda}{r}\left(1-e^{-(S_0/\sigma_{\rm N})\sqrt{2r}}\right)$$

(c) By equating the results of (a) and (b),

$$\frac{\lambda}{r}\left(1-e^{-(S_0/\sigma_{\rm N})\sqrt{2r}}\right) = Me^{-(S_0/\sigma_{\rm N})\sqrt{2r}},$$

we obtain λ as

$$\lambda = \frac{rMe^{-(S_0/\sigma_{\rm N})\sqrt{2r}}}{1-e^{-(S_0/\sigma_{\rm N})\sqrt{2r}}} = \frac{rM}{e^{(S_0/\sigma_{\rm N})\sqrt{2r}}-1}.$$

6. **[2016ME(ASP), Asian option]** Asian option is an option where the payoff at maturity T is derived from the average of the underlying prices at a given set of times before and at the maturity,

$$\left(\frac{1}{N}\sum_{k=1}^{N} S(t_k) - K\right)^+ \quad \text{for} \quad 0 < t_1 < \cdots < t_N = T.$$

When $N = 4$ and $t_k = k/4$ ($T = 1$) (a quarterly averaged Asian option), find the price of the Asian option. Assume the underlying stock price follows BM process, $dS(t) = \sigma\, dB(t)$, and the option price is given by $C = 0.4\,\sigma\sqrt{T}$ (at-the-money strike, zero interest rate and zero divided rate). How much the price of this Asian option is cheaper (or more expensive) to that of the European option with the same maturity and the same volatility?

Solution:

$$\begin{aligned}
&\mathrm{Var}\Big(\frac{\sigma}{4}(B_{1/4}+B_{2/4}+B_{3/4}+B_{4/4})\Big) = \frac{\sigma^2}{16}\mathrm{Var}(B_{1/4}+B_{2/4}+B_{3/4}+B_{4/4})\\
&= \frac{\sigma^2}{16}E\Big((B_{1/4}+B_{2/4}+B_{3/4}+B_{4/4})^2\Big)\\
&= \frac{\sigma^2}{16}E\Big(B_{1/4}^2+B_{2/4}^2+B_{3/4}^2+B_{4/4}^2+2B_{1/4}(B_{2/4}+B_{3/4}+B_{4/4})\\
&\qquad\qquad + 2B_{2/4}(B_{3/4}+B_{4/4}) + 2B_{3/4}B_{4/4}\Big)\\
&= \frac{\sigma^2}{16}\Big(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{4}{4}+2\frac{1}{4}\cdot 3+2\frac{2}{4}\cdot 2+2\frac{3}{4}\cdot 1\Big) = \frac{\sigma^2}{16}\frac{30}{4} = \frac{15\sigma^2}{32}.
\end{aligned}$$

The option price is given by

$$\text{Price of Asian option} = 0.4\sqrt{\frac{15}{32}}\,\sigma.$$

Asian option is about 31.5% ($= 1 - \sqrt{15/32}$) cheaper than European option with the same expiry, 0.4σ.

7. **[2017ME(ASP), Asian option]**
Asian option is an option where the payoff at maturity T is derived from the average of the underlying prices at a given set of times before and at the maturity,

$$\left(\frac{1}{N}\sum_{k=1}^{N} S(t_k) - K\right)^+ \quad \text{for} \quad 0 < t_1 < \cdots < t_N = T.$$

When $N = 3$ and $t_k = (k+3)/3$ ($T = 2$), find the price of the at-the-money Asian option. Assume the underlying stock price follows an arithmetic BM process (Bachelier model), $dS_t = \sigma\, dW_t$, and the option price is given by $C = 0.4\,\sigma\sqrt{T}$ (at-the-money strike, zero interest rate and zero divided rate). How much the price of this Asian option is cheaper (or more expensive) than that of the European option with the same maturity ($T = 2$) and the same volatility?

Solution:

$$\begin{aligned}\mathrm{Var}\Big(\frac{1}{3}(B_{4/3} + B_{5/3} + B_2)\Big) &= \frac{1}{9}E\Big((B_{4/3} + B_{5/3} + B_2)^2\Big)\\ &= \frac{1}{9}E\Big(B_{4/3}^2 + B_{5/3}^2 + B_2^2 + 2B_{4/3}(B_{5/3} + B_2) + 2B_{5/3}B_2\Big)\\ &= \frac{1}{9}\Big(\frac{4}{3} + \frac{5}{3} + 2 + 2\cdot\frac{4}{3}\cdot 2 + 2\cdot\frac{5}{3}\Big) = \frac{1}{9}\cdot\frac{41}{3} = \frac{41}{27}.\end{aligned}$$

The option price is given by

$$\text{Price of Asian option} = 0.4\sqrt{\frac{41}{27}}\,\sigma.$$

Asian option is about 12.8% ($= 1 - \sqrt{41/54}$) cheaper than European option with the same expiry, $0.4\sqrt{2}\,\sigma$.

8. **[2017ME, Basket/Spread option under the Bachelier model]** Let W_t and Z_t are standard **BM**s with correlation ρ, i.e., $E(W_t\, Z_t) = \rho\, t$. For (b) and (c), assume that two stock prices follow

$$S_{1t} = 100 + 20W_t, \quad S_{2t} = 80 + 10Z_t.$$

and you may use the simple ATM option price formula, $C = P = 0.4\sigma\sqrt{T}$.

(a) Calculate $\mathrm{Var}(a\, W_T + b\, Z_T)$ for some constants a and b.

(b) Consider a spread option, whose payout is on *the difference of the prices*, $\max\big(S_{1T} - S_{2T} - K, 0\big)$. What is the price of the ATM option ($K = 20$) as a function of ρ? What is the min and max price and when do you get the those values?

(c) Consider a basket option, whose payout is on *the weighted average of the prices*, $\max\big((S_{1T} + S_{2T})/2 - K, 0\big)$. What is the price of the ATM option ($K = 90$) as a function of ρ? What is the min and max price and when do you get the those values?

Solution:

(a)
$$\mathrm{Var}(a\,W_T + b\,Z_T) = (a^2 + 2\rho ab + b^2)T$$

(b) $\mathrm{Var}(S_{1T} - S_{2T}) = \mathrm{Var}(a\,W_T + b\,Z_T)$ with $a = 20$ and $b = -10$. Therefore,
$$C = 0.4\,\sqrt{(a^2 + 2\rho ab + b^2)T} = 4\sqrt{(5 - 4\rho)T},$$
and
$$4\sqrt{T}\ \ (\rho = 1,\text{ i.e., correlated}) \le C \le 12\sqrt{T}\ \ (\rho = -1,\text{ i.e., anti-correlated}).$$

(c) $\mathrm{Var}((S_{1T} + S_{2T})/2) = \mathrm{Var}(a\,W_T + b\,Z_T)$ with $a = 10$ and $b = 5$. Therefore,
$$C = 0.4\,\sqrt{(a^2 + 2\rho ab + b^2)T} = 2\sqrt{(5 + 4\rho)T},$$
and
$$2\sqrt{T}\ \ (\rho = -1,\text{ i.e., anti-correlated}) \le C \le 6\sqrt{T}\ \ (\rho = 1,\text{ i.e., correlated}).$$

9. **[2018ME, Merton's model]** From an accounting standpoint, a firm's equity (stock) value, S, is equal to $A - D$, where A is total asset value and D is total debt. When A goes below D, the firm defaults with equity value $S = 0$. In 1974, Merton proposed a model to price the current equity value S_0 as the expected asset value A_T above the constant debt value D, i.e., the call option on the asset A_T stuck at D, at some time T (expiry):
$$S_0 = E(\max(A_T - D, 0)) \quad \text{where} \quad A_0 > D.$$
If A_t follows a geometric BM, the resulting equity price is same as the call option price formula by Black and Scholes (1973). Thus, the formula is called Black-Scholes-Merton formula (and Merton was awarded Nobel prize with Scholes in 1997).

Instead, in this problem, assume that A_t follows an arithmetic BM with volatility σ:
$$A_t = A_0 + \sigma B_t \quad (\text{assume} \quad r = 0).$$

(a) What should be the current equity value S_0? You may use the result from class.

(b) Under this framework, we can also derive the corporate bond (issued by the firm) maturing at T. At the maturity $t = T$, the firm pays \$1 to the bond holder. If the firm defaults before T, however, the bond value becomes zero. Assume that the risk-free interest rate is zero. (Hint: use the result on the probability of the first hitting time, $P(\tau_a > T)$)

Solution:

(a) It is same as the call option value under the Bachelier model:
$$S_0 = (A_0 - D)N(d_{\mathrm{N}}) + \sigma\sqrt{T}\,n(d_{\mathrm{N}}) \quad \text{where} \quad d_{\mathrm{N}} = \frac{A_0 - D}{\sigma\sqrt{T}}$$

(b) For a standard BM, the probability for the first time hitting the level a is given by

$$P(\tau_a > T) = 2N\left(\frac{|a|}{\sqrt{T}}\right) - 1$$

The bond price is same as the probability of A_t hitting D happening later than T. Therefore,

$$\text{Corporate bond price} = 2N\left(\frac{A_0 - D}{\sigma\sqrt{T}}\right) - 1$$

10. **[2019ME, Equity Linked Note]** An equity-linked note (ELN) is a debt instrument, usually a bond, that differs from a standard fixed-income security in that the final payout is based on the return of the underlying equity, which can be a single stock, basket of stocks, or an equity index. Equity-linked notes are a type of structured products (WIKIPEDIA). We (as a security firm) want to design and sell an ELN based on a stock following a BM:

$$S_t = S_0 + \sigma\, B_t.$$

This note has coupon N periods, $t = k\Delta t$ for $k = 1, 2, \cdots, N$. At $t = 0$, investors buy this note at the price of P for the notional value of \$1. At the end of the k-th period, $t = k\Delta t$, it pays coupon μ if the stock price did not fall more than δ (i.e., if $S_{k\Delta t} - S_{(k-1)\Delta t} \geq -\delta$) and continues to the next period. At the maturity $t = N\Delta t$, it pays $1 + \mu$ (1 is the redemption of the notional value). If the price falls more than δ at $t = k\Delta t$, (i.e., $S_{k\Delta t} - S_{(k-1)\Delta t} < -\delta$), the note terminates immediately by redeeming $(1 - L)$ (at the loss of L).

Assume that the discounting rate for one period, Δt, is r. So, the present value of \$1 paid after time Δt is $1/(1+r)$. To simplify notation, you can use $D = 1/(1+r)$.

The basic design of this ELN is that investors receive coupon μ higher than the risk-free rate r if stock market does not crash. However, they take a risk of heavy loss L if market crashes.

(a) Obtain the price P by calculating the expected value of the payout of this ELN.

(b) Assume the following specific parameters:

$$N = 8,\ \Delta t = 0.25,\ S_0 = 100,\ \sigma = 10,\ \delta = 10,\ D = 0.97\ (r \approx 3\%),\ L = 0.5.$$

That is, this ELN observes the price every 3 months and the maturity is 2 years. Given the parameters, determine the return μ to make the price of this ELN par (i.e., $P = 1$). You may use spreadsheet. How does it compares to the risk-free rate $r \approx 3\%$?

(c) Right after clients buy this ELN at the price $P = 1$, the volatility of the underlying stock suddenly increased to $\sigma = 20$ due to the spread of the Corvid-19 virus. (Assume that $S_0 = 100$ is unchanged.) What is client's loss?

Solution:

(a) From the independent increment of BM, the probability p to pay the coupon μ is same at every period. The probability, p, and $q = 1 - p$ is given by

$$\begin{aligned} p &= P(S_{k\Delta t} - S_{(k-1)\Delta t} \geq -\delta) = P(\sigma B_{\Delta t} \geq -\delta) \\ &= 1 - N\left(-\frac{\delta}{\sigma\sqrt{\Delta t}}\right) = N\left(\frac{\delta}{\sigma\sqrt{\Delta t}}\right). \end{aligned}$$

The present value of the ELN is decomposed into three components:

- The coupon μ paid at the k-th period ($k = 1, \cdots, n$):

$$\text{Present Value} = \mu D^k, \quad \text{Probability} = p^k$$

  The expectation is given by

$$\sum_{k=1}^{n} \mu D^k \cdot p^k = \mu(pD)\frac{1-(pD)^n}{1-pD}$$

- The early terminated redemption with loss, $(1 - L)$, at the k-th period ($k = 1, \cdots, n$):

$$\text{Present Value} = (1-L)D^k, \quad \text{Probability} = p^{k-1}q.$$

  The expectation is give by

$$\sum_{k=1}^{n}(1-L)D^k \cdot p^{k-1}q = (1-L)\frac{q}{p}\sum_{k=1}^{n}(pD)^k = (1-L)(qD)\frac{1-(pD)^n}{1-pD}$$

- The redemption of the notional \$1 at the n-th period (maturity):

$$\text{Present Value} = D^n, \quad \text{Probability} = p^n.$$

  The expectation is $(pD)^n$.

The price P is the sum of the three expected values:

$$\begin{aligned} P &= \mu(pD)\frac{1-(pD)^n}{1-pD} + (1-L)(qD)\frac{1-(pD)^n}{1-pD} + (pD)^n \\ &= \Big(\mu p + (1-L)q\Big)\, D\, \frac{1-(pD)^n}{1-pD} + (pD)^n. \end{aligned}$$

(b) The coupon μ to satisfy $P = 1$ is obtained as

$$\begin{aligned} \Big(\mu p + (1-L)q\Big)\, D\, \frac{1-(pD)^n}{1-pD} + (pD)^n &= 1 \\ \mu p + (1-L)q &= \frac{1-pD}{D} \\ \mu &= \frac{1}{p}\left(\frac{1-pD}{D} - (1-L)q\right) \end{aligned}$$

Using the parameter values, the coupon should be $\mu = 4.234\%$. The intermediate values are

$$p = N(2) = 97.725\% \quad \text{and} \quad q = 1 - p = 2.275\%.$$

(c) When σ is suddenly jump to $\sigma = 20$, the price drops down to $P = 0.6867$. The loss is about 31%. The intermediate values are

$$p = N(1) = 84.134\% \quad \text{and} \quad q = 1 - p = 15.866\%.$$

11. **[2020ME, Forward-starting option]** A forward-starting option with expiry T is an option whose strike price is set relative to the stock price at time $t = T'$ $(< T)$ not at time $t = 0$. Suppose that the strike will be set as $K = S_{T'} + \Delta$ at $t = T'$. Therefore, the payout of the forward-starting call option at expiry T is given by

$$\text{Payout} = \max(S_T - K, 0) = \max(S_T - S_{T'} - \Delta, 0).$$

Assume that the underlying stock price follows a BM, $S_t = S_0 + \sigma B_t$ (and $r = q = 0$). Derive the price of the forward-starting call option. You may use the Bachelier model option formula without proof.

Solution: Since $S_T - S_{T'} = \sigma(B_T - B_{T'}) \sim \sigma\sqrt{T - T'}\, Z$ for a standard normal Z, we can use the Bachelier option price formula with time-to-expiry $T - T'$, spot price 0, and strike price Δ:

$$C = -\Delta N(d_{\text{N}}) + \sigma\sqrt{T - T'}\, n(d_{\text{N}}), \quad d_{\text{N}} = -\frac{\Delta}{\sigma\sqrt{T - T'}}.$$

5 Richness of Paths

1. **[An extra problem]** Derive the probability results on the running maximum and the first hitting time to the BM with the volatility σ. Using the scaling, $\sigma B_t = B_{\sigma^2 t}$, you are going to replace t with $\sigma^2 t$.

Solution: For the CDF (on both time and space) and the PDF on space, the simple replacement works.

$$P(\sigma B_t^* < x) = P(\tau_x > t) = 2N(x/\sigma\sqrt{t}) - 1.$$

$$f_{\sigma B_t^*}(x) = \frac{2}{\sigma\sqrt{t}} n\left(\frac{x}{\sigma\sqrt{t}}\right)$$

For the PDF on time, however, we need to consider the normalization because the time is scaled. The original PDF $f_{\tau_x}(t)$ satisfies $\int_0^\infty f_{\tau_x}(t)dt = 1$. After the scaling,

$$\int_0^\infty f_{\tau_x}(\sigma^2 t)dt = \frac{1}{\sigma^2},$$

so the new PDF should be

$$\sigma^2 f_{\tau_x}(\sigma^2 t) = \frac{x}{\sigma t^{3/2}}\, n\left(\frac{x}{\sigma\sqrt{t}}\right).$$

2. **[2016ME(ASP). Maximum of a BM with drift]**

Proposition The maximum of a BM with drift $\mu < 0$, $B^* = \max_{0\le t\le\infty}(B_t + \mu t)$, has exponential distribution

$$P(B^* > x) = e^{2\mu x} \quad (x \ge 0)$$

(a) Using the above proposition, prove that

$$P(B_t \le at + b \text{ for all } t > 0) = 1 - e^{-2ab} \quad \text{for} \quad a, b > 0$$

(b) Using a proper change of variable, scaling of BM and etc, extend the result of (a) for BM with volatility, σB_t.

Solution:

(a)

$$\begin{aligned} P(B_t \le at + b \text{ for all } t \ge 0) &= P(B_t - at \le b \text{ for all } t \ge 0) \\ &= 1 - P(B_t - at > b \text{ for some } t \ge 0) \\ &= 1 - P(B^* > b) = 1 - e^{-2ab} \end{aligned}$$

(b) Method 1:

$$P(\sigma B_t \le at + b) = P\left(B_t \le \frac{a}{\sigma}t + \frac{b}{\sigma}\right) = 1 - e^{-2ab/\sigma^2}$$

Method 2: We use $\sigma B_t \sim B_{\sigma^2 t}$ ($t' = \sigma^2 t$):

$$P(\sigma B_t \leq at + b) = P(B_{\sigma^2 t} \leq at + b) = P\left(B_{t'} \leq \frac{a}{\sigma^2}t' + b\right) = 1 - e^{-2ab/\sigma^2}$$

3. **[2018ME, Call option on maximum]** Assume that a stock price follows a BM, $S_t = S_0 + \sigma B_t$. As in the text book, assume that $S_t^* = \max_{0 \leq s \leq t} S_s$. Calculate the call option price whose payout at expiry $t = T$ is given by the maximum value on the path

$$\max(S_T^* - K, 0) \quad \text{where} \quad K > S_0$$

Intuitively, this option should be more expensive that the regular call option whose payout is given by the final price S_T. By how much more is it more expensive? (Hint: In class and textbook, we derived the PDF of B_T^*. Properly adjust σ.)

Solution: The PDF for the maximum of BM, B_t^*, is given by

$$f(x) = \frac{2}{\sqrt{t}}\, n\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x \geq 0$$

This is equivalent to the normal distribution, $N(0, t)$, defined on the positive side only (that is why factor 2 is multiplied). Therefore, $S_T^* = S_0 + \sigma\sqrt{T}\, z$ where the PDF of z is $2n(z)$ and $z \geq 0$. The option price is twice as expensive as that of the regular call option:

$$C(K) = 2(S_0 - K)N(d_{\text{N}}) + 2\sigma\sqrt{T}\, n(d_{\text{N}}) \quad \text{where} \quad d_{\text{N}} = \frac{S_0 - K}{\sigma\sqrt{T}}$$

4. **[2019ME, Put option on minimum]** Assume that the stock price follows a BM, $S_t = S_0 + \sigma B_t$. Assume that $S_t^m = \min\{S_s : 0 \leq s \leq t\}$. Calculate the put option price whose payout at expiry $t = T$ is given by the minimum value along the path

$$\max(K - S_T^m, 0) \quad \text{where} \quad K < S_0$$

Intuitively, this option should be more expensive that the regular put option whose payout is given by the final price S_T. By how much is it more expensive?

Solution: Let $W_t = -B_t$, then W_t is also a standard BM. The minimum of B_t is the negative of the maximum of W_t:

$$B_t^m = \min\{B_s : 0 \leq s \leq t\} = -\max\{W_s : 0 \leq s \leq t\} = -W_t^M$$

Since the PDF for W_t^M, is given by

$$f(x) = \frac{2}{\sqrt{t}}\, n\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x \geq 0,$$

the PDF for B_t^m is same as

$$f(x) = \frac{2}{\sqrt{t}}\, n\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x \le 0.$$

because $n(x) = n(-x)$. This is two times the PDF for B_t defined on the negative side only. The minimum put option price is twice as expensive as that of the regular put option:

$$P(K) = 2(K - S_0)N(-d_{\text{N}}) + 2\sigma\sqrt{T}\, n(d_{\text{N}}) \quad \text{where} \quad d_{\text{N}} = \frac{S_0 - K}{\sigma\sqrt{T}}$$

5. **[2016HW 3-3, Bachelier knock-out (up-and-out) option]**
Using the joint distribution of B_t and $B_t^* = \max_{0\le s\le t} B_s$, derive the price of the call option struck at K and knock-out at H ($> K$). First, generalize the joint CDF function $P(u < B_t, v < B_t^*)$ to σB_t. Next, derive the pdf on u by taking derivative on u. Then, integrate the payoff $(S_T - K)^+$ from K to H. (Assume $r = q = 0$. Otherwise the problem is too complicated.)

Solution:

$$\begin{aligned} P(S_T^* < v, S_T < u) &= P(\sigma B_T^* < v - S_0,\ \sigma B_T < u - S_0) \\ &= P(B_T^* < (v - S_0)/\sigma,\ B_T < (u - S_0)/\sigma) \\ &= N\Big(\frac{u - S_0}{\sigma\sqrt{T}}\Big) - N\Big(\frac{u - 2v + S_0}{\sigma\sqrt{T}}\Big) \end{aligned}$$

The probability density function on u conditional on $S_T^* < H$ is obtained from the partial derivative w.r.t. u,

$$f(u) = \frac{1}{\sigma\sqrt{T}}\left(n\Big(\frac{u - S_0}{\sigma\sqrt{T}}\Big) - n\Big(\frac{u - 2H + S_0}{\sigma\sqrt{T}}\Big)\right) \quad \text{for} \quad -\infty < u \le v$$

With variables, $z = (u - S_0)/\sigma\sqrt{T}$. $d = (S_0 - K)/\sigma\sqrt{T}$ $d^* = (S_0 - H)/\sigma\sqrt{T}$, the knock-out call option price is given by

$$
\begin{aligned}
C(K,H) = &\int_K^H (u-K)\, f(u) du = \int_{-d}^{-d^*} (S_0 - K + \sigma\sqrt{T}\, z)\big(n(z) - n(z+2d^*)\big) dz \\
=&(S_0-K)\int_{-d}^{-d^*} \big(n(z) - n(z+2d^*)\big) dz \\
&+ \sigma\sqrt{T}\int_{-d}^{-d^*} \big(z\, n(z) - (z+2d^*)\, n(z+2d^*) + 2d^*\, n(z+2d^*)\big) dz \\
=&(S_0-K)\big(N(-d^*) - N(-d) - N(d^*) + N(-d+2d^*)\big) + \sigma\sqrt{T}\Big(- n(-d^*) \\
&+ n(-d) + n(d^*) - n(-d+2d^*) + 2d^* N(d^*) - 2d^* N(-d+2d^*)\Big) \\
=&(S_0-K)\big(N(d) - 2N(d^*) + N(2d^*-d)\big) \\
&+ \sigma\sqrt{T}\big(n(d) - n(2d^*-d) + 2d^* N(d^*) - 2d^* N(2d^*-d)\big)
\end{aligned}
$$

We can verify two trivial cases:

1. If $K = H$ $(d = d^*)$, the option price should be zero

$$C(K, H=K) = 0.$$

2. If $H = \infty$ $(d^* = -\infty)$, the price is same as the price of the regular call option

$$C(K, H=\infty) = (S_0 - K)N(d) + \sigma\sqrt{T}\, n(d).$$

We also note that the difference in the prices of the knock-out option and the regular option is given by

$$
\begin{aligned}
C(K) - C(K,H) =&(S_0-K)\big(2N(d^*) - N(2d^*-d)\big) \\
&+ \sigma\sqrt{T}\big(n(2d^*-d) - 2d^* N(d^*) + 2d^* N(2d^*-d)\big).
\end{aligned}
$$

6. **[2016ME(ASP), Down-and-out digital option, Joint distribution of B_t^m and B_t]** We are going to derive the price of the binary call option with knock-out (down-and-out) feature under the Bachelier model. Assume that the underlying stock follows the process $S_t = S_0 + \sigma B_t$. The option will pay you \$1 at the expiry T if $S_T > K$ for a strike price K **and** the stock price S_t has never been below L for $L < \min(S_0, K)$ anytime before the expiry, $0 \le t \le T$. In other words, this option knocks out (expires worthless) if S_t falls below L any time before the expiry T. Let the running maximum and minimum of B_t

$$B_T^M = \max_{0\le t\le T} B_t \quad \text{and} \quad B_T^m = \min_{0\le t\le T} B_t.$$

(a) In the class (and in the textbook), we derived the joint CDF for B_T and B_T^M,

$$P(B_T^M < v,\ B_T < u) = N\Big(\frac{u}{\sqrt{T}}\Big) - N\Big(\frac{u-2v}{\sqrt{T}}\Big) \quad \text{for} \quad v \ge \max(0, u)$$

Using this result, derive the joint CDF for BM with volatility, σB_t,

$$P(\sigma B_T^M < v,\ \sigma B_T < u).$$

(b) Using the symmetry that $-\sigma B_t$ has the same distribution as σB_t, drive the probability

$$P(\sigma B_T^m > v, \sigma B_T > u) \quad \text{for} \quad v \le \min(0, u).$$

(c) Finally find the price of the binary call option with knock-out feature? Assume that interest rate and divided rate is zero. How much is this derivative cheaper (or more expensive) than the regular binary call option **without** knock-out feature?

Solution:

(a)

$$P(\sigma B_T^M < v,\ \sigma B_T < u) = P(B_T^M < v/\sigma,\ B_T < u/\sigma) = N\Big(\frac{u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{u-2v}{\sigma\sqrt{T}}\Big)$$

(b) Under the reflection, $B_t \to -B_t$, we get

$$(-B)_T^m = \min_{0\le t\le T}(-B_t) = -\max_{0\le t\le T} B_t = -B_T^M.$$

Applying the reflection to the probability,

$$\begin{aligned}
P(\sigma B_T^m > v,\ \sigma B_T > u) &= P(\sigma(-B)_T^m > v,\ \sigma(-B_T) > u)\\
&= P(-B_T^M > v/\sigma,\ -B_T > u/\sigma)\\
&= P(B_T^M < -v/\sigma,\ B_T < -u/\sigma)\\
&= N\Big(\frac{-u}{\sigma\sqrt{t}}\Big) - N\Big(\frac{2v-u}{\sigma\sqrt{t}}\Big).
\end{aligned}$$

(c) Plug in $K - S_0 \to u$ and $L - S_0 \to v$ to get

$$\text{Price with Knockout} = N\Big(\frac{S_0-K}{\sigma\sqrt{t}}\Big) - N\Big(\frac{2L-K-S_0}{\sigma\sqrt{t}}\Big)$$

The price without knockout is the first term, so the knockout feature is cheapening the price by $N\Big(\frac{2L-K-S_0}{\sigma\sqrt{t}}\Big)$.

7. **[2017HW 2-3, Bachelier knock-out (down-and-out) option]** Derive the price of down-and-out call option with knock-out strike L and option strike K. (Obviously, $L < S_0$ and $L < K$) See the derivation for up-and-out call option (2016HW 3-3) and down-and-out digital option (2016ME(ASP)).

Solution: Assume $B_T^M = \max_{0\le t\le T} B_t$ and $B_T^m = \min_{0\le t\le T} B_t$. From textbook and class, we know

$$P(\sigma B_T^M < v,\ \sigma B_T < u) = P(B_T^M < v/\sigma,\ B_T < u/\sigma) = N\Big(\frac{u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{u-2v}{\sigma\sqrt{T}}\Big),$$

where $N(\cdot)$ is the normal distribution CDF. Using the reflection, $B_t \to -B_t$, we get

$$(-B)_T^m = \min_{0\le t\le T}(-B_t) = -\max_{0\le t\le T} B_t = -B_T^M$$

and

$$P(\sigma B_T^m > v,\ \sigma B_T > u) = N\Big(\frac{-u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{2v-u}{\sigma\sqrt{T}}\Big).$$

As the stock price is given by $S_T = S_0 + \sigma B_T$,

$$P(S_T^m > v,\ S_T > u) = N\Big(\frac{S_0-u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{2v-u-S_0}{\sigma\sqrt{T}}\Big).$$

The probability density function on u with the joint condition, $\sigma B_T^m > v$ is obtained from the partial derivative w.r.t. u (with negative sign),

$$f(u) = \frac{1}{\sigma\sqrt{T}}\left(n\Big(\frac{S_0-u}{\sigma\sqrt{T}}\Big) - n\Big(\frac{2v-u-S_0}{\sigma\sqrt{T}}\Big)\right) \quad \text{for} \quad -\infty < v \le u.$$

Let $z = (u-S_0)/\sigma\sqrt{T}$, $d = (S_0-K)/\sigma\sqrt{T}$ and $d^* = (S_0-L)/\sigma\sqrt{T}$. Then, the down-and-out call option price is given by

$$\begin{aligned} C(K,L) = \int_{u=K}^{\infty}(u-K)\,f(u)du &= \int_{z=-d}^{\infty}(S_0-K+\sigma\sqrt{T}\,z)(n(z)-n(z+2d^*))dz \\ &= (S_0-K)N(d) + \sigma\sqrt{T}\,n(d) - (S_0-K-2d^*\sigma\sqrt{T})N(d-2d^*) \\ &\quad -\sigma\sqrt{T}\,n(d-2d^*). \end{aligned}$$

The first two terms are exactly the regular call option price, $C(K) = (S_0-K)N(d) + \sigma\sqrt{T}\,n(d)$. Therefore, the down-and-out option is cheaper than the regular option by $(S_0-K-2d^*\sigma\sqrt{T})N(d-2d^*) + \sigma\sqrt{T}\,n(d-2d^*)$.

We can verify two cases:

1. If $L\to-\infty$ ($d^*\to\infty$), $C(K,L) = C(K)$ because the probably of being knocked out is zero. It is indeed the case because $N(d-2d^*) = n(d-2d^*) = 0$.
2. If $L \to S_0$ from below ($d^*\to 0$) on the other hand, the knock-out probability approaches to 100%, so the price should be zero. This is also the case from the formula.

8. **[2021ME, Knock-out (up-and-out) option under the Bachelier model]**
In 2016HW 3-3, we derived the price of the up-and-out call option with strike price K and knock-out barrier H ($> K, > S_0$). In the problem, the option expires worthless when the stock price S_t hits H. Instead, suppose that the option holder receive a payout of $H-K$ when $S_t = H$ (and the option expires).
   (a) How should you modify the price obtained in 2016HW 3-3?
   (b) The knock-out payout $H-K$ makes this option very similar to the regular European option in the sense that the option pays European option's payout $(S_t-K)^+ = H-K$ at an early expiry $t<T$. Therefore, the price from (a) is similar, but not equal, to

the European option price. Is the price from (a) more expensive or cheaper than the European option price? Intuitively explain why.

Solution:

(a) From 2016HW 3-3, we obtained the knock-out option price as

$$\begin{aligned}C(K,H) =&(S_0-K)\big(N(d)-2N(d^*)+N(2d^*-d)\big)\\&+\sigma\sqrt{T}\big(n(d)-n(2d^*-d)+2d^*N(d^*)-2d^*N(2d^*-d)\big),\end{aligned}$$

where $d=(S_0-K)/\sigma\sqrt{T}$ and $d^*=(S_0-H)/\sigma\sqrt{T}$. In the old problem, the option knocks out worthless when $S_T^*>H$, where S_T^* is the running maximum, $S_T^*=\max_{0\le t\le T}S_t$. Therefore, the new knock-out option (with the payout at knock-out) is more valuable by the payout $(H-K)$ times the knock-out probability. The probability is given by

$$P(S_T^*>H)=P\left(B_T^*>\frac{H-S_0}{\sigma}\right)=2-2N\left(\frac{H-S_0}{\sigma\sqrt{T}}\right)=2N(d^*).$$

Therefore, the new option price is adjusted to

$$C'(K,H)=C(K,H)+2(H-K)N(d^*)$$

(b) The price obtained in (a) is further simplified to

$$\begin{aligned}C'(K,H)&=(S_0-K)\big(N(d)+N(2d^*-d)\big)+\sigma\sqrt{T}\big(n(d)-n(2d^*-d)-2d^*N(2d^*-d)\big)\\&=C(K)-\sigma\sqrt{T}\big(n(2d^*-d)+(2d^*-d)N(2d^*-d)\big),\end{aligned}$$

where $C(K)$ is the regular call option price under the Bachelier model. The last term,

$$\sigma\sqrt{T}\big(n(2d^*-d)+(2d^*-d)N(2d^*-d)\big)$$

is understood as the call option price with strike $2H-K$ because

$$2d^*-d=\frac{S_0-(2H-K)}{\sigma\sqrt{T}}$$

Since option price is positive, we conclude that $C(K)\ge C'(K,H)$

Another intuitive method of comparing the two option values is to consider the knock-out moment (i.e., $S_t=H$). (i) if you hold the knock-out option, you receive $H-K$. (ii) if you hold the regular European option, you may receive the option value (by selling the option at the moment). Or course, the European option value is larger then the payout. Therefore, $C(K)\ge C'(K,H)$.

9. **[2022ME, Knock-out (up-and-out) digital option]** We are going to derive the price of the binary call option with knock-out (up-and-out) feature under the Bachelier model. Assume that the underlying stock follows the process $S_t=S_0+\sigma B_t$. The option will pay you \$1 at the expiry T if $S_T>K$ for a strike price K **and** the stock price S_t has never gone above $H>\max(S_0,K)$ anytime $0\le t\le T$. In other words, this option knocks out (expires

worthless) if S_t goes above H any time before the expiry T. Let the running maximum of B_t

$$B_T^M = \max_{0 \le t \le T} B_t.$$

(a) In the class (and in the textbook), we derived the joint CDF for B_T and B_T^M,

$$P(B_T^M < v,\ B_T < u) = N\Big(\frac{u}{\sqrt{T}}\Big) - N\Big(\frac{u-2v}{\sqrt{T}}\Big) \quad \text{for} \quad v \ge \max(0,u)$$

Using this result, derive the joint CDF for BM with volatility, σB_t,

$$P(\sigma B_T^M < v,\ \sigma B_T > u).$$

(b) Finally find the price of the binary call option with the up-and-out feature? Assume that interest rate and divided rate are zero. How much is this derivative cheaper (or more expensive) than the regular binary call option **without** knock-out feature?

Solution:

(a) We first derive $P(B_T^M < v,\ B_T > u)$:

$$P(B_T^M < v,\ B_T > u) = P(B_T^M < v) - P(B_T^M < v,\ B_T < u).$$

Since (or you can just use the result from class)

$$P(B_T^M < v) = P(B_T^M < v,\ B_T < v) = N\Big(\frac{v}{\sqrt{T}}\Big) - N\Big(\frac{-v}{\sqrt{T}}\Big) = 2N\Big(\frac{v}{\sqrt{T}}\Big) - 1,$$

Therefore,

$$\begin{aligned} P(B_T^M < v,\ B_T > u) &= 2N\Big(\frac{v}{\sqrt{T}}\Big) - 1 - N\Big(\frac{u}{\sqrt{T}}\Big) + N\Big(\frac{u-2v}{\sqrt{T}}\Big) \\ &= N\Big(\frac{-u}{\sqrt{T}}\Big) - 2N\Big(\frac{-v}{\sqrt{T}}\Big) + N\Big(\frac{u-2v}{\sqrt{T}}\Big). \end{aligned}$$

Finally

$$P(\sigma B_T^M < v,\ \sigma B_T > u) = N\Big(\frac{-u}{\sigma\sqrt{T}}\Big) - 2N\Big(\frac{-v}{\sigma\sqrt{T}}\Big) + N\Big(\frac{u-2v}{\sigma\sqrt{T}}\Big).$$

(b) The price of the up-and-out binary call option is obtained by plugging in $u = K - S_0$ and $v = H - S_0$:

$$D(K,H) = N\Big(\frac{S_0 - K}{\sigma\sqrt{T}}\Big) - 2N\Big(\frac{S_0 - H}{\sigma\sqrt{T}}\Big) + N\Big(\frac{K + S_0 - 2H}{\sigma\sqrt{T}}\Big).$$

This price makes sense because

$$D(K, H = K) = N\Big(\frac{S_0 - K}{\sigma\sqrt{T}}\Big) - 2N\Big(\frac{S_0 - K}{\sigma\sqrt{T}}\Big) + N\Big(\frac{K + S_0 - 2K}{\sigma\sqrt{T}}\Big) = 0.$$

The price of the regular binary call option is

$$D(K) = D(K, \infty) = N\Big(\frac{S_0 - K}{\sigma\sqrt{T}}\Big).$$

Therefore, the price is cheaper by

$$D(K) - D(K,H) = 2N\Big(\frac{S_0 - H}{\sigma\sqrt{T}}\Big) - N\Big(\frac{K + S_0 - 2H}{\sigma\sqrt{T}}\Big).$$

10. **[2019ME, First hitting time of a BM with drift]** From SCFA Chapter 5, we derived that the probability density of the first-hitting time τ to the level $\delta > 0$ is given by

$$f_\tau(t) = \frac{\delta}{\sqrt{2\pi t^3}} e^{-\delta^2/2t} = \frac{\delta}{\sqrt{t^3}}\, n\left(\frac{\delta}{\sqrt{t}}\right). \tag{1}$$

We also know from **Hitting time of a level** in Chapter 4.5 that the Laplace transform of τ is given by

$$E(e^{-r\tau}) = e^{-\delta\sqrt{2r}}$$

and that the result is interpreted as the present value (discounted with the interest rate r) of a derivative paying \$1 at the event. We are going to generalize the result to the first hitting time of a drifted BM, $B_t + \gamma t$ ($\gamma > 0$). The probability density for τ,

$$\tau = \min\{t : B_t + \gamma t = \delta\} \quad (\delta > 0)$$

is given by

$$f_\tau(t) = \frac{\delta}{\sqrt{2\pi t^3}} \exp\left(-\frac{(\gamma t - \delta)^2}{2t}\right) = \frac{\delta}{\sqrt{t^3}}\, n\left(\frac{\delta - \gamma t}{\sqrt{t}}\right), \tag{2}$$

and you can use this result without proof.

(a) What is the price of the derivative, $E(e^{-r\tau})$?
Hint: Consider the Laplace transform of Eq. (1).

(b) Obtain the mean and variance of τ, $E(\tau)$ and $\mathrm{Var}(\tau)$.
Hint: if $L(r) = E(e^{-r\tau})$ is the Laplace transform of τ, $L(-r)$ is the moment generating function because $L(-r) = E(e^{r\tau})$.

(c) We further generalize the results to derivative pricing. Assume that a stock follows the process, $S_t = S_0 + \mu t + \sigma B_t$ and the derivative pays \$1 when S_t hits K ($> S_0$) for the first time, i.e., $S_\tau = K$. What is the price of the derivative at $t = 0$? (Modify the result from (a).) Assume that you sold the derivative to clients and that you need to hedge the position using the underlying stock. How many shares of the underlying stock do you need to long or short?

(d) From class we know that the CDF for Eq. (1) is

$$P(\tau \le t) = 2 - 2N(\delta/\sqrt{t}),$$

where $N(\cdot)$ is the normal cumulative distribution function. Can you derive the CDF for the generalized density function, Eq. (2)? (This question might be challenging. Try it only when you have extra time.)

Solution: The distribution of τ, Eq. (2), is known as the inverse Gaussian distribution (WIKIPEDIA). The inverse Gaussian distribution is typically parameterized by μ and λ:

$$f_\tau(t) = \sqrt{\frac{\lambda}{2\pi t^3}} \exp\left(-\frac{\lambda(t-\mu)^2}{2\mu^2\, t}\right).$$

The parameters (μ, λ) are related to our parameters (γ, δ) by

$$\mu = \delta/\gamma, \quad \lambda = \delta^2.$$

Any established results for the inverse Gaussian distribution can be expressed in terms of (γ, δ) using the above formula. The distribution is called *inverse* Gaussian because it describes the (first-hitting) time of BM at a fixed location, whereas the Gaussian distribution describes the location at a fixed time. The answers to this problem is well-known properties of the inverse Gaussian distribution. But we can derive the solution from the knowledge obtained in class except (d).

(a) The density function, Eq. (1) and its Laplace transform are expressed as

$$E(e^{-r\tau}) = \int_{t=0}^{\infty} e^{-rt} f_\tau(t)dt = \int_{t=0}^{\infty} \frac{\delta}{\sqrt{2\pi t^3}} \exp\left(-\frac{\delta^2}{2t} - rt\right) dt = e^{-\delta\sqrt{2r}}$$

Based on this, we can derive the Laplace transform of Eq. (2):

$$\begin{aligned} E(e^{-r\tau}) &= \int_{t=0}^{\infty} \frac{\delta}{\sqrt{2\pi t^3}} \exp\left(-\frac{(\gamma t-\delta)^2}{2t} - rt\right) dt \\ &= \int_{t=0}^{\infty} \frac{\delta e^{\gamma\delta}}{\sqrt{2\pi t^3}} \exp\left(-\frac{\delta^2}{2t} - (r+\frac{\gamma^2}{2})t\right) dt \\ &= \int_{t=0}^{\infty} \frac{\delta e^{\gamma\delta}}{\sqrt{2\pi t^3}} \exp\left(-\frac{\delta^2}{2t} - r't\right) dt \quad \left(r' = r + \frac{\gamma^2}{2}\right) \\ &= \exp\left(\gamma\delta - \delta\sqrt{2r'}\right) \\ &= \exp\left(\gamma\delta\left(1-\sqrt{1+2r/\gamma^2}\right)\right). \end{aligned}$$

(b) From Taylor's expansion,

$$1 - \sqrt{1+\frac{2r}{\gamma^2}} = 1 - \left(1 + \frac{r}{\gamma^2} - \frac{r^2}{2\gamma^4} + \cdots\right) = -\frac{r}{\gamma^2} + \frac{r^2}{2\gamma^4} + \cdots,$$

the Laplace transform is expanded into

$$E(e^{-r\tau}) = 1 + \gamma\delta\left(-\frac{r}{\gamma^2} + \frac{r^2}{2\gamma^4} + \cdots\right) + \frac{1}{2}\left(\frac{\delta^2}{\gamma^2}r^2 + \cdots\right) + \cdots.$$

Therefore, the first two moments and the variance are given by

$$M_1 = \frac{\delta}{\gamma}, \quad M_2 = \frac{\delta}{\gamma^3} + \frac{\delta^2}{\gamma^2}, \quad \text{Var} = \frac{\delta}{\gamma^3}.$$

(c) From

$$S_\tau = S_0 + \mu\tau + \sigma B_\tau = K \quad \Longrightarrow \quad (\mu/\sigma)\tau + B_\tau = (K-S_0)/\sigma,$$

we can use

$$\delta = \frac{K-S_0}{\sigma}, \quad \gamma = \frac{\mu}{\sigma}.$$

From the result of (a),

$$P = E(e^{-r\tau}) = \exp\left(\frac{\mu(K-S_0)}{\sigma^2}\left(1-\sqrt{1+\frac{2r\sigma^2}{\mu^2}}\right)\right)$$

The amount of the underlying stock to hold for hedging the position is given by

$$\frac{\partial P}{\partial S_0} = -\frac{\mu}{\sigma^2}\left(1-\sqrt{1+\frac{2r\sigma^2}{\mu^2}}\right)\exp\left(\frac{\mu(K-S_0)}{\sigma^2}\left(1-\sqrt{1+\frac{2r\sigma^2}{\mu^2}}\right)\right)$$

(d) The CDF of the inverse Gaussian distribution is give by

$$F(t) = N\left(\gamma\sqrt{t}-\frac{\delta}{\sqrt{t}}\right) + e^{2\gamma\delta}N\left(-\gamma\sqrt{t}-\frac{\delta}{\sqrt{t}}\right).$$

This CDF was originally found by

Shuster, J. (1968). On the inverse Gaussian distribution function. *Journal of the American Statistical Association*, 63(324), 1514–1516. https://doi.org/10.1080/01621459.1968.10480942

It can be also derived using Girsanov theorem in Chapter 13.

When $\gamma = 0$, the CDF $F(t)$ is reduced to the case we learned in class:

$$F(t) = P(\tau \le t) = N\left(-\frac{\delta}{\sqrt{t}}\right) + N\left(-\frac{\delta}{\sqrt{t}}\right) = 2 - 2N\left(\frac{\delta}{\sqrt{t}}\right)$$

6 Itô Integration

1. **[SCFA 6.1]**

Solution: The mean of the two expressions are zero.

$$\text{Var}\left(\int_0^t |B_s|^{\frac{1}{2}} dB_s\right) = E\left(\int_0^t |B_s| ds\right) = \int_0^t E(|B_s|)ds = \int_0^t \sqrt{\frac{2s}{\pi}} ds = \frac{2}{3}\sqrt{\frac{2}{\pi}}\, t^{\frac{3}{2}}$$

$$\begin{aligned}\text{Var}\left(\int_0^t (B_s+s)^2 dB_s\right) &= E\left(\int_0^t (B_s+s)^4 ds\right) \\ &= \int_0^t E(B_s^4 + 4sB_s^3 + 6s^2B_s^2 + 4s^3B_s + s^4)ds \\ &= \int_0^t (3s^2 + 0 + 6s^2 \cdot s + 0 + s^4)ds = \frac{1}{5}t^5 + \frac{3}{2}t^4 + t^3\end{aligned}$$

2. **[SCFA 6.2]**

Solution: For I_1,

$$E(I_1) = \int_0^t E(B_s)ds = \int_0^t 0\, ds = 0.$$

Using Itô's lemma applied to sB_s, $d(sB_s) = sdB_s + B_s ds$, we can express I_1 as

$$I_1 = tB_t - \int_0^t sdB_s = t\int_0^t dB_s - \int_0^t sdB_s = \int_0^t (t-s)dB_s,$$

where we used a trick of $B_t = \int_0^t dB_s$ in order to make the expression suitable for Ito's isometry. We get

$$\text{Var}(I_1) = \int_0^t (t-s)^2 ds = \frac{1}{3}t^3$$

For I_2,

$$E(I_2) = \int_0^t E(B_s^2)ds = \int_0^t s\, ds = \frac{t^2}{2}.$$

Using Itô's lemma applied to sB_s^2, $d(sB_s^2) = B_s^2 ds + 2sB_s dB_s + sds$, we can express I_2 as

$$I_2 = tB_t^2 - 2\int_0^t sB_s dB_s - \frac{t^2}{2}.$$

We apply a similar trick, $d(B_s^2) = 2B_s dB_s + ds$, to replace B_t^2 with a more suitable expression for Itô's isometry,

$$I_2 = t\left(2\int_0^t B_s dB_s + t\right) - 2\int_0^t sB_s dB_s - \frac{t^2}{2} = 2\int_0^t (t-s)B_s dB_s + \frac{t^2}{2},$$

where we can reconfirm that $E(I_2) = t^2/2$. Finally,

$$\mathrm{Var}(I_2) = E\left(\left(I_2 - \frac{t^2}{2}\right)^2\right) = 4\int_0^t E\Big((t-s)^2 B_s^2\Big)ds = 4\int_0^t (t-s)^2 s\, ds = 4\cdot\frac{t^4}{12} = \frac{t^4}{3}$$

3. **[SCFA 6.3]**

Solution: At any time s, X_s and B_s has the same distribution, normal distribution with mean 0 and variance s, so $E(f(B_s)) = E(f(X_s))$ and

$$E(U_t) = \int_0^t E(f(B_s))ds = \int_0^t E(f(X_s))ds = E(V_t)$$

For variance, simply let $f(x) = x$. Using that $V_t = \int_0^t \sqrt{s}\, Z ds = \frac{2}{3}t^{\frac{3}{2}}\, Z$,

$$\mathrm{Var}(V_t) = \frac{4}{9}t^3.$$

According to SCFA 6.2, however,

$$\mathrm{Var}(U_t) = \frac{1}{3}t^3 \neq \mathrm{Var}(V_t).$$

4. **[2016HW 4-1, Itô's isometry]** Find the mean and variance of the following stochastic integral

$$Y_t = \int_0^t e^{B_s} dB_s$$

Solution: The mean is zero because an Itô's integral is a martingale. Alternatively, it is because $dY_t = e^{B_t}dB_t$ has no drift term (i.e., dt). The variance can be calculated using Itô's isometry:

$$\mathrm{Var}(Y_t) = E\Big[\Big(\int_0^t e^{B_s}dB_s\Big)^2\Big] = E\Big[\int_0^t e^{2B_s}ds\Big] = \int_0^t E(e^{2B_s})ds = \int_0^t e^{2s}ds = \frac{1}{2}(e^{2t}-1)$$

5. **[2016ME(ASP), Volatility on holidays]**
In the class, we covered the relation between the daily volatility σ_d and the annual volatility σ_y as

$$\sigma_y = \sqrt{256}\,\sigma_d = 16\,\sigma_d,$$

where we assume the stock price is moving only on the 256 trading days in one year.

In this problem, we want to make it slightly more complicated. The stock price is usually more volatile on Mondays because new information, e.g., news related to the stock, economy, etc, is accumulated over the weekend and make the price move on Monday when the stock market is open. What would be the relation between the daily volatility σ_d and the annual volatility σ_y if we assume Monday's price is 50% more volatile than the other trading days, i.e., $1.5\,\sigma_d$ on Mondays? Assume there are 52 Mondays in one year (so the rest of the trading days are $256 - 52 = 204$).

Solution:

$$\sigma_y^2 = \left(204 \times 1 + 52 \times 1.5^2\right)\sigma_d^2 = 321\sigma_d^2$$

Therefore,

$$\sigma_y = \sqrt{321}\sigma_d \approx 17.92\,\sigma_d$$

6. **[2017ME, Exponentially decaying volatility]** Assume that a stock follows BM with time-varying volatility:

$$dS_t = \sigma(t)dB_t \quad \text{for} \quad \sigma(t) = a + be^{-\lambda t} \quad (a, b > 0)$$

What is the ATM call option price at expiry T? What is the equivalent Bachelier model volatility? In other words, what value of σ_N gives the same option price when the stock price follows $dS_t = \sigma_\mathrm{N}\, dW_t$ at $t = T$.

Solution: The variance of S_t computed as

$$\begin{aligned}\mathrm{Var}(S_T) &= \int_0^T \sigma^2(t)dt = \int_0^T (a^2 + 2ab\, e^{-\lambda t} + b^2\, e^{-2\lambda t})dt \\ &= a^2 T + \frac{2ab}{\lambda}(1 - e^{-\lambda T}) + \frac{b^2}{2\lambda}(1 - e^{-2\lambda T}).\end{aligned}$$

The option price is given by

$$C = 0.4\sqrt{a^2 T + \frac{2ab}{\lambda}(1 - e^{-\lambda T}) + \frac{b^2}{2\lambda}(1 - e^{-2\lambda T})}.$$

From $\sigma_\mathrm{N}^2 T = \mathrm{Var}(S_T)$,

$$\sigma_\mathrm{N} = \sqrt{a^2 + \frac{2ab}{\lambda T}(1 - e^{-\lambda T}) + \frac{b^2}{2\lambda T}(1 - e^{-2\lambda T})}$$

7. **[2017ME, Stochastic integral]** Assume that you follow a trading strategy based on *momentum*, where you long more stock if the stock price is up and short more if the stock price is down. If S_t is the process for the stock price, the amount you long/short is $S_t - S_0$.

   (a) If X_T is the profit and loss from this strategy at time $t = T$, express X_T using stochastic integral.

   (b) When the stock follows a BM, $S_t = S_0 + \sigma B_t$, what is X_T? You may directly use the result from the class.

(c) Imagine a scenario where the stock price goes up a lot but loses later recovering the original price, $S_T = S_0$. How much do you profit or lose at $t = T$? Intuitively explain why you profit or lose? (The opposite is called *mean-reversion* strategy, where you long/short by $S_0 - S_t$.)

(d) Calculate the variance of X_T. You can either use the result from (b) or use Itô's isometry.

Solution:

(a)

$$X_T = \int_0^T (S_t - S_0) dS_t$$

(b)

$$X_T = \sigma^2 \int_0^T B_t \, dB_t = \frac{\sigma^2}{2}(B_T^2 - T)$$

(c) Since $B_T = B_0 = 0$, the momentum strategy loses by $\sigma^2 T/2$ (where as the mean-reversion strategy profits the same amount). The strategy profits when the stock price goes up but loses when it goes down. Intuitively, loss is bigger than profit because you start from zero long position when the price is up where as you already have some short position when the price is down.

(d) From Itô's isometry,

$$\mathrm{Var}(X_T) = \sigma^4 \int_0^T E(B_t^2) dt = \sigma^4 \int_0^T t \, dt = \frac{\sigma^4}{2} T^2$$

From (b),

$$\begin{aligned}\mathrm{Var}(X_T) &= \frac{\sigma^4}{4}\mathrm{Var}(B_T^2 - T) = \frac{\sigma^4}{4} E(B_T^4 - 2T\, B_T^2 + T^2) \\ &= \frac{\sigma^4}{4} E(3T^2 - 2T \cdot T + T^2) = \frac{\sigma^4}{2} T^2\end{aligned}$$

8. **[2018ME, Time-dependent volatility]** The at-the-money options on Meituan Dianping (IPO in September 2018 on Hong Kong stock exchange) with three-month ($T = 1/4$) and one-year ($T = 1$) maturities are currently trading at the prices of 4.0 and 6.4 Hong Kong dollars, respectively. Assume that the stock price follows $dS_t = f(t)\, dB_t$ and that the option price can be approximated with 0.4 stdev(S_T). Find the piece-wise constant instantaneous volatility $f(t)$ that satisfies the observed option prices.

Solution: We need to find

$$f(t) = \begin{cases} a & \text{if} \quad 0 \le t \le 0.25 \\ b & \text{if} \quad 0.25 \le t \end{cases}$$

For the two options,

$$4 = 0.4\sqrt{0.25\,a^2}$$
$$6.4 = 0.4\sqrt{0.25\,a^2 + 0.75\,b^2}$$

We get $a = 20$ and $b = \sqrt{208} \approx 14.42$

9. **[2020ME, Stochastic integral]** Based on the highschool calculus, $\int_0^x e^{-x}dx = 1 - e^{-x}$, I make the statement on the following stochastic integral:

$$\int_0^T e^{-B_t} dB_t = 1 - e^{-B_T}.$$

We will check if this is true or false.

(a) What is the mean and variance of the left-hand side (LHS)?

(b) What is the mean and variance of the right-hand side (RHS)?

(c) Is the statement true or false?

Solution:

(a) The mean of LHS is zero by symmetry. Using the Itô's isometry,

$$\text{Var(LHS)} = E\left(\int_0^T e^{-2B_t} dt\right) = \int_0^T E(e^{-2B_t})dt = \int_0^T e^{2t}dt = \frac{e^{2T}-1}{2}$$

(b) Regarding RHS,

$$\begin{aligned}
E(\text{RHS}) &= E(1 - e^{-B_T}) = 1 - E(e^{-B_T}) = 1 - e^{T/2},\\
\text{Var(RHS)} &= E\left((e^{-B_T} - e^{T/2})^2\right) = E(e^{-2B_T}) - 2e^{T/2}E(e^{-B_T}) + e^T\\
&= e^{2T} - 2e^{T/2}\cdot e^{T/2} + e^T = e^T(e^T - 1).
\end{aligned}$$

(c) Because the means and variance are not same, the statement is false.

7 Localization and Itô's integral

1. **[SCFA 7.1]**

Solution: The function τ_t is given by

$$\tau_t = \text{Var}(B_{\tau_t}) = \text{Var}(Y_t) = \text{Var}(X_t) = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1).$$

$E(X_t^2) = E(Y_t^2)$ because

$$E(X_t^2) = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1), \quad E(Y_t^2) = \tau_t = \frac{1}{2}(e^{2t} - 1).$$

$E(X_t^4)$ and $P(X_t \geq 1)$ are given by

$$E(X_t^4) = E(Y_t^4 = B_{\tau_t}^4) = 3\tau_t^2 = \frac{3}{4}(e^{2t} - 1)^2$$
$$P(X_t \geq 1) = P(Y_t = B_{\tau_t} \geq 1) = 1 - N(1/\sqrt{\tau_t}) = 1 - N(\sqrt{2/(e^{2t} - 1)}).$$

Note the difference between this problem and **SCFA Corollary 7.1** (time-changed BM). Given B_t is a standard BM,

$$X_t = \int_0^t f(s) dB_s, \quad \text{and} \quad \tau(t) = v = \text{Var}(X_t) = \int_0^t f^2(s) ds,$$

this exercise problem is effectively stating that X_t and $B_{\tau(t)}$ are same processes. Whereas, the Corollary 7.1 states that $X_{\tau^{-1}(v)}$ and B_v are same processes where $\tau^{-1}(\cdot)$ is the inverse function of $\tau(\cdot)$, i.e., $t = \tau^{-1}(v)$. Although they look different in forms, the intuitions behind them are same in that the variance of X_t can be used as a new *time scale* of a standard BM.

8 Itô's Formula

1. **[SCFA 8.2]** (An Integration by Parts). Use Itô'ss formula to prove that if $h \in \mathcal{C}^1(\mathbb{R}^+)$ then

$$\int_0^t h(s)dB_s = h(t)B_t - \int_0^t h'(s)B_s\,ds.$$

This formula shows that the Itô integral of a deterministic function can be calculated just in terms of traditional integrals, a fact that should be compared to formula (8.10), where a similar result is obtained for integrands that are functions of B_t alone.

Solution: (The notation $h \in C^1(\mathbb{R}^+)$ means that the function $h(s)$ is differentiable for $s > 0$.) From the SDE of $h(t)B_t$,

$$d(h(t)B_t) = h(t)dB_t + h'(t)B_t dt$$

we get

$$\int_0^t h(s)dB_s = h(t)B_t - \int_0^t h'(s)B_s\,ds.$$

Using that $B_T = \int_0^T dB_t$, we also get another useful result:

$$\int_0^T h'(s)B_s\,ds = h(T)B_T - \int_0^T h(t)dB_t = \int_0^T (h(T) - h(t))dB_t$$

2. **[SCFA 8.4]** (More Charms of the Third Martingale).
   (a) The martingale $M_t = \exp(\alpha B_t - \alpha^2 t/2)$ is separable in the sense that it is of form $f(t, B_t)$, where $f(t,x) = \phi(t)\psi(x)$. Solve the equation $f_t = -\frac{1}{2}f_{xx}$ under the assumption that $f(t,x) = \phi(t)\psi(x)$ and see if you can find any other separable martingales.
   (b) By expanding $M_t = \exp(\alpha B_t - \alpha^2 t/2)$ as a Taylor series in α, we can write

$$M_t = \sum_{k=0}^{\infty} \alpha^k H_k(t, B_t),$$

where the $H_k(t,x)$ are polynomials. Find the first four of the $H_k(t,x)$, and show that for each $k \in \mathbb{Z}^+$ the process defined by

$$M_t(k) = H_k(t, B_t)$$

is a martingale. You will recognize the first three of these martingales as old friends.

Solution:

(a) If $f(t,x) = \phi(t)\psi(x)$, the condition $f_t = -\frac{1}{2}f_{xx}$ yields to

$$-2\frac{\phi_t(t)}{\phi(t)} = \frac{\psi_{xx}(x)}{\psi(x)} = \lambda,$$

where λ is a constant. If $\lambda > 0$ ($\lambda = \alpha^2$ for some α), we get the GBM solution,

$$\phi(t) = \phi(0)e^{-\alpha^2 t/2} \quad \text{and} \quad \psi(x) = Ce^{\alpha x} + De^{-\alpha x} \quad \text{for some constants} \quad C, D$$

$$M_t = (Ce^{\alpha B_t} + De^{-\alpha B_t})e^{-\alpha^2 t/2}$$

If $\lambda < 0$ ($\lambda = -\alpha^2$ for some α), we get

$$M_t = (C\cos(\alpha B_t) + D\sin(\alpha B_t))e^{\alpha^2 t/2}$$

If $\lambda = 0$, we get $\phi(t) = \phi(0)$ and $\psi(x) = Cx + D$, therefore we have

$$M_t = CB_t + D.$$

(b) The sub-problem is better understood after solving 2017HW 3-2.

$$\begin{aligned} M_t =& 1 + (\alpha B_t - \alpha^2 t/2) + \frac{1}{2}(\alpha B_t - \alpha^2 t/2)^2 + \frac{1}{6}(\alpha B_t - \alpha^2 t/2)^3 \\ &+ \frac{1}{24}(\alpha B_t - \alpha^2 t/2)^4 + \cdots \\ =& 1 + (B_t)\alpha + \frac{1}{2}(B_t^2 - t)\alpha^2 + \frac{1}{6}(B_t^3 - 3tB_t)\alpha^3 + \frac{1}{24}(B_t^4 - 6tB_t^2 + 3t^2)\alpha^4 + \cdots \end{aligned}$$

We get the first five martingales as below:

$$\begin{aligned} H_0(t, B_t) &= 1 \\ H_1(t, B_t) &= B_t \\ H_2(t, B_t) &= \frac{1}{2}(B_t^2 - t) \\ H_3(t, B_t) &= \frac{1}{6}(B_t^3 - 3tB_t) \\ H_4(t, B_t) &= \frac{1}{24}(B_t^4 - 6tB_t^2 + 3t^2). \end{aligned}$$

3. **[2016HW 4-3. SDE]** For the following functions $f(t, x)$, find the stochastic differential equation (SDE) of the stochastic process $Y_t = f(t, B_t)$ where B_t is a standard BM. If $f(x) = x^2$, for example, the SDE is

$$dY_t = d\,f(B_t) = d\,B_t^2 = 2B_t dB_t + dt.$$

(a) $f(t, x) = x^3 - 3tx$

(b) $f(t, x) = e^{t/2}\sin(x)$

Solution: The function $f(t, x)$ was constructed in such a way that $dY_t = df(t, B_t)$ becomes a martingale.

(a)

$$df(t, x = B_t) = (3x^2 - 3t)dB_t + \frac{1}{2}(6x)(dB_t)^2 - 3x dt = 3(B_t^2 - t)dB_t$$

(b)

$$\begin{aligned} df(t, x = B_t) &= e^{t/2}\cos(x)dB_t - \frac{1}{2}e^{t/2}\sin(x)(dB_t)^2 + \frac{1}{2}e^{t/2}\sin(x)dt \\ &= e^{t/2}\cos(B_t)dB_t \end{aligned}$$

4. **[2017HW 3-2]** For a standard BM B_t, let

$$N_t = B_t^3 - 3t\,B_t.$$

(i) Prove that N_t is a martingale. (Hint: use **SCFA Proposition 8.1**) (ii) By applying Itô's lemma, express N_t as a stochastic integration. (iii) Calculate the variance of N_t.

Solution: We set $N_t = f(t, B_t)$ where $f(t, x) = x^3 - 3tx$. Applying Itô's lemma,

$$dN_t = 3(B_t^2 - t)dB_t + 3B_t(dB_t)^2 - 3B_t dt = 3(B_t^2 - t)dB_t$$

As there is no drift term, N_t is a martingale and is represented as a stochastic integral:

$$N_t = \int_0^t 3(B_s^2 - s)dB_s.$$

The variance is calculated as

$$\begin{aligned} \mathrm{Var}(N_t) &= \int_0^t E[3^2(B_s^2 - s)^2]ds = 9\int_0^t (E(B_s^4) - 2sE(B_s^2) + s^2)ds \\ &= 9\int_0^t (3s^2 - 2s^2 + s^2)ds = 6t^2 \end{aligned}$$

5. **[2016ME(ASP), Itô's calculus]** Which of the following quantity is same as $(dB_t)^2$?
A. $\sqrt{dt}$ B. dt C. $dt/2$ D. $(dt)^2/2$

Solution: B.

6. **[2016FE, Stochastic calculus]** Find **all** surviving terms in stochastic calculus
(a) $dB_t \cdot dt$
(b) $(dB_t)^2$
(c) $dx \cdot dt$
(d) $dB_t^1 \cdot dB_t^2$ for the two independent BMs, B_t^1 and B_t^2

Solution: (b) $(dB_t)^2 = dt$

7. **[2017FE, Stochastic calculus]** Give **True or False**.

(a) $(dB_t)^2 = dt$

(b) $(dB_t)^4 = (dt)^2$

(c) $d(B_t^2) = 2(B_t dB_t + dt)$

(d) $d\sinh(B_t) = \cosh(B_t)dB_t + \frac{1}{2}\sinh(B_t)dt$ where $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$

Solution:

(a) True. $(dB_t)^2 = dt$.

(b) False. $(dB_t)^4 = 3(dt)^2$ from $E(B_t^4) = 3t^2$.

(c) False. $d(B_t^2) = 2B_t dB_t + dt$

(d) True. $d\sinh(B_t) = \cosh(B_t)dB_t + \frac{1}{2}\sinh(B_t)dt$ since $\sinh'(x) = \cosh(x)$ and $\sinh''(x) = \sinh(x)$.

8. **[2018FE, Stochastic calculus]** Choose **all** surviving terms (i.e., non-zero terms) in stochastic calculus. Assume that $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$ for some functions μ and σ.

(a) $dX_t \cdot dt$

(b) $(dX_t)^2$

(c) $dX_t \cdot dB_t$

(d) $dB_t^1 \cdot dB_t^2$ for the two independent BMs, B_t^1 and B_t^2

Solution: (b) $(dX_t)^2 = \sigma(t, X_t)^2\, dt$ and (c) $dX_t \cdot dB_t = \sigma(t, X_t)\, dt$

9. **[2020FE, Stochastic calculus]** Give **True or False**. If false, give the correct answer.

(a) $(dB_t)^2 = dt$

(b) $d(B_t^2) = 2(B_t dB_t + dt)$

(c) $d\sin(B_t) = \cos(B_t)dB_t + \frac{1}{2}\sin(B_t)dt$

(d) $d\left(\dfrac{1}{1 - B_t}\right) = \dfrac{dB_t}{(1 - B_t)^2} + \dfrac{dt}{(1 - B_t)^3}$

Solution:

(a) True.

(b) False. $d(B_t^2) = 2B_t dB_t + dt$

(c) False. $d\sin(B_t) = \cos(B_t)dB_t - \frac{1}{2}\sin(B_t)dt$ since $\sin'(x) = \cos(x)$ and $\sin''(x) = -\sin(x)$.

(d) True.

10. **[2021FE, Stochastic calculus]** Calculate the following stochastic derivatives.

(a) $d\left((T-t)B_t^2\right)$ (with respect to the time variable t)

(b) $d\,(B_t^3 - 3t\,B_t)$

(c) $d\left(\frac{1}{B_t}\right)$

(d) $d\left(e^{-aB_t}B_t\right)$

Solution: Applying Itô's lemma,

$$df(t,B_t) = \left(f_t(t,B_t) + \frac{1}{2}f_{xx}(t,B_t)\right)dt + f_x(t,B_t)dB_t,$$

we obtain the following:

(a)
$$d\left((T-t)B_t^2\right) = (T-t-B_t^2)dt + 2(T-t)B_t dB_t$$

(b)
$$d\,(B_t^3 - 3t\,B_t) = 3(B_t^2 - t)dB_t$$

(c)
$$d\left(\frac{1}{B_t}\right) = -\frac{dB_t}{B_t^2} + \frac{dt}{B_t^3}$$

(d)
$$d\left(e^{-aB_t}B_t\right) = (1-aB_t)e^{-aB_t}dB_t - \frac{1}{2}(2a - a^2B_t)e^{-aB_t}dt$$

11. **[2022FE, Stochastic calculus]** Calculate the following stochastic derivatives.

(a) $d\,(B_t^4 - 6t\,B_t^2 + 3t^2)$

(b) $d\left(\frac{1}{1+e^{-B_t}}\right)$

(c) $d\left(W_t\,e^{\lambda B_t}\right)$ (W_t is another BM correlated with B_t by $dB_t\,dW_t = \rho dt$.)

Solution: Applying Itô's lemma, we obtain the following stochastic derivatives:

(a)
$$d\,(B_t^4 - 6t\,B_t^2 + 4t^2) = 4(B_t^3 - 3tB_t)dB_t$$

(b)
$$\begin{aligned} d\left(\frac{1}{1+e^{-B_t}}\right) &= \frac{e^{-B_t}\,dB_t}{(1+e^{-B_t})^2} + \frac{e^{-2B_t}\,dt}{(1+e^{-B_t})^3} - \frac{e^{-B_t}\,dt}{2(1+e^{-B_t})^2} \\ &= \frac{e^{-B_t}}{(1+e^{-B_t})^2}\left(dB_t - \frac{(1-e^{-B_t})}{2(1+e^{-B_t})}\,dt\right) \end{aligned}$$

(c)

$$\begin{aligned} d\left(W_t\, e^{\lambda B_t}\right) &= e^{\lambda B_t}\, dW_t + \lambda W_t\, e^{\lambda B_t}\, dB_t + \frac{\lambda^2}{2} W_t\, e^{\lambda B_t}\, (dB_t)^2 + \lambda e^{\lambda B_t}(dB_t\, dW_t) \\ &= e^{\lambda B_t}(dW_t + \lambda W_t\, dB_t) + \lambda e^{\lambda B_t}\left(\frac{\lambda}{2} W_t + \rho\right) dt \end{aligned}$$

12. **[2019FE, Itô's lemma]** Assume that a stochastic process X_t follows

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

for some functions, μ and σ. Find the stochastic differential equation (SDE) for the process $Y_t = \exp(X_t)$.

Solution: Applying, Itô's lemma,

$$\begin{aligned} dY_t &= \exp(X_t)dX_t + \frac{1}{2}\exp(X_t)(dX_t)^2 \\ &= Y_t\left(\mu(t, X_t)dt + \sigma(t, X_t)dB_t\right) + \frac{1}{2} Y_t \sigma^2(t, X_t)(dB_t)^2. \end{aligned}$$

Finally, we obtain the SDE as

$$\frac{dY_t}{Y_t} = \left(\mu(t, X_t) + \frac{1}{2}\sigma^2(t, X_t)\right) dt + \sigma(t, X_t)dB_t.$$

13. **[2021FE, Itô's lemma]** For a given stochastic process X_t, find a transformation $Y_t = f(X_t)$ that makes the SDE on Y_t in the form of

$$d\,Y_t = \sigma dB_t + \mu(t, Y_t)\,dt,$$

and also find $\mu(t, Y_t)$.

(a) $dX_t = \sigma\sqrt{X_t}\,dB_t$.

(b) $dX_t = \sigma\, e^{-\lambda X_t}\, dB_t$

(c) $dX_t = \sigma \tanh(X_t)\, dB_t \quad \left(\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$.
You may use other hyperbolic functions, such as $\sinh x$, $\cosh x$, $\sinh^{-1} x$, and $\cosh^{-1} x$.

Solution: When the SDE for X_t is given by

$$dX_t = \sigma h(X_t)dB_t,$$

The transformation, $f(x) = \int 1/h(x)\,dx$ ($f'(x) = 1/h(x)$), makes the required result:

$$dY_t = \frac{dX_t}{h(X_t)} - \frac{h'(X_t)}{2h^2(X_t)}(dX_t)^2 = \sigma d\,B_t - \frac{\sigma^2}{2} h'(X_t)dt$$

(a)

$$f(x) = \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} \quad \Rightarrow \quad Y_t = 2\sqrt{X_t}$$

$$dY_t = d\left(2\sqrt{X_t}\right) = \sigma dB_t - \frac{\sigma^2\, dt}{4\sqrt{X_t}} = \sigma dB_t - \frac{\sigma^2}{2Y_t}dt \quad \left(\mu(t, Y_t) = -\frac{\sigma^2}{2Y_t}\right)$$

(b)

$$f(x) = \int e^{\lambda x}dx = \frac{1}{\lambda}e^{\lambda x} \quad \Rightarrow \quad Y_t = \frac{1}{\lambda}e^{\lambda X_t}$$

$$dY_t = d\left(\frac{e^{\lambda X_t}}{\lambda}\right) = \sigma dB_t + \frac{\lambda\sigma^2}{2}e^{-\lambda X_t}dt = \sigma dB_t + \frac{\sigma^2}{2Y_t}dt \quad \left(\mu(t, Y_t) = \frac{\sigma^2}{2Y_t}\right)$$

(c) The derivative of $h(x) = \tanh(x)$ is derived as

$$h'(x) = \tanh'(x) = 1 - \tanh^2(x) = \frac{1}{\cosh^2(x)} = \frac{1}{1 + \sinh^2(x)}.$$

Therefore,

$$f(x) = \int \frac{dx}{\tanh x} = \int \frac{\cosh x}{\sinh x}dx = \log(\sinh x) \quad \Rightarrow \quad Y_t = \log(\sinh X_t)$$

$$dY_t = d\log(\sinh X_t) = \sigma dB_t - \frac{\sigma^2\, dt}{2(1 + \sinh^2 X_t)} = \sigma dB_t - \frac{\sigma^2\, dt}{2(1 + e^{2Y_t})}$$

$$\left(\mu(t, Y_t) = -\frac{\sigma^2}{2(1 + e^{2Y_t})}\right)$$

14. **[2016FE, Volatility of a stock denominated in difference currency]** If you invest in a stock listed in a foreign country, you are exposed to the risk of both the stock price (in the foreign currency) and the foreign exchange rate. We are going to see volatility of the value of the stock in the domestic currency. Assume that you invested in Amazon's stock (NASDAQ ticker `AMZN`, which is currently about 900 USD) and the volatility is 20%. Also assume that the volatility of the foreign exchange rate (currency code `USDCNY`, X CNY = 1 USD, which is 6.89 recently) and the volatility is 8%. Also assume that the correlation between the stock price change and the FX rate change is given by ρ (not necessarily independent!). The SDEs for the processes can be written as

$$\text{Stock price:} \quad \frac{dS_t}{S_t} = r_S dt + \sigma_S dB_t^S \quad \text{where} \quad \sigma_S = 20\%$$

$$\text{FX rate:} \quad \frac{dF_t}{F_t} = r_F dt + \sigma_F dB_t^F \quad \text{where} \quad \sigma_F = 8\%$$

$$\text{Correlation:} \quad dB_t^S \cdot dB_t^F = \rho\, dt.$$

(a) Derive the SDE for the stock value in CNY. That is, calculate $d(F_t\, S_t)$.

(b) Re-write the SDE with a single BM, let's say B_t^{FS}, and find the volatility of the stock price in CNY. This is the extension of our mid-term exam problem. You need to find

c such that

$$c\,dB_t^{\text{FS}} = a\,dB_t^S + b\,dB_t^F \quad \text{with} \quad dB_t^S \cdot dB_t^F = \rho\,dt$$

(c) What is the minimum/maximum value of the combined volatility? Under which scenario?

Solution:

(a)

$$\begin{aligned}
d(F_t\,S_t) &= S_t dF_t + F_t dS_t + dF_t \cdot dS_t \\
&= S_t \cdot F_t(r_F dt + \sigma_F dB_t^F) + F_t \cdot S_t(r_S dt + \sigma_S dB_t^S) + \sigma_F F_t dB_t^F \cdot \sigma_S S_t dB_t^S \\
\frac{d(F_t\,S_t)}{F_t\,S_t} &= (r_F + r_S + \rho\sigma_F\sigma_S)dt + \sigma_F dB_t^F + \sigma_S dB_t^S
\end{aligned}$$

(b) Since,

$$(\sigma_F dB_t^F + \sigma_S dB_t^S)^2 = (\sigma_F^2 + \sigma_S^2 + 2\rho\sigma_F\sigma_S)\,dt,$$

we can re-write the SDE as

$$\frac{d(F_t\,S_t)}{F_t\,S_t} = (r_F + r_S + \rho\sigma_F\sigma_S)dt + \sqrt{\sigma_F^2 + \sigma_S^2 + 2\rho\sigma_F\sigma_S}\;dB_t^{\text{FS}}.$$

The volatility is

$$\sigma_{\text{FS}} = \sqrt{\sigma_F^2 + \sigma_S^2 + 2\rho\sigma_F\sigma_S}.$$

(c) The maximum is 20%+8%=28% when $\rho = 100\%$ and the minimum is 20%-8%=12% when $\rho = -100\%$.

15. **[2020FE, Stochastic Calculus]** Let us consider the two stochastic processes, X_t and Y_t:

$$X_t = \sinh(B_t) \quad \text{and} \quad Y_t = e^{B_t}\int_0^t e^{-B_s} dW_s,$$

where B_t and W_t are two independent standard BMs. Reminded that $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.

(a) What are the mean and variance of X_t and Y_t respectively?

(b) Prove that $X_t = \sinh(B_t)$ satisfies the SDE:

$$dX_t = \sqrt{1 + X_t^2}\;dB_t + \frac{1}{2}X_t dt.$$

(c) Derive the SDE for Y_t. What can you say about the distributions of X_t and Y_t ?

Solution:

(a) The means of X_t and Y_t are zero,

$$E(X_t) = E(Y_t) = 0$$

because $\sinh(-x) = -\sinh(x)$ and the symmetry of Y_t.

The variance of X_t is

$$\begin{aligned}\mathrm{Var}(X_t) &= E\left[\sinh^2(B_t)\right] = \frac{1}{4}E\left[\left(e^{2B_t} + e^{-2B_t} - 2\right)\right] \\ &= \frac{1}{4}\left(e^{2t} + e^{2t} - 2t\right) = \frac{1}{2}\left(e^{2t} - 1\right).\end{aligned}$$

The variance of Y_t is

$$\begin{aligned}\mathrm{Var}(Y_t) &= \int_0^t E[e^{2(B_t - B_s)}]ds = \int_0^t e^{2(t-s)}ds \\ &= e^{2t} \cdot \frac{1}{2}\left(1 - e^{-2t}\right) = \frac{1}{2}\left(e^{2t} - 1\right)\end{aligned}$$

where we used Itô's isometry and the property, $B_t - B_s \sim N(0, t-s)$. Therefore, X_t and Y_t have the same mean and variance.

(b)

$$\begin{aligned}dX_t &= \cosh(B_t)\, dB_t + \frac{1}{2}\sinh(B_t)dt \\ &= \sqrt{1 + X_t^2}\, dB_t + \frac{1}{2}X_t dt.\end{aligned}$$

Here, we used

$$1 + \sinh^2(x) = 1 + \frac{1}{4}(e^x - e^{-x})^2 = \frac{1}{4}(e^x + e^{-x})^2 = \cosh^2(x).$$

(c)

$$\begin{aligned}dY_t &= \left(e^{B_t}dB_t + \frac{1}{2}e^{B_t}dt\right)\int_0^t e^{-B_s}dW_s + e^{B_t}\cdot e^{-B_t}dW_t \\ &= Y_t dB_t + dW_t + \frac{1}{2}Y_t dt \\ &= \sqrt{1 + Y_t^2}\, dB_t' + \frac{1}{2}Y_t dt.\end{aligned}$$

In the last step, we combined $Y_t dB_t + dW_t$ into a new BM, B_t'. The two stochastic processes, X_t and Y_t, start from the same point, $X_0 = Y_0 = 0$, and they have the same SDE from (a) and (b). Therefore, the distributions of X_t and Y_t are the same.

16. **[2022FE]** Consider the following integral over time t:

$$I_T = \int_0^T e^{2B_t}dt.$$

(a) (2 points) Apply Itô's lemma to derive the stochastic differentiation of e^{2B_t}:

(b) (3 points) Based on (a), find the mean and variance of I_T.

Solution: See 2016HW 4-1, Itô's isometry and 2020ME, Stochastic integral for comparison.

(a)

$$d\left(e^{2B_t}\right) = 2e^{2B_t}dB_t + 2e^{2B_t}dt$$

(b) From (a), I_T can be alternatively expressed by

$$I_T = \int_0^T e^{2B_t}dt = \frac{1}{2}\left(e^{2B_T} - 1\right) + \int_0^T e^{2B_t}dB_t$$

Therefore, we have the mean:

$$E(I_T) = \frac{1}{2}E\left(e^{2B_T} - 1\right) = \frac{1}{2}\left(e^{2T} - 1\right).$$

For variance, we apply a trick similar to Gaussian integral:

$$\begin{aligned} E(I_T^2) = E(I_T \cdot I_T) &= E\left(\int_0^T e^{2B_t}dt \cdot \int_0^T e^{2B_s}ds\right) \\ &= \int_0^T \int_0^T E\left(e^{2(B_t+B_s)}\right) dtds. \end{aligned}$$

The integral on (s,t) can be divided into the two regions: (i) $s > t$ and (ii) $t > s$. If (i) $s > t$ holds,

$$\begin{aligned} E\left(e^{2(B_t+B_s)}\right) &= E\left(e^{2(B_s-B_t)+4B_t}\right) \\ &= E\left(e^{2(B_s-B_t)}\right)E\left(e^{4B_t}\right) = \exp\left(2(s-t)+8t\right). \end{aligned}$$

Because of the symmetry, we perform the integral by using (i) only and multiplying 2:

$$\begin{aligned} E(I_T^2) &= 2\int_0^T \int_0^s \exp\left(2(s-t)+8t\right) dtds = 2\int_0^T e^{2s}\int_0^s e^{6t}dtds \\ &= \frac{1}{3}\int_0^T (e^{8s} - e^{2s})ds = \frac{1}{3}\left(\frac{e^{8T}}{8} - \frac{e^{2T}}{2} + \frac{3}{8}\right) = \frac{e^{8T} - 4e^{2T} + 3}{24}. \end{aligned}$$

Finally, the variance is obtained by

$$\mathrm{Var}(I_T) = E(I_T^2) - E(I_T)^2 = \frac{e^{8T} - 6e^{4T} + 8e^{2T} - 3}{24}.$$

9 Stochastic Differential Equations

1. **[SCFA 9.1]** Solve the SDE

$$dX_t = (-\alpha X_t + \beta)dt + \sigma dB_t \quad \text{where} \quad X_0 = x_0 \text{ and } \alpha > 0,$$

and verify that the solution can be written as

$$X_t = e^{-\alpha t}\left(x_0 + \frac{\beta}{\alpha}(e^{\alpha t} - 1) + \sigma \int_0^t e^{\alpha s} dB_s\right).$$

Use the representation to show that X_t converges in distribution as $t \to \infty$, and find the limiting distribution. Finally, find the covariance $\mathrm{Cov}(X_s, X_t)$.

Solution: This is slightly modified from the OU process with the extra βdt term. We use the same initial guess, $e^{\alpha t}X_t$, for the OU process.

$$\begin{aligned} d\big(e^{\alpha t}X_t\big) &= \alpha e^{\alpha t}X_t dt + e^{\alpha t}dX_t + \frac{1}{2}0 \cdot (dX_t)^2 \\ &= e^{\alpha t}\big(\alpha X_t dt - \alpha X_t dt + \beta dt\big) + \sigma e^{\alpha t} dB_t \\ &= \beta e^{\alpha t} dt + \sigma e^{\alpha t} dB_t. \end{aligned}$$

Therefore, we get

$$\begin{aligned} e^{\alpha t}X_t - x_0 &= \frac{\beta}{\alpha}\big(e^{\alpha t} - 1\big) + \sigma \int_0^t e^{\alpha s} dB_s \\ X_t &= e^{-\alpha t}x_0 + \frac{\beta}{\alpha}\big(1 - e^{-\alpha t}\big) + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dB_s. \end{aligned}$$

2. **[SCFA 9.2]** Solve the SDE

$$dX_t = tX_t dt + e^{t^2/2} dB_t \quad \text{with} \quad X_0 = 1.$$

Solution: We first guess from the traditional calculus. We let $x = X_t$ and solve

$$e^{-t^2/2}dx = txe^{-t^2/2} \quad \Rightarrow \quad \frac{dx}{x} = t\, dt$$

Luckily we can solve this to have $x = Ce^{t^2/2}$ or $e^{-t^2/2}x = C$ for some constant C, so we stochastically differentiate $e^{-t^2/2}X_t$ to get

$$d\big(e^{-t^2/2}X_t\big) = -te^{-t^2/2}X_t dt + e^{-t^2/2}\big(tX_t\, dt + e^{t^2/2}dB_t\big) = dB_t.$$

We can solve the SDE as

$$e^{-t^2/2}X_t - X_0 = B_t \quad \Rightarrow \quad X_t = e^{t^2/2}(B_t + X_0).$$

3. **[SCFA 9.3]** Use appropriate coefficient matching to solve the SDE

$$X_0 = 0 \quad dX_t = -2\frac{X_t}{1-t}dt + \sqrt{2t(1-t)}\, dB_t \quad 0 \le t < 1.$$

Show that the solution X_t is a Gaussian process. Find the covariance function $\mathrm{Cov}(X_s, X_t)$. Compare this covariance function to the covariance function for the Brownian bridge.

Solution: The guess from the traditional calculus is

$$\frac{dx}{x} = \frac{-2}{1-t}\, dt \quad \Rightarrow \quad x = C(1-t)^2.$$

Therefore we start by differentiating $(1-t)^{-2}X_t$:

$$d\left(\frac{X_t}{(1-t)^2}\right) = 2\frac{X_t}{(1-t)^3}dt + \frac{1}{(1-t)^2}\big(-2\frac{X_t}{1-t}dt + \sqrt{2t(1-t)}\, dB_t\big) = \frac{\sqrt{2t}}{(1-t)^{3/2}}dB_t$$

and finally solve the SDE as

$$X_t = (1-t)^2 \int_0^t \frac{\sqrt{2u}}{(1-u)^{3/2}} dB_u.$$

Since the integrand $\sqrt{2u}(1-u)^{-3/2}$ depends only on the time variable u, X_t is a Gaussian process with the variance

$$\begin{aligned}
\mathrm{Var}(X_t) &= (1-t)^4 \int_0^t \frac{2u}{(1-u)^3}du = (1-t)^4 \int_{1-t}^1 \frac{2(1-u')}{u'^3}du \quad (u' = 1-u) \\
&= (1-t)^4 \left(1 - \frac{2}{1-t} + \frac{1}{(1-t)^2}\right) = (1-t)^4 \frac{t^2}{(1-t)^2} = t^2(1-t)^2
\end{aligned}$$

The covariance can be obtained similarly. Assuming that $s < t$,

$$\begin{aligned}
\mathrm{Cov}(X_s, X_t) = E(X_s X_t) &= (1-s)^2(1-t)^2\, E\left[\int_0^s \frac{\sqrt{2u}}{(1-u)^{3/2}}dB_u \int_0^t \frac{\sqrt{2v}}{(1-v)^{3/2}}dB_v\right] \\
&= (1-s)^2(1-t)^2\, E\left[\left(\int_0^s \frac{\sqrt{2u}}{(1-u)^{3/2}}dB_u\right)^2\right] \\
&= (1-s)^2(1-t)^2 \cdot \frac{s^2}{(1-s)^2} = s^2(1-t)^2.
\end{aligned}$$

The covariance is square of that of the Brownian bridge $\mathrm{Cov}(X_s, X_t) = s(1-t)$.

4. **[SCFA 9.6]** (Estimation of Real-world σ's).

Solution: We first derive the mean and the variance of lognormal distribution, $Y \sim \exp(\mu + \sigma Z)$, where Z is a standard normal (see the same result at (WIKIPEDIA)):

$$E(Y) = e^{\mu+\sigma^2/2}$$

and

$$\begin{aligned}\mathrm{Var}(Y) &= e^{2\mu}E((e^{\sigma Z} - e^{\sigma^2/2})^2) = e^{2\mu}E(e^{2\sigma Z} - 2e^{\sigma Z+\sigma^2/2} + e^{\sigma^2}) \\ &= e^{2\mu}(e^{2\sigma^2} - 2e^{\sigma^2} + e^{\sigma^2}) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).\end{aligned}$$

Back to the problem, for $t = kh$ and $s = (k-1)h$,

$$R_k(h) + 1 = \frac{X_t}{X_s} = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)h + \sigma(B_t - B_s)\right) = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)h + \sigma\sqrt{h}Z\right),$$

where Z is standard normal distribution. Since $R_k(h) + 1$ is a lognormal distribution with $\mu := (\mu - \sigma^2/2)h$ and $\sigma := \sigma\sqrt{h}$, we obtain the mean and the variance of $R_k(h)+1$ as

$$\begin{aligned}E(R_k(h) + 1) &= \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)h + \frac{\sigma^2}{2}h\right) = e^{\mu h} \\ \mathrm{Var}(R_k(h) + 1) &= e^{2\mu h}(e^{\sigma^2 h} - 1).\end{aligned}$$

Therefore,

$$E(R_k(h)) = e^{\mu h} - 1 \quad \text{and} \quad \mathrm{Var}(R_k(h)) = e^{2\mu h}(e^{\sigma^2 h} - 1)$$

and it follows that

$$\begin{aligned}\mathrm{Var}(R_k(h)) &= (1 + E(R_k(h))^2(e^{\sigma^2 h} - 1) \\ \sigma^2 &= \frac{1}{h}\log\left(1 + \frac{\mathrm{Var}(R_k(h))}{(1 + E(R_k(h))^2}\right).\end{aligned}$$

From sample data, we can estimate the mean and the variance as

$$E(R_k(h)) = \frac{1}{n}\sum_{k=1}^{n} R_k(h), \quad \mathrm{Var}(R_k(h)) = \frac{1}{n-1}\sum_{k=1}^{n}(R_k(h) - E(R_k(h))).$$

Below are the s and σ values for various values of $E(R_k(h))$ and $\mathrm{Stdev}(R_k(h))$:

(All numbers are in the unit of %. $h = 1/12$.)

$\mathrm{Stdev}(R_k(h)) \to$		5	10	15	20	25
$E(R_k(h))$	s	17.3	34.6	52.0	69.3	86.6
-2	σ	17.7	35.3	52.7	70.0	87.0
0		17.3	34.6	51.7	68.6	85.3
2		17.0	33.9	50.7	67.3	83.7
4		16.6	33.2	49.7	66.0	82.1
6		16.3	32.6	48.8	64.8	80.6

The values of s and σ are not significantly different unless the average return $E(R_k(h))$ is high.

5. **[2018HW 3-4, Exponential Ornstein-Uhlenbeck process]** Solve the following SDE:

$$\frac{dP_t}{P_t} = \alpha(\mu - \log P_t)dt + \sigma dB_t.$$

What are $E(P_t)$ and $\text{Var}(P_t)$ as $t \to \infty$? (Hint: use $X_t = \log P_t$.)

Solution: The SDE for X_t satisfies

$$dX_t = \frac{dP_t}{P_t} - \frac{(dP_t)^2}{2P_t^2} = \alpha\left(\mu - \frac{\sigma^2}{2\alpha} - X_t\right)dt + \sigma dB_t.$$

This is the Ornstein-Uhlenbeck with $X_\infty = \mu - \frac{\sigma^2}{2\alpha}$. Therefore,

$$X_t = X_\infty + e^{-\alpha t}(X_0 - X_\infty) + \frac{\sigma e^{-\alpha t}}{\sqrt{2\alpha}} B_{e^{2\alpha t}-1}$$
$$\log P_t = \mu - \frac{\sigma^2}{2\alpha} + e^{-\alpha t}\left(\log P_0 - \mu + \frac{\sigma^2}{2\alpha}\right) + \frac{\sigma e^{-\alpha t}}{\sqrt{2\alpha}} B_{e^{2\alpha t}-1}.$$

The mean and variance as $t \to \infty$ are given by

$$E(X_t) = X_\infty = \mu - \frac{\sigma^2}{2\alpha}, \quad \text{Var}(X_t) = \frac{\sigma^2}{2\alpha}.$$

For P_t, we apply the properties of the lognoraml distributions:

$$E(P_t) = \exp\left(X_\infty + \frac{1}{2}\text{Var}(X_t)\right) = \exp\left(\mu - \frac{\sigma^2}{4\alpha}\right).$$

$$\text{Var}(P_t) = \left(\exp\left(\frac{\sigma^2}{2\alpha}\right) - 1\right)\exp\left(2\mu - \frac{\sigma^2}{2\alpha}\right) = \left(1 - \exp\left(-\frac{\sigma^2}{2\alpha}\right)\right)e^{2\mu}$$

6. **[2020HW 2-1]** The OU process is given by

$$dX_t = -\alpha X_t dt + \sigma dB_t \quad \text{for} \quad \alpha > 0.$$

(a) Find $\text{Cov}(X_s, X_t)$ and $\text{Corr}(X_s, X_t)$.

(b) When $\alpha \to 0$, the OU process converges to the BM with volatility σ. Therefore, show that

$$\lim_{\alpha\to 0} \text{Cov}(X_s, X_t) \to \sigma^2 \min(s, t).$$

Solution:

(a) The OU process is represented as

$$X_t = X_0 e^{-\alpha t} + \sigma e^{-\alpha t}\int_0^t e^{\alpha u} dB_u = X_0 e^{-\alpha t} + \frac{\sigma e^{-\alpha t}}{\sqrt{2\alpha}} B'_{e^{2\alpha t}-1},$$

where B'_t is another BM independent from B_t. Therefore, the covariance can be obtained as

$$\begin{aligned}\mathrm{Cov}(X_s, X_t) &= \frac{\sigma^2}{2\alpha} e^{-\alpha s} e^{-\alpha t}\,\mathrm{Cov}\left(B'_{e^{2\alpha s}-1}, B'_{e^{2\alpha t}-1}\right)\\ &= \frac{\sigma^2\, e^{-\alpha(s+t)}}{2\alpha} \min(e^{2\alpha s}-1, e^{2\alpha t}-1) = \frac{\sigma^2\, e^{-\alpha(s+t)}}{2\alpha} (e^{2\alpha \min(s,t)}-1)\\ &= \frac{\sigma^2}{2\alpha}\left(e^{-\alpha|t-s|} - e^{-\alpha(t+s)}\right)\end{aligned}$$

(b) When $\alpha \to 0$, the covariance of the OU process converges to that of the BM with volatility σ.

$$\begin{aligned}\lim_{\alpha\to 0}\mathrm{Cov}(X_s, X_t) &= \frac{\sigma^2}{2\alpha}\left(e^{-\alpha|t-s|} - e^{-\alpha(t+s)}\right) = \frac{\sigma^2}{2\alpha}\left(-\alpha|t-s| + \alpha(t+s)\right)\\ &= \frac{\sigma^2}{2}\left((t+s) - |t-s|\right) = \sigma^2 \min(s,t).\end{aligned}$$

7. **[2018FE, Mean-reversion]** The stochastic variance in Heston model is given by

$$dV_t = \alpha(V_\infty - V_t)dt + \sigma\sqrt{V_t}\, dB_t.$$

This process is also know as Cox-Ingersoll-Ross(CIR) model for stochastic interest rate. Additionally, the stochastic variance under the 3/2 model is also given by

$$dV_t = \lambda V_t(V_\infty - V_t)\, dt + \xi V_t\sqrt{V_t}\, dB_t.$$

(a) Derive $E(V_t \,|\, \mathcal{F}_0)$ under the Heston model. (Hint: The transformation, $y_t = e^{\alpha t}(V_t - V_\infty)$, used in the Ornstein-Uhlenbeck process is also useful in this problem. Then, use the martingale property.) What is $E(V_t \,|\, \mathcal{F}_0)$ as $t \to \infty$.

(b) To solve the 3/2 model, the inverse variance, $Y_t = 1/V_t$, is helpful. Find the SDE satisfied by Y_t.

(c) What is $E(1/V_t \,|\, \mathcal{F}_0)$ under the 3/2 model? (Hint: You should recognize a similarity between the SDEs for V_t under Heston model and Y_t under the 3/2 model.)

Solution: See Cox-Ingersoll-Ross model (Wikipedia).

(a) The transformation y_t is a martingale because

$$dy_t = \alpha e^{\alpha t}(V_t - V_\infty) + e^{\alpha t} dV_t = e^{\alpha t}\sigma\sqrt{V_t} dB_t.$$

Therefore, $y_0 = E(y_t|\mathcal{F}_0)$:

$$\begin{aligned}V_0 - V_\infty &= E(e^{\alpha t}(V_t - V_\infty)\,|\,\mathcal{F}_0)\\ E(V_t|\mathcal{F}_0) &= V_\infty + e^{-\alpha t}(V_0 - V_\infty) = (1-e^{-\alpha t})V_\infty + e^{-\alpha t} V_0.\end{aligned}$$

$E(V_t|\mathcal{F}_0)$ goes to V_∞ as $t \to \infty$.

(b) Applying Itô's lemma:

$$\begin{aligned}
dY_t &= -(1/V_t^2)dV_t + (1/V_t^3)(dV_t)^2 \\
&= -Y_t\left(\lambda(V_\infty - V_t)dt + \xi\sqrt{V_t}\,dB_t\right) + Y_t^3\xi^2V_t^3dt \\
&= (\lambda + \xi^2 - \lambda V_\infty Y_t)dt - \xi\sqrt{Y_t}dB_t. \\
&= \lambda V_\infty\left(\frac{\lambda+\xi^2}{\lambda V_\infty} - Y_t\right)dt - \xi\sqrt{Y_t}dB_t.
\end{aligned}$$

(c) The process Y_t follows an CIR model with the parameters,

$$\alpha = \lambda V_\infty, \quad Y_\infty = \frac{\lambda+\xi^2}{\lambda V_\infty}, \quad \sigma = -\xi$$

Therefore,

$$\begin{aligned}
E(1/V_t\,|\,\mathcal{F}_0) = E(Y_t\,|\,\mathcal{F}_0) &= (1-e^{-\alpha\,t})Y_\infty + e^{-\alpha\,t}Y_0 \\
&= \left(1 - e^{-\lambda V_\infty\,t}\right)\frac{\lambda+\xi^2}{\lambda V_\infty} + e^{-\lambda V_\infty\,t}\frac{1}{V_0}
\end{aligned}$$

8. **[2022FE, Mean-reversion]** The stochastic variance V_t in the GARCH diffusion model is given by

$$dV_t = \alpha(V_\infty - V_t)dt + \sigma V_t\,dB_t.$$

This stochastic process has both geometric BM and mean-reversion, and it is known as the inhomogeneous geometric Brownian motion (IGBM).

Derive $E(V_t\,|\,\mathcal{F}_0)$ under this model. (Hint: The transformation, $y_t = e^{\alpha\,t}(V_t - V_\infty)$, used in the Ornstein-Uhlenbeck process is also useful in this problem. Then, use the martingale property.) What is $E(V_t\,|\,\mathcal{F}_0)$ as $t \to \infty$.

Solution:

(a) The transformation y_t is a martingale because

$$dy_t = \alpha e^{\alpha\,t}(V_t - V_\infty) + e^{\alpha\,t}dV_t = e^{\alpha\,t}\sigma V_t dB_t.$$

Therefore, $y_0 = E(y_t|\mathcal{F}_0)$:

$$\begin{aligned}
V_0 - V_\infty &= E(e^{\alpha\,t}(V_t - V_\infty)\,|\,\mathcal{F}_0) \\
E(V_t|\mathcal{F}_0) &= V_\infty + e^{-\alpha\,t}(V_0 - V_\infty) = (1-e^{-\alpha\,t})V_\infty + e^{-\alpha\,t}V_0.
\end{aligned}$$

$E(V_t|\mathcal{F}_0)$ goes to V_∞ as $t \to \infty$.

9. **[2021FE, OU process]**. A stochastic process X_t starts from $X_0 = 1$ and follows the process,

$$dX_t = \alpha(\theta - X_t)dt + \sigma dB_t.$$

You made two observations about X_t:

(i) At $t = 0$, the average speed of X_t was 4. That is, $E(X_t) = 1 + 4t$ for very short time after $t = 0$.
(ii) After long enough time t, X_t is normally distributed with mean 2 and variance 1/2.

Based on these observations, can you estimate the parameters (i.e., θ, α, and σ)?

Solution: At $t = 0$,

$$E(X_t) = X_0 + \alpha(\theta - X_0)t.$$

At $t \gg 1$,

$$X_t \sim N\left(\theta, \frac{\sigma^2}{2\alpha}\right).$$

Therefore, the parameters are estimated as $\alpha = 4$, $\theta = 2$, and $\sigma = 2$.

10. **[2020HW 2-2, IGBM]** The inhomogeneous geometric Brownian motion (IGBM) is the geometric BM with mean reversion. The SDE is given by

$$dX_t = \lambda(X_\infty - X_t)dt + \sigma X_t dB_t \quad \text{for} \quad \lambda,\ \sigma > 0.$$

Prove that the solution for X_t is given by

$$X_t = e^{-\left(\lambda + \frac{\sigma^2}{2}\right)t + \sigma B_t}\left(X_0 + \lambda X_\infty \int_0^t e^{\left(\lambda + \frac{\sigma^2}{2}\right)s - \sigma B_s} ds\right).$$

What happens if $\lambda = 0$? **Hint:** consider the stochastic derivative (i.e., dY_t) of

$$Y_t = X_t\, e^{\left(\lambda + \frac{\sigma^2}{2}\right)t - \sigma B_t}.$$

Solution: Following the hint, we take the stochastic derivative of Y_t:

$$\begin{aligned}
dY_t &= e^{\left(\lambda + \frac{\sigma^2}{2}\right)t - \sigma B_t}\left(dX_t + X_t\left((\lambda + \frac{\sigma^2}{2})dt - \sigma dB_t + \frac{\sigma^2}{2}(dB_t)^2\right) - \sigma(dB_t)(dX_t)\right) \\
&= e^{\left(\lambda + \frac{\sigma^2}{2}\right)t - \sigma B_t}\left(\lambda(X_\infty - X_t)dt + \sigma X_t dB_t + X_t\left(\left(\lambda + \sigma^2\right)dt - \sigma dB_t\right) - \sigma(dB_t)(\sigma X_t dB_t)\right) \\
&= \lambda X_\infty e^{\left(\lambda + \frac{\sigma^2}{2}\right)t - \sigma B_t} dt.
\end{aligned}$$

Here, you should not forget to include the second-order cross term (red). Integrating both sides, we obtain

$$\begin{aligned}
Y_t &= Y_0 + \lambda X_\infty \int_0^t e^{\left(\lambda + \frac{\sigma^2}{2}\right)s - \sigma B_s} ds \\
X_t\, e^{\left(\lambda + \frac{\sigma^2}{2}\right)t - \sigma B_t} &= X_0 + \lambda X_\infty \int_0^t e^{\left(\lambda + \frac{\sigma^2}{2}\right)s - \sigma B_s} ds \\
X_t &= e^{-\left(\lambda + \frac{\sigma^2}{2}\right)t + \sigma B_t}\left(X_0 + \lambda X_\infty \int_0^t e^{\left(\lambda + \frac{\sigma^2}{2}\right)s - \sigma B_s} ds\right).
\end{aligned}$$

11. **[2016ME(ASP), SDE modified from SCFA 9.2]** Solve the SDE for the starting value X_0

$$dX_t = -t\,X_t\,dt + \sigma e^{-t^2/2}\,dB_t.$$

What is the mean and the variance of the process when $t \to \infty$?

Solution: From $e^{t^2/2}(dx + tdx) = d(e^{t^2/2}x)$, we apply Itô's lemma to our initial guess of $e^{t^2/2}X_t$;

$$d\Big(e^{t^2/2}X_t\Big) = te^{t^2/2}\,X_t\,dt + e^{t^2/2}dX_t = e^{t^2/2}\Big(tX_tdt + dX_t\Big) = \sigma dB_t$$
$$e^{t^2/2}X_t - X_0 = \sigma B_t$$
$$X_t = e^{-t^2/2}(X_0 + \sigma B_t),$$
$$E(X_t) = e^{-t^2/2}X_0 \to 0 \text{ as } t \to \infty$$
$$\mathrm{Var}(X_t) = e^{-t^2}\sigma^2\,t \to 0 \text{ as } t \to \infty$$

12. **[2016ME(ASP), SDE between normal and lognormal]** Solve the SDE for the starting value X_0 and $0 < \beta < 1$,

$$\frac{dX_t}{X_t^\beta} = \sigma\,dB_t + \frac{\sigma^2\beta}{2}X_t^{\beta-1}\,dt$$

Solution: We apply Itô's lemma to $X_t^{1-\beta}$,

$$\begin{aligned}
d(X_t^{1-\beta}) &= (1-\beta)X_t^{-\beta}dX_t - \frac{1}{2}(1-\beta)X_t^{-1-\beta}\beta(dX_t)^2 \\
&= (1-\beta)X_t^{-\beta}dX_t - \frac{1}{2}(1-\beta)\beta X_t^{-1-\beta}\cdot X_t^{2\beta}\sigma^2 dt \\
&= (1-\beta)\Big(\sigma\,dB_t + \frac{\sigma^2\beta}{2}X_t^{\beta-1}\,dt\Big) - \frac{1}{2}(1-\beta)\beta X_t^{-1-\beta}\big(X_t^\beta\sigma\big)^2 dt \\
&= (1-\beta)\sigma dB_t.
\end{aligned}$$

Therefore,

$$X_t^{1-\beta} = X_0^{1-\beta} + (1-\beta)\sigma B_t$$
$$X_t = \Big(X_0^{1-\beta} + (1-\beta)\sigma B_t\Big)^{1/(1-\beta)}.$$

13. **[2016FE, Solving SDE]** Solve the modified Cox-Ingersoll-Ross (CIR) model.

$$dX_t = \frac{\sigma^2}{4}dt + \sigma\sqrt{X_t}\,dB_t.$$

The original CIR model has the drift term $a(b - X_t)dt$ instead of $(\sigma^2/4)dt$ and was originally proposed for the movements of interest rates.

Solution: First we remove $\sqrt{X_t}$ from the coefficient of dB_t

$$\frac{dX_t}{\sqrt{X_t}} = \frac{\sigma^2}{4\sqrt{X_t}}dt + \sigma\, dB_t$$

and we solve either $dx/\sqrt{x}$ or $dx/\sqrt{x} = \sigma^2 dt/(4\sqrt{x})$ for our guess. It turns out that the first candidate $dx/\sqrt{x}$ works. Since $\int dx/\sqrt{x} = 2\sqrt{x}$,

$$d\left(2\sqrt{X_t}\right) = \frac{dX_t}{\sqrt{X_t}} - \frac{(dX_t)^2}{4X_t\sqrt{X_t}} = \frac{\sigma^2}{4\sqrt{X_t}}dt + \sigma\, dB_t - \frac{\sigma^2 X_t}{4X_t\sqrt{X_t}}dt = \sigma dB_t$$

$$2\sqrt{X_t} - 2\sqrt{X_0} = \sigma B_t \quad \Rightarrow \quad X_t = \left(\sqrt{X_0} + \frac{\sigma B_t}{2}\right)^2$$

14. **[2017FE, Solving SDE]** Solve the modified "3/2" volatility process defined by

$$dV_t = \frac{3\sigma^2}{4}V_t^2\, dt + \sigma V_t^{3/2}\, dB_t.$$

The original "3/2" model has the drift term $a(b - V_t)V_t\, dt$ instead of $(3\sigma^2/4)V_t^2\, dt$ and was originally proposed for a stochastic model for volatility.

Solution: First we remove $\sqrt{V_t}$ from the coefficient of dB_t

$$\frac{dV_t}{V_t^{3/2}} = \frac{3\sigma^2}{4}\sqrt{V_t}\, dt + \sigma\, dB_t.$$

It turns out that the first candidate $\dfrac{dx}{x^{3/2}}$ works. Since $\displaystyle\int \frac{dx}{x^{3/2}} = -\frac{2}{\sqrt{x}}$,

$$-d\left(\frac{2}{\sqrt{V_t}}\right) = \frac{dV_t}{V_t^{3/2}} - \frac{3(dV_t)^2}{4V_t^{5/2}} = \frac{3\sigma^2}{4}\sqrt{V_t}\, dt + \sigma\, dB_t - \frac{3\sigma^2}{4}\sqrt{V_t}dt = \sigma dB_t$$

$$\frac{2}{\sqrt{V_0}} - \frac{2}{\sqrt{V_t}} = \sigma B_t \quad \Rightarrow \quad V_t = \left(\frac{1}{\sqrt{V_0}} - \frac{\sigma B_t}{2}\right)^{-2}$$

10 Arbitrage and SDEs

1. **[2016FE, Option pricing under the BSM and Bachelier models]** Assume that a stock's daily price change is 1.5% of the current price. What is the annual volatility of the stock -Scholes-Merton model and the Bachelier model? What is the price of the at-the-money call option maturing in 3 months under the two models? Assume that $S_0 = 100$, $r = q = 0$ and there are 256 trading days in one year. You may use the following CDF values for the standard normal distribution $N(z)$.

z	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
$N(z)$	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564

Solution:

$$\sigma_{\text{BS}} = 1.5\% \times \sqrt{265} = 24\%, \quad \sigma_{\text{N}} = 24\% \times S_0 = 24$$

Under the BSM model:

$$d_1 = \frac{\sigma_{\text{BS}}\sqrt{1/4}}{2} = 0.06, \quad d_2 = -0.06$$
$$C_0 = S_0 N(d_1) - KN(d_2) = 100N(0.06) - 100(1 - N(0.06)) = 4.8$$

Under the Bachelier model:

$$C_0 = 0.4 \times \sigma_{\text{N}} \times \sqrt{1/4} = 4.8$$

2. **[2018FE, Option price and delta under the BSM model]** You hold a call option with $K = 100$ maturing in 3 months. Assume that a stock's annual volatility is 32% of the current price. Also assume that $r = q = 0$. You may use the following CDF values for the standard normal distribution $N(z)$.

z	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
$N(z)$	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564

(a) The stock's current spot price is $S_0 = 100$. What is the price of the call option under the BSM model?

(b) What is the delta (i.e., the sensitivity to S_0) of the option?

(c) If the spot price changed to $S_0 = 100.5$, what is the new option price under the BSM model? Approximate the price with Taylor's expansion using the results from (a) and (b).

Solution:

(a) Under the BSM model:

$$d_1 = \frac{\sigma_{\text{BS}}\sqrt{1/4}}{2} = 0.08, \quad d_2 = -0.08$$
$$C_0 = S_0 N(d_1) - KN(d_2) = 100N(0.08) - 100(1 - N(0.08)) = 6.4$$

(b)

$$\frac{\partial C}{\partial S_0} = N(d_1) = 0.532$$

(c)

$$C_0' = C_0 + (S_0' - S_0)N(d_1) = 6.4 + 0.5 \times 0.532 = 6.666$$

3. **[2016FE, Option delta under the BSM model]** By directly computing the derivative, show that the delta of a call option (i.e., sensitivity with respect to the underlying stock price S_0) is

$$D = \frac{\partial C_0}{\partial S_0} = N(d_1) \quad \text{with} \quad d_1 = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

Since the terms d_1 and d_2 are defined through S_0, you should also differentiate d_1 and d_2 rather than treating them as constants.

Solution: Using the properties,

$$\frac{\partial d_1}{\partial S_0} = \frac{\partial d_2}{\partial S_0} = \frac{1}{S_0\sigma\sqrt{T}}$$

and

$$d_1^2 - d_2^2 = (A+B)^2 - (A-B)^2 = 4AB = 2\log(S_0\, e^{rT}/K)$$
$$\text{for} \quad A = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} \quad \text{and} \quad B = \frac{\sigma\sqrt{T}}{2},$$

we compute the delta as

$$\begin{aligned}
D &= \frac{\partial}{\partial S_0}\left(S_0 N(d_1) - e^{-rT} K N(d_2)\right) \\
&= N(d_1) + S_0 n(d_1)\frac{\partial d_1}{\partial S_0} - e^{-rT} K n(d_2)\frac{\partial d_2}{\partial S_0} \\
&= N(d_1) + \frac{n(d_1)}{\sigma\sqrt{T}}\left(1 - e^{(d_1^2 - d_2^2)/2}\frac{K}{S_0 e^{rT}}\right) \\
&= N(d_1) + \frac{n(d_1)}{\sigma\sqrt{T}}\left(1 - \frac{S_0 e^{rT}}{K} \cdot \frac{K}{S_0 e^{rT}}\right) = N(d_1).
\end{aligned}$$

4. **[2018ME(ASP), 2019FE, Option vega under the BSM model]** Derive that the vega of a call option (i.e., sensitivity with respect to the volatility σ) is

$$V = \frac{\partial C}{\partial \sigma} = S_0\, n(d_1)\sqrt{T} = K e^{-rT} n(d_2)\sqrt{T}.$$

Remind that the call option price under the BSM model is

$$C = S_0 N(d_1) - e^{-rT} K N(d_2) \quad \text{where} \quad d_{1,2} = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}$$

Since the terms d_1 and d_2 are implicit functions of σ, you should also differentiate d_1 and d_2.

Solution: Using the following properties:
(i)
$$\frac{\partial d_{1,2}}{\partial \sigma} = -\frac{\log(S_0\, e^{rT}/K)}{\sigma^2\sqrt{T}} \pm \frac{1}{2}\sqrt{T} = -\frac{d_{2,1}}{\sigma}$$
(ii) if
$$A = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} \quad \text{and} \quad B = \frac{\sigma\sqrt{T}}{2},$$
$$d_1^2 - d_2^2 = (A+B)^2 - (A-B)^2 = 4AB = 2\log(S_0\, e^{rT}/K) \quad \Rightarrow \quad \frac{n(d_2)}{n(d_1)} = \frac{S_0\, e^{rT}}{K}.$$
Then, we compute the vega as
$$\begin{aligned} V &= \frac{\partial}{\partial \sigma}\big(S_0 N(d_1) - e^{-rT} K N(d_2)\big) = S_0 n(d_1)\frac{-d_2}{\sigma} - e^{-rT} K n(d_2)\frac{-d_1}{\sigma} \\ &= S_0 n(d_1)\left(-\frac{d_2}{\sigma} + \frac{K n(d_2)}{S_0 e^{rT}\, n(d_1)}\frac{d_1}{\sigma}\right) = S_0 n(d_1)\left(-\frac{d_2}{\sigma} + \frac{d_1}{\sigma}\right) \\ &= S_0 n(d_1)\sqrt{T} = K e^{-rT} n(d_2)\sqrt{T}. \end{aligned}$$

5. **[2020FE, Option theta under the BSM model]** Remind that the present value of the call option under the BSM model is
$$C = S_0 N(d_1) - e^{-rT} K N(d_2) \quad \text{where} \quad d_{1,2} = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}.$$
Derive the theta, that is the derivative of C with respect to the (decreasing) time-to-maturity:
$$\Theta = -\frac{\partial C}{\partial T}.$$
Hint: Since d_1 and d_2 are the functions of T, you should also differentiate d_1 and d_2 with respect to T.

Solution: Since
$$\frac{\partial d_{1,2}}{\partial T} = -\frac{\log(S_0/K)}{2\sigma T\sqrt{T}} + \frac{r}{2\sigma\sqrt{T}} \pm \frac{\sigma}{4\sqrt{T}},$$
let
$$A = -\frac{\log(S_0/K)}{2\sigma T\sqrt{T}} + \frac{r}{2\sigma\sqrt{T}}, \quad B = \frac{\sigma}{4\sqrt{T}}.$$
From **[2016FE, Option delta under the BSM model]** and **[2016FE, Option vega under the BSM model]**, we also know that
$$S_0\, e^{rT} n(d_1) = K n(d_2)$$

Now we can calculate Θ as

$$\begin{aligned}\Theta = -\frac{\partial C}{\partial T} &= -S_0 n(d_1)\frac{\partial d_1}{\partial T} + e^{-rT}Kn(d_2)\frac{\partial d_2}{\partial T} - rKe^{-rT}N(d_2)\\ &= -S_0 n(d_1)(A+B) + e^{-rT}Kn(d_2)(A-B) - rKe^{-rT}N(d_2)\\ &= -S_0 n(d_1)(2B) - rKe^{-rT}N(d_2)\\ &= -\frac{S_0\, n(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2).\end{aligned}$$

Alternatively, Θ is expressed as

$$\Theta = -\frac{Ke^{-rT}n(d_2)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) = -Ke^{-rT}\left(\frac{n(d_2)\sigma}{2\sqrt{T}} + rN(d_2)\right).$$

6. **[2021FE, Black-Scholes Greeks]** Remind that the call option price under the Black–Scholes model (assuming $r = q = 0$) is

$$C = S_0 N(d_1) - KN(d_2) \quad \text{where} \quad d_{1,2} = \frac{\log(S_0/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}.$$

For the Greeks, consider the following expressions.

$$\begin{gathered}(1)\ N(d_1), \quad (2)\ Kn(d_2)\sqrt{T}, \quad (3)\ S_0\, n(d_1)\sqrt{T}, \quad (4)\ S_0\, n(d_2)\sqrt{T}\\ (5)\ N(d_2), \quad (6)\ \frac{n(d_1)}{S_0\,\sigma\sqrt{T}}, \quad (7)\ -\frac{\sigma\, S_0\, n(d_1)}{2\sqrt{T}}, \quad (8)\ \frac{\sigma\, S_0\, n(d_1)}{2\sqrt{T}}\end{gathered}$$

(a) From above, choose Delta, $D = \frac{\partial C}{\partial S_0}$.

(b) From above, choose Vega, $V = \frac{\partial C}{\partial \sigma}$.

(c) From above, choose Gamma, $G = \frac{\partial^2 C}{\partial S_0^2}$.

(d) From above, choose Theta, $\Theta = -\frac{\partial C}{\partial T}$.

(e) Obtain the corresponding Greeks for the put option by properly modifying your answers in (a)–(d). (The answers are not necessarily in the list above.)

Solution:

(a) (1)

(b) (2) or (3)

(c) (6)

(d) (7)

(e) From the Greeks of put options, we use the put–call parity:

$$P = C - (S_0 - K).$$

Regarding Delta, we know $\frac{\partial P}{\partial S_0} = \frac{\partial C}{\partial S_0} - 1$. Therefore, the Delta of put option is

$$D = N(d_1) - 1 = -N(-d_1).$$

Vega, Gamma, and Theta are same as those of call option.

7. **[2022FE, Homogeneous function]** A multi-variate function F is *homogeneous* if

$$F(\lambda x_1, \cdots, \lambda x_n) = \lambda F(x_1, \cdots, x_n).$$

Euler's homogeneous function theorem states that, if F is a homogeneous function,

$$F(x_1, \cdots, x_n) = \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k} F(x_1, \cdots, x_n).$$

(a) Show that the Black–Scholes (call or put) option price is a homogeneous function of the strike price K and spot price S_0:

$$C(\lambda K, \lambda S_0) = \lambda C(K, S_0).$$

Show it using that S_t follows a geometric BM, but **do not** use the Black–Scholes formula.

(b) Is Bachelier option price a homogeneous function of K and S_0? Explain why. For this question, you may use the Bachelier formula.

(c) If (a) is true, the BS option price satisfies:

$$C(K, S_0) = S_0 \frac{\partial C}{\partial S_0} + K \frac{\partial C}{\partial K}.$$

By comparing to the BS call option price formula (you can use it), find the two partial derivatives, $\frac{\partial C}{\partial S_0}$ and $\frac{\partial C}{\partial K}$. ($\frac{\partial C}{\partial S_0}$ is the option delta. This is another way of deriving $\frac{\partial C}{\partial S_0}$ and $\frac{\partial C}{\partial K}$.)

Solution:

(a) The stock price follows a geometric BM, so the process is multiplicative. If the spot price S_0 is multiplied by λ, so is the terminal price S_T:

$$(\lambda S_T) = (\lambda S_0)\, e^{\sigma B_T - \sigma^2 T/2}.$$

Therefore,

$$C(\lambda K, \lambda S_0) = e^{-rT} E\left((\lambda S_T - \lambda K)^+\right) = \lambda\, e^{-rT} E\left((S_T - K)^+\right) = \lambda\, C(K, S_0).$$

(b) The stock price under the Bachelier model,

$$S_T = S_0 + \sigma T,$$

is not multiplicative. The spot price λS_0 does not guarantee λS_T. It can be verified from the Bachelier formula:

$$C_{\mathrm{N}}(\lambda K, \lambda S_0) = (\lambda S_0 - \lambda K)N(d_{\mathrm{N}}) + \sigma\sqrt{T}\, n(d_{\mathrm{N}}) \neq \lambda\, C(K, S_0) \quad \text{for} \quad d_{\mathrm{N}} = \frac{\lambda S_0 - \lambda K}{\sigma_{\mathrm{N}}\sqrt{T}}.$$

Instead, it is translative; $S_0 + \lambda \;\Rightarrow\; S_T + \lambda$:

$$C_{\mathrm{N}}(K + \lambda, S_0 + \lambda) = (S_0 - K)N(d_{\mathrm{N}}) + \sigma\sqrt{T}\, n(d_{\mathrm{N}}) = C(K, S_0),$$

where d_{N} remains the same.

(c) By comparing to the BS call option formula, we have

$$\frac{\partial C}{\partial S_0} = N(d_1) \quad \text{and} \quad \frac{\partial C}{\partial K} = e^{-rT} N(d_2),$$

where

$$d_{1,2} = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}.$$

8. **[2019FE, Modified BSM]** Assume that the stock price S_t follows a BM with the stochastic volatility σ_t following a GBM. The price and volatility are driven by the same standard BM B_t.

$$dS_t = \sigma_t dB_t \quad \text{and} \quad \frac{d\sigma_t}{\sigma_t} = -\nu dB_t \quad (\nu > 0).$$

(a) Express the final price S_T in terms of S_0, σ_0, ν, T, and B_T by solving this SDE.

(b) What is the call option price with strike price K and time-to-maturity T?

(c) If $\nu = 0$, then $\sigma_t = \sigma_0$ for all t, that is, the volatility is no longer stochastic. Prove that, in the limit of $\nu \to 0$, the call option price from (b) converges to that of the Bachelier model with normal volatility σ_0. You may need Taylor's expansion: $\log(1+\varepsilon) \approx \varepsilon - \varepsilon^2/2$ when ε is very small.

Solution:

(a)

$$\sigma_T = \sigma_0 \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right)$$

$$\sigma_T - \sigma_0 = -\nu \int_0^T \sigma_t dB_t$$

$$S_T - S_0 = \int_0^T \sigma_t dB_t = \frac{1}{\nu}(\sigma_0 - \sigma_T) = \frac{\sigma_0}{\nu}\left(1 - \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right)\right)$$

(b) The final stock price S_T is expressed by a standard normal variable z:

$$S_T - S_0 = \frac{\sigma_0}{\nu}\left(1 - \exp\left(-\nu z\sqrt{T} - \frac{1}{2}\nu^2 T\right)\right).$$

If $z=-d_1$ is the root of $S_T=K$, d_1 is obtained as

$$\log\left(1-\frac{\nu}{\sigma_0}(K-S_0)\right)=\nu d_1\sqrt{T}-\frac{1}{2}\nu^2 T$$
$$d_1=\frac{\log\left(1-\frac{\nu}{\sigma_0}(K-S_0)\right)}{\nu\sqrt{T}}+\frac{1}{2}\nu\sqrt{T}.$$

The call option price can be obtained as

$$\begin{aligned}
C &= E\Big((S_T-K)^+\Big)=\int_{-d_1}^{\infty}\left[\frac{\sigma_0}{\nu}\left(1-\exp\left(-\nu z\sqrt{T}-\frac{1}{2}\nu^2 T\right)\right)+S_0-K\right]n(z)dz\\
&=\left(S_0-K+\frac{\sigma_0}{\nu}\right)\int_{-d_1}^{\infty}n(z)dz-\frac{\sigma_0}{\nu}\int_{-d_1}^{\infty}n(z+\nu\sqrt{T})dz\\
&=\left(S_0-K+\frac{\sigma_0}{\nu}\right)(1-N(-d_1))-\frac{\sigma_0}{\nu}(1-N(-d_1+\nu\sqrt{T}))\\
&=\left(S_0-K+\frac{\sigma_0}{\nu}\right)N(d_1)-\frac{\sigma_0}{\nu}N(d_2)\\
&\quad\text{where}\quad d_{1,2}=\frac{\log\left(1-\frac{\nu}{\sigma_0}(K-S_0)\right)}{\nu\sqrt{T}}\pm\frac{1}{2}\nu\sqrt{T}
\end{aligned}$$

(c) Let

$$d=\frac{(S_0-K)}{\sigma_0\sqrt{T}},$$

then d_1 and d_2 converge to d as $\nu\to 0$:

$$d_{1,2}=\frac{\log\left(1-\frac{\nu}{\sigma_0}(K-S_0)\right)}{\nu\sqrt{T}}\pm\frac{1}{2}\nu\sqrt{T}=\frac{(S_0-K)}{\sigma_0\sqrt{T}}+A\nu\pm\frac{1}{2}\nu\sqrt{T}\to d\quad\text{for some } A.$$

Therefore, the call option price converges to

$$\begin{aligned}
C &= (S_0-K)N(d_1)+\frac{\sigma_0}{\nu}(N(d_1)-N(d_2))\\
&= (S_0-K)N(d)+\sigma_0\sqrt{T}\,n(d),
\end{aligned}$$

where we used the L'Hopital's rule,

$$\lim_{\nu\to 0}\frac{N(d_1)-N(d_2)}{\nu}=\lim_{\nu\to 0}\frac{N(d+A\nu+\nu\sqrt{T}/2)-N(d+A\nu-\nu\sqrt{T}/2)}{\nu}=n(d)\sqrt{T}$$

9. **[2021FE, Displaced Black-Scholes (DBS) model]** For a stock price S_t, assume that the 'displaced' price,

$$D(S_t)=\beta S_t+(1-\beta)A\quad\text{for some constants}\quad A>0\text{ and }0\le\beta\le 1,$$

follows a geometric BM with volatility $\beta\sigma_{\text{D}}$:

$$\frac{dD(S_t)}{D(S_t)}=\beta\sigma_{\text{D}}dB_t\quad\text{or}\quad\frac{dS_t}{D(S_t)}=\sigma_{\text{D}}dB_t.$$

$D(S_t)$ can be understood as a liner interpolation between S_t and A as β changes. The DBS model offers a "bridge" between the Bachelier and BSM models. For the questions, assume $r = q = 0$.

(a) Express S_T as a function of B_T and other parameters.

(b) Derive the price of the call option with strike price K and time to maturity T under the DBS model. (Hint: you should recover the BS formula as $\beta \to 1$.)

(c) In the limit $\beta \to 0$, show that the price from (b) converges to the Bachelier model price. What is the Bachelier volatility σ_{N} in the limit?

Solution:

(a) Because $D(S_t)$ follows a geometric BM, $D(S_T)$ is given by

$$D(S_T) = D(S_0)\exp\left(\beta\sigma_{\text{D}}B_T - \beta^2\sigma_{\text{D}}^2T/2\right).$$

Therefore, S_T is

$$S_T = \left(S_0 + \frac{1-\beta}{\beta}A\right)\exp\left(\beta\sigma_{\text{D}}B_T - \frac{\beta^2\sigma_{\text{D}}^2T}{2}\right) - \frac{1-\beta}{\beta}A.$$

(b) The call option price is given by

$$C_{\text{D}}(K) = E((S_T-K)^+) = \frac{E((D(S_T)-D(K))^+)}{\beta} = \frac{D(S_0)N(d_1) - D(K)N(d_2)}{\beta}$$
$$\text{where}\quad d_{1,2} = \frac{\log\left(D(S_0)/D(K)\right)}{\beta\sigma_{\text{D}}\sqrt{T}} \pm \frac{\beta\sigma_{\text{D}}\sqrt{T}}{2}.$$

Note $D(S_t) = S_t$ when $\beta = 1$. Therefore, it is obvious that $C_{\text{D}}(K)$ converges to the Black–Scholes option price with volatility σ_{D} when $\beta = 1$.

(c) For small β, we have the following approximations:

$$\log\left(\frac{D(S_0)}{D(K)}\right) = \frac{\beta(S_0-K)}{(1-\beta)A}\left(1+\frac{\beta(S_0+K)}{2(1-\beta)A}\right) + O(\beta^2),$$
$$d_{1,2} = \frac{S_0-K}{(1-\beta)A\sigma_{\text{D}}\sqrt{T}}\left(1+\frac{\beta(S_0+K)}{2(1-\beta)A}\right) \pm \frac{\beta\sigma_{\text{D}}\sqrt{T}}{2} + O(\beta).$$

The DBS model option price converges to the Bachelier model price as $\beta \downarrow 0$:

$$\begin{aligned}
C_{\text{D}}(K) &= \frac{D(S_0)-D(K)}{\beta}N(d_2) + \frac{D(S_0)}{\beta}\left(N(d_1)-N(d_2)\right) \\
&= (S_0-K)N(d_2) + \frac{D(S_0)}{\beta}(d_1-d_2)\,n(d_1) + O(\beta) \\
&\to (S_0-K)N(d_{\text{N}}) + \sigma_{\text{N}}\sqrt{T}\,n(d_{\text{N}}) = C_{\text{N}}(K) \quad \text{with} \quad \sigma_{\text{N}} = A\sigma_{\text{D}}.
\end{aligned}$$

10. **[2020FE, Implied volatility bound]** Suppose that a call option with strike K and time-to-maturity T is traded at the price C in the market. The implied volatility is σ such that

the Black–Scholes formula with σ yields the observed price C. Because there is no analytic solution for $\sigma = \sigma(C, K, S_0, T)$, we solve σ using numerical methods such as bisection or Newton's method. For such numerical root-finding, knowing the bounds of σ can reduce computation. While we can use a simple lower bound $\sigma_L = 0$, we are going to find an upper bound σ_U in this question so that $0 \le \sigma \le \sigma_U$. Assume that $r = q = 0$.

(a) Before finding σ_U, let us consider a special case first. If C is the market price of the at-the-money option ($K = S_0$), it is possible to obtain the implied volatility σ directly. Find $\sigma = \sigma(C, S_0, T)$. You can use the inverse normal CDF, $N^{-1}(\cdot)$, in your answer. **Hint:** $N(x) + N(-x) = 1$.

(b) When you hold a call option with strike K, you can maximize the option value by exercising it when $S_T \ge K$. But, imagine that you **incorrectly** exercise the option when $S_T \ge S_0$. In this way, your payout out can be even negative. Derive the option value (i.e., the expected payout) under the incorrect exercise policy.

(c) Using the result from (b), find an upper bound σ_U. **Hint:** (i) The price from (b) is always lower than (or equal to) the usual BSM price with the same volatility because of the incorrect exercise policy. (ii) The option value is an increasing function of volatility.

Solution:

(a) Because $d_{1,2} = \pm\sigma\sqrt{T}/2$ when $K = S_0$, the at-the-money call option price under the BSM model is

$$C = S_0(N(d_1) - N(-d_1)) = S_0(2N(d_1) - 1)$$

Therefore, the implied volatility is given by

$$\sigma = \frac{2}{\sqrt{T}} N^{-1}\left(\frac{C + S_0}{2S_0}\right).$$

(b) If you (incorrectly) exercise the call option when $S_T \ge K^*$ in general, the option value is

$$C^* = S_0 N(d_1^*) - K N(d_2^*) \quad \text{for} \quad d_{1,2}^* = \frac{\log(S_0/K^*)}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}.$$

Note that the only modifications from the BSM formula are d_1 and d_2 because they are related to the exercise boundary. If $K^* = S_0$, the price is given by

$$C^* = S_0 N(d_1^*) - K N(-d_1^*) \quad \text{for} \quad d_1^* = \frac{\sigma\sqrt{T}}{2}.$$

(c) The implied volatility of the incorrect price C^* from (b) is given by

$$C^* = S_0 N(d_1^*) - K N(-d_1^*) = (S_0 + K)N(d_1^*) - K$$
$$\sigma* = \frac{2}{\sqrt{T}} N^{-1}\left(\frac{C^* + K}{S_0 + K}\right).$$

Because of the exercise decision is suboptimal, this price C^* is lower than the BSM price (with correct exercise). In other words, if $C^* = C$, the implied volatility σ^*

under the incorrect exercise should be higher than the implied volatility σ under the correct exercise. Therefore, σ^* for C is an upper bound of σ:

$$\sigma \le \sigma_U = \frac{2}{\sqrt{T}} N^{-1}\left(\frac{C+K}{S_0+K}\right).$$

Here we note that (i) when C approaches the maximum call option value S_0, $\sigma_U \to \infty$ as expected, and (ii) when $K = S_0$, the upper bound is the exact implied volatility ($\sigma = \sigma_U$) in (a).

11 The Diffusion Equation

There is no question in this chapter.

12 Representation Theorems

1. **[2017HW 3-5, Martingale representation theory]**
For a standard BM B_t ($0 \le t \le T$), find the martingale representation of $X_t = E(B_T^3|\mathcal{F}_t)$. (In class, we did the same for $X_t = E(B_T^2|\mathcal{F}_t)$)

Solution: Using the short notation $E_t(\,\cdot\,) = E(\,\cdot\,|\mathcal{F}_t)$,

$$\begin{aligned} X_t &= E_t((B_t + B_T - B_t)^3) = B_t^3 + 3B_t^2 E_t(B_T - B_t) + 3B_t\, E_t((B_T - B_t)^2) + E_t((B_T - B_t)^3) \\ &= B_t^3 + 0 + 3(T-t)B_t + 0 = B_t^3 + 3(T-t)B_t. \end{aligned}$$

In particular, $X_0 = 0$. From the SDE,

$$dX_t = 3B_t^2 dB_t + 3B_t(dB_t)^2 + 3(T-t)dB_t - 3B_t dt = 3(B_t^2 + T - t)dB_t,$$

the martingale representation is

$$X_T = \int_0^T 3(B_t^2 + T - t)\; dB_t$$

2. **[2018FE, SDE and martingale representation theorem]**
   (a) Apply Itô calculus to find the stochastic differentiation of $\cosh(B_t)$. Reminded that $\cosh x = (e^x + e^{-x})/2$.
   (b) Find λ such that $X_t = e^{\lambda t}\cosh(B_t)$ is a martingale (i.e., dX_t has no dt term.)
   (c) Using (b), find the martingale representation of $\cosh(B_T)$. In other words, find V_0 and ϕ_t satisfying
   $$\cosh(B_T) = V_0 + \int_0^T \phi_t\, dB_t$$
   (d) Assume a stock price S_t follows $dS_t = \sigma dB_t$ and $r = 0$. What is the price of a derivative that pays $\cosh(S_T - S_0)$ at time $t = T$?

Solution:

(a)

$$d\cosh(B_t) = \sinh(B_t)dB_t + \frac{1}{2}\cosh(B_t)dt.$$

(b) The SDE for X_t is

$$dX_t = e^{\lambda t}\sinh(B_t)dB_t + e^{\lambda t}\left(\lambda + \frac{1}{2}\right)\cosh(B_t)dt.$$

Therefore X_t is a martingale when $\lambda = -1/2$.

(c)

$$X_T - X_0 = e^{-T/2}\cosh(B_T) - 1 = \int_0^T e^{-t/2}\sinh(B_t)\,dB_t$$

$$\cosh(B_T) = e^{T/2} + \int_0^T e^{(T-t)/2}\sinh(B_t)\,dB_t$$

Therefore, we obtain $V_0 = e^{T/2}$ and $\phi_t = e^{(T-t)/2}\sinh(B_t)$.

(d) With the presence of σ, the results above becomes

$$X_t = e^{-\sigma^2 t/2}\cosh(\sigma B_t)$$

and

$$\cosh(S_T - S_0) = \cosh(\sigma B_T) = e^{\sigma^2 T/2} + \int_0^T e^{\sigma^2(T-t)/2}\sinh(\sigma B_t)\,dB_t$$

The price is $V_0 = e^{\sigma^2 T/2}$. It can also directly computed as

$$V_0 = E(\cosh(\sigma B_T)) = \frac{1}{2}E\left(e^{\sigma\sqrt{T}z} + e^{-\sigma\sqrt{T}z}\right) = \frac{1}{2}\left(e^{\sigma^2 T/2} + e^{\sigma^2 T/2}\right) = e^{\sigma^2 T/2}.$$

3. **[2022FE, Martingale representation theory]** Consider the following martingale:

$$X_t = E\left(e^{\lambda B_T} \,\middle|\, \mathcal{F}_t\right).$$

(a) Directly calculate X_t as a function of t, T, and B_t.

(b) Find the martingale representation of X_t. In other words, find ϕ_t that satisfies

$$dX_t = \phi_t\,dB_t \quad \text{or} \quad X_T = X_0 + \int_0^T \phi_t\,dB_t.$$

Solution: This question was inspired from this quant finance StackExchange post.

(a) X_t is calculated as

$$X_t = E\left(e^{\lambda B_T} \,\middle|\, \mathcal{F}_t\right) = e^{\lambda B_t} E\left(e^{\lambda(B_T - B_t)} \,\middle|\, \mathcal{F}_t\right) = e^{\lambda B_t} e^{\lambda^2(T-t)/2} = e^{\lambda B_t + \lambda^2(T-t)/2}.$$

Basically, X_t is a geometric BM: $X_t = e^{\lambda^2 T/2}\, e^{\lambda B_t - \lambda^2 t/2}$.

(b) Applying Itô's lemma, we get the SDE:

$$dX_t = \lambda e^{\lambda B_t + \lambda^2 (T-t)/2} dB_t \quad (= \lambda X_t\, dB_t).$$

Therefore, $\phi_t = \lambda X_t$:

$$\phi_t = \lambda e^{\lambda B_t + \lambda^2 (T-t)/2}.$$

4. **[2019FE, Black-Scholes and martingale representation theorem]** For this question, assume that the current time is $t = t$ (instead of $t = 0$) and the option expiry is $t = T$. Therefore, the time-to-maturity is $T - t$ (instead of T). Also assume that $r = 0$ to make the problem simple. Then, the underlying stock price follows a geometric BM, $dS_t = \sigma S_t\, dB_t$, and the call option price at time t is given by the Black-Scholes formula,

$$C_t = S_t N(d_1) - K\, N(d_2), \quad \text{where} \quad d_{1,2} = \frac{\log(S_t/K)}{\sigma\sqrt{T-t}} \pm \frac{1}{2}\sigma\sqrt{T-t}.$$

From the derivation of the BS formula in class, we know

$$dC_t = D_t\, dS_t,$$

where D_t is the delta of the option, i.e., the amount of the underlying stock to hold at time t to hedge the option:

$$D_t = \frac{\partial C_t}{\partial S_t} = N(d_1).$$

The martingale representation also tells us that the option premium, C_0, and the P&L from the hedge position from $t = 0$ to T will exactly add up to the payoff of the option, $C_T = (S_T - K)^+$:

$$C_T = (S_T - K)^+ = C_0 + \int_0^T N(d_1) dS_t.$$

In this question, we are going to show $dC_t = D_t\, dS_t$ by direct computation.

(a) Find the gamma, the second derivative with respect to the spot price S_t, and theta, the derivative with respect to time t:

$$G_t = \frac{\partial^2 C_t}{\partial S_t^2} \quad \text{and} \quad \Theta_t = \frac{\partial C_t}{\partial t}$$

(b) Apply Itô's lemma to find the stochastic differential equation (SDE) for C_t. Use the result of (a).

(c) Imagine that a situation where the volatility for pricing option is different from that for the underlying stock. You price and risk-manage option using volatility σ_i (i for *implied* volatility). That is, C_t and D_t is evaluated with

$$d_{1,2} = \frac{\log(S_t/K)}{\sigma_i\sqrt{T-t}} \pm \frac{1}{2}\sigma_i\sqrt{T-t}.$$

But the underlying stock has volatility σ_r (r for *realized* volatility),

$$dS_t = \sigma_r S_t\, dB_t.$$

Derive the SDE for C_t again under this new situation. In this situation, does the option premium and hedging P&L amount to the option payout? Compare the cases, $\sigma_i > \sigma_r$ and $\sigma_i > \sigma_r$.

Solution:

(a) Gamma and theta are obtained as

$$G_t = \frac{n(d_1)}{S_t\,\sigma\sqrt{T-t}} \quad \text{and} \quad \Theta_t = -\frac{\sigma\, S_t\, n(d_1)}{2\sqrt{T-t}}.$$

(b) The SDE for C_t is computed as

$$\begin{aligned} dC_t &= \frac{\partial C_t}{\partial t}dt + \frac{\partial C_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial C_t^2}{\partial^2 S_t}(dS_t)^2 \\ &= \left(\Theta_t + \frac{\sigma^2 S_t^2}{2}G_t\right) dt + D_t\, dS_t \\ &= \left(-\frac{\sigma\, S_t\, n(d_1)}{2\sqrt{T-t}} + \frac{\sigma^2 S_t^2}{2}\frac{n(d_1)}{S_t\,\sigma\sqrt{T-t}}\right) dt + N(d_1)\, dS_t \\ &= N(d_1)\, dS_t. \end{aligned}$$

(c) Using $(dS_t)^2 = \sigma_r^2\, S_t^2\, dt$ instead, we obtain

$$\begin{aligned} dC_t &= \left(-\frac{\sigma_i\, S_t\, n(d_1)}{2\sqrt{T-t}} + \frac{\sigma_r^2 S_t^2}{2}\frac{n(d_1)}{S_t\,\sigma_i\sqrt{T-t}}\right) dt + N(d_1)\, dS_t \\ &= \frac{\sigma_r^2 - \sigma_i^2}{2} \cdot \frac{S_t\, n(d_1)}{\sigma_i\sqrt{T-t}} dt + N(d_1)\, dS_t \\ &= \frac{S_t^2}{2}(\sigma_r^2 - \sigma_i^2) G_t dt + N(d_1)\, dS_t. \end{aligned}$$

The sum of the premium and hedging P&L is

$$C_0 + \int_0^T N(d_1) dS_t = (S_T - K)^+ + \frac{\sigma_i^2 - \sigma_r^2}{2}\int_0^T S_t^2\, G_t\, dt.$$

Notice that $S_t^2 G_t$ is always positive. If $\sigma_i > \sigma_r$, the value you hold at $t = T$ is bigger than the option payout $(S_T - K)^+$. If $\sigma_r > \sigma_i$, the value is less than the option payout $(S_T - K)^+$. This is consistent with observation,

$$C_0 > C_0^r \quad \text{if} \quad \sigma_i > \sigma_r \quad \text{and} \quad C_0 < C_0^r \quad \text{if} \quad \sigma_i < \sigma_r,$$

where C_0^r is the *correct* option price evaluated with the realized volatility σ_r,

$$d_{1,2}^r = \frac{\log(S_t/K)}{\sigma_r\sqrt{T-t}} \pm \frac{1}{2}\sigma_r\sqrt{T-t}.$$

(Option price is a monotonically increasing function of volatility.)

13 Girsanov Theory

1. **[2017FE, Black-Scholes and Importance sampling]** Assume that $r = q = 0$ in this problem.

   (a) Suppose that a stock price follows $dS_t/S_t = \sigma dB_t$ and you want to price a call option with strike price K and maturity T. How do you get the option price by Monte-Carlo simulation? In other words, what should be the expression for $f(Z)$ in the following Monte-Carlo formula:

   $$C = \frac{1}{N}\sum_{k=1}^{N} f(Z_k),$$

   where Z_k is a sequence of i.i.d. standard normal RVs.

   (b) Using Black-Scholes formula, find the call option price for $S_0 = 100, K = 2000, T = 1, \sigma = 0.5$. For computation, use approximation $20 \approx e^3$. You may also use the following CDF values for the standard normal distribution $N(z)$.

z	-6.5	-6.25	-6.0	-5.75	-5.5
$N(z)$	4.0×10^{-11}	2.1×10^{-10}	9.9×10^{-10}	4.5×10^{-9}	1.9×10^{-8}

   (c) As observed in (b), the call option value can be very small if K is extremely high compared to S_0 ($K \gg S_0$). Therefore, the Monte-Carlo price from (a) would be zero because no Z_k will make the stock price in-the-money. Apply important sampling and express the option price. (Hint: Select the shift amount to make the stock price at-the-money.)

Solution:

(a) The security price can be simulated as

$$S_T = S_0 \exp(\sigma\sqrt{T}Z - \sigma^2 T/2) \quad \text{for} \quad Z \sim N(0,1).$$

and the call option price is obtained as

$$C = \frac{1}{N}\sum_{k=1}^{N}\Big(S_0 \exp(\sigma\sqrt{T}Z_k - \sigma^2 T/2) - K\Big)^+$$

for i.i.d. samples $\{Z_k\}$ (assume $r = 0$) and $(x)^+ = \max(x, 0)$.

(b) From $d_{1,2} = \log(S_0/K)/\sigma\sqrt{T} \pm \sigma\sqrt{T}/2 = -6 \pm 0.25 = -5.75,\ -6.25$,

$C = 100N(-5.75) - 2000N(-6.25) = 100\times 4.5\times 10^{-9} - 2000\times 2.1\times 10^{-10} = 3\times 10^{-8}$

(c) Remind that

$$E[f(Z)] = E\big[f(Z+\mu)\, e^{-\mu Z - \mu^2/2}\big].$$

Therefore,

$$C = \frac{1}{N}\sum_{k=1}^{N}\Big(S_0 \exp(\sigma\sqrt{T}(Z_k+\mu) - \sigma^2 T/2) - K\Big)^+ e^{-\mu Z_k - \mu^2/2}$$

where μ is determined as

$$S_0 \exp(\sigma\sqrt{T}\mu - \sigma^2 T/2) = K \quad \Rightarrow \quad \mu = \frac{\log(K/S_0)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}.$$

Notice that $\mu = -d_2$ in the Black-Scholes formula.

14 Arbitrage and Martingales

1. **[2020FE, Equivalent Martingale Measure]** Let β_t ($\beta_0 = 1$) be the value of the saving account and $P(t,T)$ be the price of the zero-coupon bond maturing at T, observed at time t. Give **True or False**. If false, briefly explain why.

   (a) For a stock price S_t and the risk-neutral measure Q,

   $$(S_0 - K)^+ = E^Q\left(\frac{(S_T - K)^+}{\beta_T}\right)$$

   (b) For the stochastic process of a security price X_t,

   $$X_0 = P(0,T)E^Q\left(X_T\right)$$

Solution:

(a) False. If $S_0 < K$ for instance, the left-hand side is zero, but the right-hand side is positive. The statement is true for the option price X_t that pays $(S_T - K)^+$ at T.

$$X_0 = E^Q\left(\frac{X_T}{\beta_T}\right) = E^Q\left(\frac{(S_T - K)^+}{\beta_T}\right)$$

(b) False. The expectation should be under the T-forward measure:

$$X_0 = P(0,T)E^T\left(X_T\right)$$

2. **[2017FE, Siegel's paradox]** In foreign exchange market, each currency has a symbol of three characters. For example, `USD` is for US Dollar, `CNY` is for Chinese Yuan, and `EUR` is for Euro. Also, foreign exchange rate is quoted in one fixed direction such that the exchange rate is bigger than one. For example, the exchange between `USD` and `CNY` is always quoted in `USD/CNY`, not in `CNY/USD`. `USD/CNY` means the price of 1 `USD` in terms of `CNY`, which is around 6.3 as of today.

   (a) Due to the recent trade war between US and China, suppose that the FX rate `USD/CNY` is going to be very volatile. Assuming `USD/CNY` = 6.0 today and it can change to either 4.0 or 8.0 after one year. Under the risk-neutral measure Q, what are the probabilities for the two scenarios? Assume that risk-free rate is zero in both US and China. (Hint: You can think of `USD/CNY` as the price of an asset (1 `USD`) in `CNY` unit.)

   (b) Now consider the scenario from the perspective of an investor in US. Suppose a US investor invests in `CNY` by converting 1 `USD` to 6 `CNY` today and holding 6 `CNY` for one year. What is her expected return in terms of `USD` after one year? If the expected return is not zero, does it conflict with that the expected return under the risk-neutral measure Q should be same as the risk-less rate (which is zero in this problem)? (Hint: Consider the numeraire.)

   (c) We can demonstrate this using stochastic calculus. If X_t = `USD/CNY`, X_t is a martingale (no drift), so the SDE is given by

   $$\frac{dX_t}{X_t} = \sigma dB_t^Q.$$

By applying Itô's lemma, find the SDE for X_0/X_t, which is the investment value in USD. Is the drift positive, negative or zero?

(d) In class, we show that, for a numeraire asset with volatility σ_N, the relation between the two standard BMs between the probability measures, Q^N and Q (risk-neutral measure), is given by

$$dB_t^{Q^N} + \sigma_N dt = dB_t^Q.$$

What is the numeraire to make X_0/X_t a martingale? From the relation above, show that X_0/X_t is indeed a martingale under the measure associated with the numeraire.

Solution: See (WIKIPEDIA) for **Siegel's paradox**.

(a)

$$4p + 8(1-p) = 6 \ \texttt{CNY} \quad \Rightarrow \quad p = 0.5$$

(b) After one year, the investor receives in

$$\frac{6}{4} \cdot \frac{1}{2} + \frac{6}{8} \cdot \frac{1}{2} = \frac{9}{8} \ \texttt{USD}$$

Therefore the return is 12.5%. This does not contradict with the zero expected return of the risk-neutral measure because the numeraire of the risk-neutral measure is the CNY cash while the numeraire for the US investor is USD cash.

(c) In general, the SDE for the exchange rate is given by

$$\frac{dX_t}{X_t} = (r_d - r_f)\,dt + \sigma dB_t^Q,$$

where r_d is the risk-free rate of domestic currency (CNY in this case) and r_f is the foreign currency (USD in this case). We assume $r_d = r_f = 0$ for this problem, but we derive general result with r_d and r_f.

$$\begin{aligned} d\log X_t &= (r_d - r_f - \sigma^2/2)dt + \sigma dB_t^Q \\ d\log(X_0/X_t) &= (r_f - r_d + \sigma^2/2)dt - \sigma dB_t^Q \\ \frac{d(X_0/X_t)}{(X_0/X_t)} &= (r_f - r_d + \sigma^2)dt - \sigma dB_t^Q \end{aligned}$$

Assuming $r_d = r_f = 0$,

$$\frac{d(X_0/X_t)}{(X_0/X_t)} = \sigma^2\,dt - \sigma\,dB_t^Q.$$

(d) The USD cash should be the numeraire and the price of USD in terms of the risk-neutral numeraire CNY is nothing but X_t! Therefore the volatility is σ and we get

$$dB_t^{\texttt{USD}} + \sigma dt = dB_t^{\texttt{CNY}} (= dB_t^Q).$$

The SDE from (c) becomes

$$\frac{d(X_0/X_t)}{(X_0/X_t)} = \sigma\,(\sigma\,dt - dB_t^{\texttt{CNY}}) = -\sigma dB_t^{\texttt{USD}}$$

and X_0/X_t is indeed a martingale under the USD numeraire.

3. **[2022FE, S_t Numeraire]** Derive the current price of a modified call option that pays

$$h(S_T) = \frac{S_T}{S_0}(S_T - K)^+$$

at maturity T. The stock price S_t follows the Black–Scholes assumption under the risk-neutral measure:

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dB_t.$$

Hint: Evaluate this option with the numeraire $N_t = S_t$.

Solution: This question is from this quant finance StackExchange post.

Using S_t as a numeraire, the price of the modified option is expressed using the corresponding equivalent martingale measure Q^S:

$$C_0 = S_0\, E^{Q^S}\left(\frac{h(S_T)}{S_T}\right) = E^{Q^S}\left((S_T - K)^+\right).$$

This is the same as the price of the regular call option except that we use the Q^S measure, instead of the risk neutral measure Q. Since BMs in the two measures are related by

$$B_t^{Q^S} + \sigma t = B_t^Q,$$

the terminal price S_T is express in terms of $B_t^{Q^S}$:

$$\begin{aligned}
S_T &= S_0 e^{rT} \exp\left(\sigma B_T^Q - \frac{\sigma^2 T}{2}\right) = S_0 e^{rT} \exp\left(\sigma B_T^{Q^S} + \frac{\sigma^2 T}{2}\right)\\
&= S_0 e^{(r+\sigma^2)T} \exp\left(\sigma B_T^{Q^S} - \frac{\sigma^2 T}{2}\right).
\end{aligned}$$

Therefore, the call price is modified from the BS formula for undiscounted call option by replacing $S_0 e^{rT}$ with $S_0 e^{(r+\sigma^2)T}$:

$$C_0 = S_0 e^{(r+\sigma^2)T} N(d_1') - K N(d_2'),$$

where

$$\begin{aligned}
d_1' &= \frac{\log(S_0 e^{(r+\sigma^2)T}/K)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} = \frac{\log(S_0 e^{rT}/K)}{\sigma\sqrt{T}} + \frac{3\sigma\sqrt{T}}{2} = d_1 + \sigma\sqrt{T}\\
d_2' &= \frac{\log(S_0 e^{(r+\sigma^2)T}/K)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2} = \frac{\log(S_0 e^{rT}/K)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} = d_2 + \sigma\sqrt{T} = d_1.
\end{aligned}$$

Here d_1 and d_2 are from the regular BS formula.

4. **[2017FE, Interest rate and bond price SDE]**

   (a) Find the mean and variance of the following expression:

$$I = \int_0^T B_t\, dt.$$

   You may find that $TB_T = \int_0^T T dB_t$ is a helpful trick. Is I normally distribution?

(b) Suppose now you live in a real world where the risk-free rate, r_t, is stochastic rather than constant. Therefore the saving account β_T exponentially grows as $\beta_T = \exp\left(\int_0^T r_t dt\right)$. Show that, in risk-neutral measure Q, the zero-coupon bond price $P(t,T)$ is given by

$$P(0,T) = E^Q\left[\exp\left(-\int_0^T r_t dt\right)\right]$$

(c) Suppose that the SDE for r_t is given by

$$r_t = r_0 + \alpha t + \beta\, B_t,$$

where B_t is a standard BM under Q ($B_t = B_t^Q$). Calculate $P(0,T)$ using the results from (a) and (b).

(d) Suppose that the SDE for the bond price $P(t,T)$ in Q is given by

$$\frac{d\,P(t,T)}{P(t,T)} = \mu_P dt + \sigma_P\, dB_t.$$

From the result of (c), find $P(t,T)$ by replacing r_0 with r_t and T with $T-t$. Then, apply Itô's lemma to $\log P(t,T)$. By matching drift and volatility, find the expressions for μ_P and σ_P. (Hint: The answers are quite simple, so check your calculation again if the expressions are complicated. Especially μ_P is almost obvious in the risk-neutral measure. Even if you cannot get the derivation right, you still earn partial credit by correctly guessing μ_P. When applying Itô's lemma to $\log P(t,T)$, t is the time variable and T should be considered as a constant.)

Solution:

(a) Applying Itô's lemma to tB_t (see SCFA 6.2), $d(tB_t) = tdB_t + B_t dt$, we can express I as

$$I = TB_T - \int_0^T tdB_t = T\int_0^t dB_t - \int_0^T tdB_t = \int_0^T (T-t)dB_t.$$

We get

$$E(I) = 0 \quad \text{and} \quad \text{Var}(I) = \int_0^T (T-t)^2 dt = \frac{1}{3}T^3.$$

and I is normally distributed.

(b) The bond price $P(0,T)$ is the $t=0$ value of \$ 1 at $t=T$. Under the risk-neutral measure Q, any security value measured in the unit of β_t is a martingale,

$$\frac{P(0,T)}{\beta_0 = 1} = E_Q\left[\frac{P(T,T)=1}{\beta_T}\right].$$

Therefore,

$$P(0,T) = E_Q\left[\exp\left(-\int_0^T r_t dt\right)\right]$$

or, more generally,

$$P(t,T) = E_Q\left[\exp\left(-\int_t^T r_s dt\right)\Big|\mathcal{F}_t\right]$$

(c) From

$$-\int_0^T r_t dt = -Tr_0 - \frac{1}{2}\alpha T^2 - \beta I,$$

we get

$$\begin{aligned} P(0,T) &= \exp\left(-Tr_0 - \frac{1}{2}\alpha T^2\right) E_Q(e^{-\beta I}) \\ &= \exp\left(-Tr_0 - \frac{1}{2}\alpha T^2 + \frac{1}{6}\beta^2 T^3\right) \end{aligned}$$

Note that this is a simplified version of the Ho–Lee model (WIKIPEDIA). In the original model, the drift α is time-dependent:

$$dr_t = \alpha_t\, dt + \sigma dB_t.$$

(d) From

$$\log P(t,T) = -(T-t)r_t - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\beta^2(T-t)^3$$

we get the SDE for $\log P(t,T)$:

$$\begin{aligned} d\log P(t,T) &= -(T-t)dr_t + \left(r_t + \alpha(T-t) - \frac{1}{2}\beta^2(T-t)^2\right) dt \\ &= \left(r_t - \frac{1}{2}\beta^2(T-t)^2\right) dt - \beta(T-t)dB_t. \end{aligned}$$

and this should be equal to

$$d\log P(t,T) = (\mu_P - \sigma_P^2/2)dt + \sigma_P dB_t.$$

Therefore we get

$$\mu_P = r_t \quad \text{and} \quad \sigma_P = -\beta(T-t).$$

Note that (i) $\mu_P = r_t$ is obvious in the risk-neutral measure, (ii) the bond volatility σ_P is negative because the bond price moves in an opposite way to the interest rate, and (iii) $\sigma_P = 0$ at maturity $t = T$ because the bond price $P(t,T)$ converges to 1.

5. **[2018FE, T-forward measure]** In class, we learned that the forward price of buying a bond maturing at $T+\Delta$ at time $t=T$,

$$F_t = \frac{B(t,T+\Delta)}{B(t,T)},$$

is a martingale under the T-forward measure. We are going to prove this under an explicit setting. From 2017FE, Interest rate and bond price SDE, we also know that

$$\frac{d\,B(t,T)}{B(t,T)} = r_t dt - \beta(T-t)\, dB_t^Q$$

when the risk-free rate changes according to $dr_t = \alpha dt + \beta\, dB_t^Q$. Here, B_t^Q is the standard BM under the risk-neutral measure.

(a) Derive the SDE for F_t. (Hint. First compute $d\log F_t$ and compute dF_t/F_t)

(b) If B_t^T is the standard BM under the T-forward measure, what is the relation between dB_t^Q and dB_t^T?

(c) From (a) and (b), finally derive the SDE for F_t under the T-forward measure. Is F_t a martingale? What is the volatility of dF_t/F_t?

Solution: Assume $\sigma = -\beta(T-t)$ and $\sigma' = -\beta(T+\Delta-t)$ so that

$$\frac{d\,B(t,T)}{B(t,T)} = r_t dt + \sigma\, dB_t^Q, \quad \frac{d\,B(t,T+\Delta)}{B(t,T+\Delta)} = r_t dt + \sigma'\, dB_t^Q$$

(a) From,

$$d\log B(t,T) = (r_t - \sigma^2/2)dt + \sigma\, dB_t^Q, \quad d\log B(t,T+\Delta) = (r_t - \sigma'^2/2)dt + \sigma'\, dB_t^Q,$$

we get

$$\begin{aligned}
d\log F_t &= -\frac{1}{2}(\sigma'^2 - \sigma^2)dt + (\sigma' - \sigma)dB_t^Q, \\
\frac{dF_t}{F_t} &= -\frac{1}{2}(\sigma'^2 - \sigma^2)dt + \frac{1}{2}(\sigma' - \sigma)^2 dt + (\sigma' - \sigma)dB_t^Q \\
&= -\sigma(\sigma' - \sigma)dt + (\sigma' - \sigma)dB_t^Q \\
&= -\Delta\beta^2(T-t)dt - \Delta\beta dB_t^Q
\end{aligned}$$

(b) Reminded that the BMs in the Q^N (equivalent martingale measure for N_t) and Q (risk-neutral measure) are related by

$$B_t^{Q^N} + \sigma_N t = B_t^Q,$$

where σ_N is the volatility of N_t. Because the volatility of the numeraire $B(t,T)$ is $\sigma = -\beta(T-t)$,

$$B_t^T - \beta(T-t)t = B_t^Q.$$

(c) Combining (a) and (b), we obtain

$$\frac{dF_t}{F_t} = -\Delta\beta dB_t^T.$$

Therefore, F_t is indeed a martingale under the T-forward measure, and the volatility is $\sigma' - \sigma = -\Delta\beta$.

6. **[2020FE, Interest rate caplet pricing]** We are going to price *caplet*, that is an option on the deposit rate L between T and $T+\Delta$. As in the class, let $L(t,T)$ be the forward rate of making a deposit from T to $T+\Delta$ observed at t. If you hold a caplet with strike price K with maturity T, you have a right (not an obligation) against a bank to make a deposit of 1 at the deposit rate K for the period Δ. Of course, you will exercise this option only when K is higher than the current deposit rate $L(T,T)$ at $t=T$ offered by the bank. So

your payout of the caplet is

$$\text{Payout} = \Delta(K - L(T,T))^+.$$

However, caplet is different from equity options is that, although the caplet expiry is at $t = T$, the payout is paid at $t = T + \Delta$ when the deposit matures.

Suppose that the SDE for r_t is given by the Ho–Lee model:

$$r_t = r_0 + \alpha t + \beta\, B_t^Q \quad (dr_t = \alpha dt + \beta\, dB_t^Q),$$

where B_t^Q is a standard BM under the risk-neutral measure. Follow the next steps to price the caplet under this model. You may use the results of 2017FE, Interest rate and bond price SDE and 2018FE, T-forward measure.

(a) From no-arbitrage replication, we know that

$$L(t,T) = \frac{P(t,T) - P(t,T+\Delta)}{\Delta \cdot P(t,T+\Delta)},$$

where $P(t,T)$ is the time t price of the zero-coupon bond maturing at T. Derive the SDE for $L(t,T)$ under the risk-neutral measure (i.e., using B_t^Q).

(b) What is the SDE for $L(t,T)$ under the $(T+\Delta)$-forward measure (i.e., with $B_t^{T+\Delta}$)?

(c) Using the $(T+\Delta)$-forward measure, derive present value of the at-the-money caplet? The at-the-money strike is $K = L(0,T)$.

Solution:

(a) Let

$$H_t = 1 + \Delta L(t,T) = \frac{P(t,T)}{P(t,T+\Delta)}$$

and we first derive the SDE of H_t. From 2018FE, T-forward measure,

$$d\log P(t,T) = (r_t - \sigma^2/2)dt + \sigma\, dB_t^Q \quad (\sigma = -\beta(T-t)),$$
$$d\log P(t,T+\Delta) = (r_t - \sigma'^2/2)dt + \sigma'\, dB_t^Q \quad (\sigma' = -\beta(T+\Delta-t)),$$

we get

$$\begin{aligned}
d\log H_t &= -\frac{1}{2}(\sigma^2 - \sigma'^2)dt + (\sigma - \sigma')dB_t^Q,\\
\frac{dH_t}{H_t} &= -\frac{1}{2}(\sigma^2 - \sigma'^2)dt + \frac{1}{2}(\sigma - \sigma')^2 dt + (\sigma - \sigma')dB_t^Q\\
&= -\sigma'(\sigma - \sigma')dt + (\sigma - \sigma')dB_t^Q\\
&= \Delta\beta^2(T+\Delta-t)dt + \Delta\beta dB_t^Q
\end{aligned}$$

Therefore,

$$\begin{aligned}
dL(t,T) &= \frac{1+\Delta L(t,T)}{\Delta}(\Delta\beta^2(T+\Delta-t)dt + \Delta\beta dB_t^Q)\\
&= (1+\Delta L(t,T))(\beta^2(T+\Delta-t)dt + \beta dB_t^Q)
\end{aligned}$$

(b) The numeraire $P(t, T+\Delta)$ has the volatility $\sigma' = -\beta(T+\Delta-t)$. The BM under the $(T+\Delta)$-forward measure and the BM under the risk-neutral measure are related by

$$dB_t^Q = dB_t^{T+\Delta} - \beta(T+\Delta-t)dt.$$

Therefore, the SDE for $L(t,T)$ under the $(T+\Delta)$-forward measure is

$$dL(t,T) = \beta(1+\Delta L(t,T))\, dB_t^{T+\Delta}.$$

Note that $L(t,T)$ is martingale under the $(T+\Delta)$-forward measure and this is already expected because of $P(t, T+\Delta)$ in the denominator of the equation for $L(t,T)$.

(c) We use H_t for the caplet pricing because H_t follows a geometric BM with volatility $\Delta\beta$:

$$\frac{dH_t}{H_t} = \Delta\beta\, dB_t^{T+\Delta}.$$

$$\begin{aligned} C &= P(0, T+\Delta)E^{T+\Delta}\left(\Delta \cdot (L(0,T) - L(T,T))^+\right) \\ &= P(0, T+\Delta)E^{T+\Delta}\left((1+\Delta L(0,T) - H_T)^+\right) \end{aligned}$$

The bond price $P(0, T+\Delta)$ is given from 2017FE, Interest rate and bond price SDE (c):

$$P(0, T+\Delta) = \exp\left(-(T+\Delta)r_0 - \frac{1}{2}\alpha(T+\Delta)^2 + \frac{1}{6}\beta^2(T+\Delta)^3\right).$$

The expectation part is obtained by the BS formula for the put option with strike and spot $(1+\Delta L(0,T))$:

$$\begin{aligned} E^{T+\Delta}\left((1+\Delta L(0,T) - H_T)^+\right) &= (1+\Delta L(0,T))N(d_1) - (1+\Delta L(0,T))N(-d_1) \\ &= (1+\Delta L(0,T))(2N(d_1) - 1) \end{aligned}$$

where

$$d_1 = \frac{\Delta\beta\sqrt{T}}{2}.$$

Finally, the caplet price is given by

$$C = \exp\left(-(T+\Delta)r_0 - \frac{1}{2}\alpha(T+\Delta)^2 + \frac{1}{6}\beta^2(T+\Delta)^3\right)(1+\Delta L(0,T))(2N(d_1)-1)$$

7. **[2021FE, Delayed payment]** In a **cash-settled** forward contact, you receive or pay (i.e., settle) the value of the contact in cash, instead of buying the underlying asset. If you hold a long forward contact at price F, for example, you receive (or pay) $S_T - F$ in cash at the maturity $t = T$. The cash-settled forward contact is usually equivalent to the regular forward contact. The present value of the forward contract is

$$P(0,T)\, E^T\left(\frac{S_T - F}{P(T,T) = 1}\right),$$

where $E^T(\cdot)$ is the expectation under the T-forward measure and $P(t,T)$ is the time t price of the zero coupon bond maturing at T. Therefore, the fair forward price F (that makes the present value zero) observed at $t = 0$ is

$$F = E^T(S_T) = S_0/P(0,T).$$

Now, consider a new rule: Although, the payout, $S_T - F$, is determined at $t = T$, the cash settlement happens (i.e, you receive or pay) at a **delayed time** $t = T + \Delta$.

(a) What is the fair forward price F' under this new rule? Express the answer using $E^T(\cdot)$, $P(t,T)$, and S_T.

(b) How does this new forward price F' compare (e.g., higher or lower) to the original price F? Make your answer based on the three scenarios: the correlation between the asset price S_T and the interest rate is positive, zero, and negative.

Solution:

(a) The present value of the new forward contract is

$$P(0,T)\,E^T\left(P(T,T+\Delta)(S_T - F')\right),$$

Since the bond price $P(0,T+\Delta)$ is expressed by

$$P(0,T+\Delta) = P(0,T)E^T(P(T,T+\Delta)),$$

Therefore, the forward price with delayed settlement is give by

$$F' = \frac{E^T(P(T,T+\Delta)S_T)}{E^T(P(T,T+\Delta))} = \frac{P(0,T)}{P(0,T+\Delta)}E^T(P(T,T+\Delta)S_T).$$

(b) Since the future bond price and the interest rate are inversely related as $P(T,T+\Delta) \approx e^{-r_T\Delta}$,

$$E^T(P(T,T+\Delta)\,S_T)\begin{cases} < E^T(P(T,T+\Delta))E^T(S_T) & \text{for positive correlation} \\ = E^T(P(T,T+\Delta))E^T(S_T) & \text{for zero correlation} \\ > E^T(P(T,T+\Delta))E^T(S_T) & \text{for negative correlation} \end{cases}.$$

Therefore,

1. When the asset price S_T and the interest rate have zero correlation,

$$F' = \frac{P(0,T)}{P(0,T+\Delta)}E^T(P(T,T+\Delta))E^T(S_T) = \frac{S_0}{P(0,T)} = F.$$

2. If they have positive correlation,

$$F' < \frac{P(0,T)}{P(0,T+\Delta)}E^T(P(T,T+\Delta))E^T(S_T) = F.$$

3. If they have negative correlation,

$$F' > \frac{P(0,T)}{P(0,T+\Delta)}E^T(P(T,T+\Delta))E^T(S_T) = F.$$