1 Model of Trading

1.1 Holdings and trades

- $p_t \in \mathbb{R}^n$ `reference prices` at the beginning of period t
- $r_t \in \mathbb{R}^{n+1}$ `vector of asset and cash returns from period t to period t+1`
- $h_t \in \mathbb{R}^{n+1}$ usd `holding values` of n assets + `cash account` at the beginning of period t
- $u_t \in \mathbb{R}^{n+1}$ usd `trading values` (at reference prices).
  - ⋆ Assume: trades are executed at the beginning of period t
  - ⋆ $(u_t)_{n+1}$ amount we put into cash account
- `NAV`(value): $v_t = \mathbf{1}^T h_t$. Assume $v_t > 0$
- `holding weights` (normalized holdings, fraction of NAV): $w_t = \frac{h_t}{v_t}$
- `normalized trades` (fraction of NAV): $z_t = \frac{u_t}{v_t}$
- `GMV`(gross exposure): $\|(h_t)_{1:n}\|_1$; `Leverage`: $\frac{GMV}{NAV} = \frac{\|(h_t)_{1:n}\|_1}{\mathbf{1}^T h_t} = \|(w_t)_{1:n}\|_1$
- `Turnover` (in usd): $\|(u_t)_{1:n}\|_1$
- `post-trade portfolio`: $h_t^+ = h_t + u_t$
- `post-trade value`: $v_t^+ = \mathbf{1}^T h_t^+$
- `normalized post-trade portfolio`[1]: $\frac{h_t^+}{v_t} = w_t + z_t$

1.2 Costs

- `trading (transaction) cost` (in usd): $\phi_t^{trade}(u_t) : \mathbb{R}^{n+1} \to \mathbb{R}$
  - ⋆ convex[2], but can be negative
  - ⋆ does not depend on $(u_t)_{n+1}$
  - ⋆ $\phi_t^{trade}(0) = 0$, i.e. "no trade $\Rightarrow$ no cost"
  - ⋆ separable into costs by individual assets $\sum_{i=1}^n (\phi_t^{trade})_i((u_t)_i)$
  - ⋆ example: $x \longrightarrow a|x| + bx + c\sigma \frac{|x|^{3/2}}{V^{1/2}}$
- `normalized trading (transaction) cost`: $\phi_t^{trade}(u_t)/v_t$
  - ⋆ same formula, just renormalize coefficients[3], e.g. $z \longrightarrow a|z| + bz + c\sigma \frac{|z|^{3/2}}{(V/v)^{3/2}}$
  - ⋆ with abuse of notation, denote $\phi_t^{trade.n}(z)$
- `holding cost` (in usd): $\phi_t^{hold}(h_t^+) : \mathbb{R}^{n+1} \to \mathbb{R}$[4]
  - ⋆ convex, but can be negative
  - ⋆ does not depend on $(h_t^+)_{n+1}$

[1] does not sum up to one
[2] example of non-convex transaction cost is constant for any non-zero trade
[3] only terms that don't scale linearly
[4] reminder, we hold post-trade portfolio h_t^+ over the period t

- $\star$ example: $x \longrightarrow s_{borrow}^T(h_t^+)_-$

- `normalized holding cost` $\phi_t^{hold}(h_t^+)/v_t$
  - $\star$ same formula
  - $\star$ with abuse of notation[5], denote $\phi_t^{hold.n}(w_t + z_t)$

1.3 Self-financing condition

Assumption: no cash is put into or taken out
Assumption: trading/holding costs are paid from cash account at the beginning of the period
The balance equation is:

$$\underbrace{\phi_t^{trade}(u_t) + \phi^{hold}(h_t^+)}_{\text{cash cost incurred}} = \underbrace{-\mathbf{1}^T u_t}_{\substack{\text{cash out of the portfolio}\\ \text{from the trades}}} \tag{1}$$

It implies that "post-trade value" is "pre-trade value" minus "transaction and holding costs":

$$v_t^+ = v_t - \underbrace{(\phi_t^{trade}(u_t) + \phi^{hold}(h_t^+))}_{\text{cash cost incurred}} \tag{2}$$

It also gives the value of `cash change amount` based on non-cash asset trades:

$$(u_t)_{n+1} = -\left(\mathbf{1}^T(u_t)_{1:n} + \underbrace{\phi_t^{trade}((u_t)_{1:n}) + \phi^{hold}((h_t + u_t)_{1:n})}_{\text{cost incurred from non-cash asset trades}}\right) \tag{3}$$

To get normalized versions, we divide by v_t and use $h_t = w_t v_t$ and $u_t = z_t v_t$:

$$\underbrace{\phi_t^{trade.n}(z_t) + \phi^{hold.n}(w_t + z_t)}_{\text{cash cost incurred in units of NAV}} = \underbrace{-\mathbf{1}^T z_t}_{\substack{\text{cash out of the portfolio}\\ \text{from the trades}}} \tag{4}$$

1.4 Investment

Assumption: post-trade portfolio and cash are invested for one period (until the beginning of the next time period)

- `next period portfolio`[6]: $h_{t+1} = h_t^+ + r_t \circ h_t^+ = (\mathbf{1} + r_t) \circ h_t^+ = (\mathbf{1} + r_t) \circ (h_t + u_t)$

- `next period portfolio value`:

$$\begin{aligned}\underbrace{v_{t+1}}_{\text{new value}} &= \mathbf{1}^T h_{t+1} = (\mathbf{1} + r_t)^T h_t^+ = \underbrace{v_t}_{\text{old value}} + \underbrace{r_t^T h_t}_{\text{hold pnl}} + \underbrace{(\mathbf{1} + r_t)^T u_t}_{\text{trade pnl}} \\ &= \underbrace{v_t}_{\text{old value}} + \underbrace{r_t^T h_t}_{\text{hold pnl}} + \underbrace{r_t^T u_t}_{\text{precost trade pnl}} - \underbrace{(\phi_t^{trade}(u_t) + \phi^{hold}(h_t^+))}_{\text{cash cost incurred}}\end{aligned}$$

- `Portfolio return` (fractional increase in portfolio value):

$$R_t^p = \frac{v_{t+1} - v_t}{v_t} = \underbrace{r_t^T w_t}_{\text{return on holdings}} + \underbrace{r_t^T z_t}_{\text{return on trades}} - \underbrace{(\phi_t^{trade.n}(z_t) + \phi^{hold.n}(w_t + z_t))}_{\text{cost in units of NAV}} \tag{5}$$

- `next period weights`: $w_{t+1} = \frac{h_{t+1}}{v_{t+1}} = \frac{(\mathbf{1}+r_t)\circ h_t^+}{v_t(1+R_t^p)} = \frac{1}{1+R_t^p}(\mathbf{1} + r_t) \circ (w_t + z_t)$

[5] for the example above, there is no abuse of notation since it only include order-1 terms
[6] "dynamics equation"; in case of non-instant trading it turns into $h_{t+1} = h_t \circ (\mathbf{1} + r_t) + u_t \circ (\mathbf{1} + r_t^{\text{avg.exec to close}})$