1. Solve the system of equations:

$$\begin{cases} x + iy - 2z = 10, \\ x - y + 2iz = 20, \\ ix + 3iy - (1+i)z = 30. \end{cases}$$

(1)

$$\begin{cases} x + iy - 2z = 10 \\ x - y + 2iz = 20 \\ ix + 3iy - (1+i)z = 30 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & i & -2 & 10 \\ 1 & -1 & 2i & 20 \\ i & 3i & -(1+i) & 30 \end{array}\right) \quad - \quad -i$$

$$\sim$$

$$\left(\begin{array}{ccc|c} 1 & i & -2 & 10 \\ 0 & -1-i & 2+2i & 10 \\ 0 & 1+3i & -1+i & 30-10i \end{array}\right)$$

$$(1+3i) + \lambda(-1-i) = 0$$

$$\lambda = \frac{-1-3i}{-1-i} = (-1-3i)\frac{(-1+i)}{2} = \frac{1-i+3i+3}{2} = \frac{4+2i}{2} = 2+i$$

$$\sim$$

$$\left(\begin{array}{ccc|c} 1 & i & -2 & 10 \\ 0 & -1-i & 2+2i & 10 \\ 0 & 0 & 1+7i & 50 \end{array}\right) \quad |:(-1-i)$$

$$\Rightarrow z = \frac{50}{1+7i} = 1-7i$$

$$\left(\begin{array}{ccc|c} 1 & i & -2 & 10 \\ 0 & 1 & 0 & -3-8i \\ 0 & 0 & 1 & 1-7i \end{array}\right) \quad -i$$

2

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3-11i \\ 0 & 1 & 0 & -3-8i \\ 0 & 0 & 1 & 1-7i \end{array}\right)$$

$$\boxed{\begin{cases} x = 3 - 11i \\ y = -3 - 8i \\ z = 1 - 7i \end{cases}}$$

answer

2. Solve the equation:

$$(x^4 + x^3 - (2-2i)x - (2-2i))(x^4 - 3x^3 + 5x^2 - 5x + 2) = 0.$$

$$\begin{cases} x^4 + x^3 - (2-2i)x - (2-2i) = 0 & (1) \\ x^4 - 3x^3 + 5x^2 - 5x + 2 = 0 & (2) \end{cases}$$

(1)

$$x^4 + x^3 - (2-2i)x - (2-2i) = 0$$

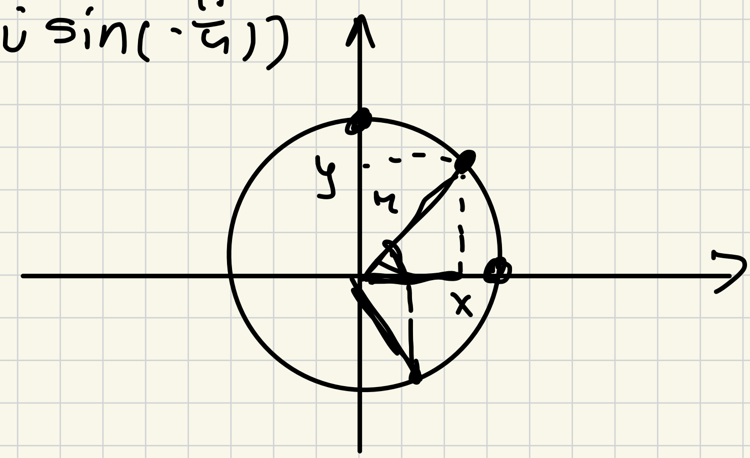
$$(x^4 + x^3) - (2-2i)(x+1) = (x+1)(x^3 - (2-2i))$$

$$\begin{cases} x = -1 \\ x^3 = 2-2i \end{cases}$$

$$x^3 = 2-2i \Rightarrow x = \sqrt[3]{2-2i}$$

$$\begin{cases} r = 2\sqrt{2} \\ \varphi = \arccos\left(\frac{2}{2\sqrt{2}}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4} \end{cases}$$

$$2-2i = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$



$$\sqrt[3]{2\sqrt{2}}\left(\cos\left(\frac{-\frac{\pi}{4}+2k\pi}{3}\right)+i\sin\left(\frac{-\frac{\pi}{4}+2k\pi}{3}\right)\right), k=0,1,2$$

$$k=0 \Rightarrow \varphi=-\frac{\pi}{12}$$

$$x_0=\sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)$$

$$\cos\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\sin\left(-\frac{\pi}{12}\right)=-\sin\frac{\pi}{12}=-\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$x_0=\sqrt{2}\left(\frac{\sqrt{6}+\sqrt{2}}{4}-i\,\frac{\sqrt{6}-\sqrt{2}}{4}\right)=\frac{\sqrt{3}+1}{2}+\frac{1-\sqrt{3}}{2}i$$

$$k=1 \Rightarrow \varphi=\frac{7\pi}{12}$$

$$x_1=\sqrt{2}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$$

$$\cos\left(\frac{7\pi}{12}\right)=-\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\sin\left(\frac{7\pi}{2}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$x_1=\sqrt{2}\left(-\frac{\sqrt{6}-\sqrt{2}}{4}+i\,\frac{\sqrt{6}+\sqrt{4}}{4}\right)$$

$$\sqrt{2}\cdot\left(-\frac{\sqrt{6}-\sqrt{2}}{4}\right)=\frac{2\sqrt{3}-2}{4}=\frac{1-\sqrt{3}}{2}$$

$$\sqrt{2}\,\frac{\sqrt{6}+\sqrt{2}}{4}=\frac{2\sqrt{3}+2}{4}=\frac{\sqrt{3}+1}{2}$$

$$x_1=\frac{1-\sqrt{3}}{2}+i\,\frac{1+\sqrt{3}}{2}$$

$k=2$

$\varphi = \frac{5\pi}{4}$

$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}, \quad \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

$x_2 = \sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -1 - i$

For First part

$\text{roots} = \left\{-1, \frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i, \frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i, -1-i\right\}$

2. Second Factor

$x^4 - 3x^3 + 5x^2 - 5x + 2 = 0$

$x = 1$

$x^4 - 3x^3 + 5x^2 - 5x + 2 = (x-1)(x^3 - 2x^2 + 3x - 2)$

	1	-3	5	-5	2
1	1	-2	3	-2	0

	1	-2	3	-2
1	1	-1	2	0

$$x^4 - 3x^3 + 5x^2 - 5x + 2 = (x-1)^2(x^2 - x + 2)$$

$$x^2 - x + 2 = 0$$

$$\mathcal{D} = 1 - 8 = -7$$

$$x = \frac{1 \pm \sqrt{-7}}{2}$$

$$\sqrt{-1 \cdot 7} = \sqrt{-1} \cdot \sqrt{7} = i\sqrt{7}$$

$$\Downarrow$$

$$x = \frac{1 \pm i\sqrt{7}}{2}$$

So

Solutions

$$x \in \left\{ -1, -1-i, \frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2} i, \frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2} i, 1, \frac{1+i\sqrt{7}}{2}, \frac{1-i\sqrt{7}}{2} \right\}$$

3. Let $\mathcal{A}$ be a linear operator on $\mathbb{C}_3[x]$, s. t. $\mathcal{A}(1) = x$, $\mathcal{A}(x) = x^2$, $\mathcal{A}(x^2) = x^3$, $\mathcal{A}(x^3) = 1$.

a) Find the matrix of $\mathcal{A}$ in the basis $(x^3, x^2, x, 1)$ and calculate $\mathcal{A}((3+i)x^2 - (3-i)x + 12i)$.

b) Prove that $(1, x-1, (x-1)^2, (x-1)^3)$ is a basis of $\mathbb{C}_3[x]$ and find the matrix of $\mathcal{A}$ with respect to this basis.

$\mathcal{A}(1) = x$

$\mathcal{A}(x) = x^2$

$\mathcal{A}(x^2) = x^3$

$\mathcal{A}(x^3) = 1$

$S = (1, x, x^2, x^3) \longrightarrow (x, x^2, x^3, 1)$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x^3 \\ x^2 \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ x^3 \\ x^2 \\ x \end{pmatrix} = [\mathcal{A}]_S$$

$$\mathcal{A}((3+i)x^2 - (3-i)x + 12i) = A \begin{pmatrix} 12i \\ -(3-i) \\ (3+i) \\ 0 \end{pmatrix} =$$

$$= (3+i)x^3 - (3-i)x^2 + 12xi$$

b) $B = (1, x-1, (x-1)^2, (x-1)^3)$

$(x-1)^2 = x^2 - 2x + 1$

$(x-1)^3 = (x-1)(x-1)^2 = x^3 - 3x^2 + 3x - 1$

in standard basis $S = (1, x, x^2, x^3)$

$$C = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\det C = 1 \neq 0 \Rightarrow$ linear indep $\Rightarrow$

$\Rightarrow$ basis

We know that matrix of d in basis B

$$[d]_B = C^{-1} [d]_S \, C$$

$$C^{-1} = \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

2

$$
\begin{array}{l}
+3 \\
+3
\end{array}
\left(\begin{array}{cccc|cccc}
1 & 0 & 0 & -1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & -3 & 0 & 1 & 2 & 0 \\
0 & 0 & 1 & -3 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
\begin{array}{l}
+ \\
+
\end{array}
$$

2

$$
\left(\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
$$

$$
C^{-1} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

$$C^{-1}[A]_S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$C^{-1}[A]_S C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -4 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 1 & -3 \end{pmatrix}}}$$