Notes on a Stationary Linear Stochastic Process

1 Setup

We consider a discrete-time stochastic process

$$x_n = \lambda x_{n-1} + \xi_n, \qquad |\lambda| < 1, \tag{1}$$

where

$$\xi_n \sim \mathcal{N}(0, \sigma_\xi^2)$$

are i.i.d. Gaussian shocks.

The process is assumed to be in its stationary regime.

2 Basic Moments

The stationary variance of x_n satisfies

$$\mathrm{Var}(x_n) = \lambda^2 \mathrm{Var}(x_{n-1}) + \sigma_\xi^2 = \frac{\sigma_\xi^2}{1-\lambda^2}. \tag{2}$$

Hence,

$$\mathbb{E}[x_n^2] = \frac{\sigma_\xi^2}{1-\lambda^2}. \tag{3}$$

3 Difference Process

Consider the increment

$$x_n - x_{n-1} = (\lambda - 1)x_{n-1} + \xi_n.$$

Its variance is

$$\mathrm{Var}(x_n - x_{n-1}) = (\lambda-1)^2 \mathrm{Var}(x_{n-1}) + \sigma_\xi^2 \tag{4}$$

$$= \sigma_\xi^2 \left(1 + \frac{(\lambda-1)^2}{1-\lambda^2}\right). \tag{5}$$

Simplifying,

$$\mathrm{Var}(x_n - x_{n-1}) = \sigma_\xi^2 \frac{2}{1+\lambda}. \tag{6}$$

Thus,

$$\mathbb{E}[(x_n - x_{n-1})^2] = \sigma_\xi^2 \frac{2}{1+\lambda}. \tag{7}$$

4 Absolute Value Expectations

Since $x_n - x_{n-1}$ is Gaussian with zero mean,

$$x_n - x_{n-1} \sim \mathcal{N}\left(0,\, \sigma_\xi^2 \frac{2}{1+\lambda}\right),$$

we use the identity

$$\mathbb{E}|Z| = \sqrt{\frac{2}{\pi}}\,\sigma, \quad Z \sim \mathcal{N}(0, \sigma^2).$$

Therefore,

$$\mathbb{E}|x_n - x_{n-1}| = \sqrt{\frac{2}{\pi}}\sqrt{\sigma_\xi^2 \frac{2}{1+\lambda}} = \frac{2}{\sqrt{\pi}} \frac{\sigma_\xi}{\sqrt{1+\lambda}}. \tag{8}$$

5 Quadratic Objective (Illustrative)

Consider the expression

$$\mathbb{E}\left[x_n \mu_n - S|x_n - x_{n-1}| - R x_n^2\right]. \tag{9}$$

Using the results above:

$$\mathbb{E}[x_n^2] = \frac{\sigma_\xi^2}{1-\lambda^2}, \tag{10}$$

$$\mathbb{E}|x_n - x_{n-1}| = \frac{2}{\sqrt{\pi}} \frac{\sigma_\xi}{\sqrt{1+\lambda}}. \tag{11}$$

Hence the objective can be written explicitly in terms of $\lambda, \sigma_\xi^2, S, R$.

6 Parameter Relation

In the notes, the parameter λ is defined as

$$\lambda = \frac{I}{I+R}. \tag{12}$$

This ensures $0 < \lambda < 1$ for $I, R > 0$.