

Analysis of Lorenz Equations

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1 Stability Analysis of Lorenz Equations

1.1 Introduction

The dynamics of a Lorenz system is described by the following differential equations,

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z\end{aligned}\tag{1}$$

where $\sigma, \rho, \beta \in \mathbb{R}$

Rewriting the above equations, for later operations,

$$\begin{aligned}\dot{x} &= \sigma(y - x) = f(x, y, z) \\ \dot{y} &= x(\rho - z) - y = g(x, y, z) \\ \dot{z} &= xy - \beta z = h(x, y, z)\end{aligned}\tag{2}$$

where $\sigma, \rho, \beta \in \mathbb{R}$

1.2 Fixed points of Lorenz equations

To find the fixed points for (2) we set, $\dot{x} = \dot{y} = \dot{z} = 0$ to arrive at the following equations,

$$\begin{aligned}\sigma(y - x) &= f(x_0, y_0, z_0) = 0 \\ x(\rho - z) - y &= g(x_0, y_0, z_0) = 0 \\ xy - \beta z &= h(x_0, y_0, z_0) = 0\end{aligned}\tag{3}$$

where (x_0, y_0, z_0) represents fixed points. Above equations simplify to the following,

$$\begin{aligned}x_0 &= y_0 \\ x_0(\rho - z_0) - y_0 &= 0 \\ x_0 y_0 - \beta z_0 &= 0\end{aligned}$$

These further simplify to,

$$\begin{aligned}x_0(\rho - z_0 - y_0) &= 0 \\ x_0^2 &= \beta z_0\end{aligned}\tag{4}$$

From the first equation of (4) we arrive at two conditions

$$x_0 = 0 \text{ and } z_0 = \rho - 1$$

which implies that $y_0 = z_0 = 0$. Thus, the point $P_1 \equiv (0, 0, 0)$ is a fixed point.

From the condition, $z_0 = \rho - 1$ and second equation of (4), we get,

$$x_0 = y_0 = \pm\sqrt{\beta(\rho - 1)}$$

Following above argument, the other two fixed points are

$$P_2 \equiv (\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1) \text{ and } P_3 \equiv (-\sqrt{\beta(\rho - 1)}, -\sqrt{\beta(\rho - 1)}, \rho - 1)$$

Finally, we arrive at three fixed points for the system defined by (2)

$$\begin{aligned}P_1 &\equiv (0, 0, 0) \\ P_2 &\equiv (\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1) \\ P_3 &\equiv (-\sqrt{\beta(\rho - 1)}, -\sqrt{\beta(\rho - 1)}, \rho - 1)\end{aligned}\tag{5}$$

1.3 Jacobian and Eigenvalue equation

Let $\delta x = x - x_0$, $\delta y = y - y_0$, $\delta z = z - z_0$ be small perturbations around the fixed points represented by (x_0, y_0, z_0) . Taylor expanding equations defined by (2) about fixed points we get,

$$\begin{aligned}\delta \dot{x} &= f(x_0, y_0, z_0) + f'_x \delta x + f'_y \delta y + f'_z \delta z + \dots \\ \delta \dot{y} &= g(x_0, y_0, z_0) + g'_x \delta x + g'_y \delta y + g'_z \delta z + \dots \\ \delta \dot{z} &= h(x_0, y_0, z_0) + h'_x \delta x + h'_y \delta y + h'_z \delta z + \dots\end{aligned}$$

But from (3), $f(x_0, y_0, z_0) = g(x_0, y_0, z_0) = h(x_0, y_0, z_0) = 0$. Therefore, above equations reduce to the following linear approximations,

$$\begin{aligned}\delta \dot{x} &= f'_x \delta x + f'_y \delta y + f'_z \delta z \\ \delta \dot{y} &= g'_x \delta x + g'_y \delta y + g'_z \delta z \\ \delta \dot{z} &= h'_x \delta x + h'_y \delta y + h'_z \delta z\end{aligned}\tag{6}$$

Above equation can be written in matrix form as follows,

$$\frac{d}{dt} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} f'_x & f'_y & f'_z \\ g'_x & g'_y & g'_z \\ h'_x & h'_y & h'_z \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}\tag{7}$$

Equation (5) can be written more compactly as,

$$\frac{d(X)}{dt} = JX \quad \text{where } J = \begin{bmatrix} f'_x & f'_y & f'_z \\ g'_x & g'_y & g'_z \\ h'_x & h'_y & h'_z \end{bmatrix}\tag{8}$$

From equation (2) and (8), we arrive at the following Jacobian matrix,

$$J = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z_0 & -1 & -x_0 \\ y_0 & x_0 & -\beta \end{bmatrix}\tag{9}$$

where (x_0, y_0, z_0) is a fixed point.

To find the eigenvalues for the Jacobian, consider the equation,

$$JX = \lambda X \quad \Rightarrow \quad \text{Det}(J - \lambda I) = 0$$

where λ represents eigenvalue and I is a 3×3 identity matrix.

$$\therefore |J - \lambda I| = 0 = \begin{vmatrix} -\sigma - \lambda & \sigma & 0 \\ \rho - z_0 & -1 - \lambda & -x_0 \\ y_0 & x_0 & -\beta - \lambda \end{vmatrix} \quad (10)$$

$$\begin{aligned} (\sigma + \lambda)[(1 + \lambda)(\beta + \lambda) + x_0^2] + \sigma[x_0 y_0 - (\beta + \lambda)(\rho - z_0)] &= 0 \\ (\sigma + \lambda)(1 + \lambda)(\beta + \lambda) + (\sigma + \lambda)x_0^2 + \sigma x_0 y_0 - \sigma(\beta + \lambda)(\rho - z_0) &= 0 \\ (\beta + \lambda)[\lambda^2 + (1 + \sigma)\lambda - \sigma(\rho - z_0) + \sigma] + x_0^2 \lambda + \sigma(x_0^2 + x_0 y_0) &= 0 \end{aligned} \quad (11)$$

Now, we determine the eigenvalue equations at fixed points P_1, P_2 and P_3 (see (5)). Substituting the values from (5) in (11), we get,

at P_1 ,

$$(\beta + \lambda)[\lambda^2 + (1 + \sigma)\lambda - \sigma(\rho - 1)] = 0 \quad (12)$$

and at P_2 and P_3 ,

$$(\beta + \lambda)[\lambda^2 + (1 + \sigma)\lambda] + \beta(\rho - 1)\lambda + 2\sigma\beta(\rho - 1) = 0 \quad (13)$$

Equations (12) and (13) are eigenvalue equations or characteristic polynomials for the eigenvalues represented by λ .

1.4 Stability of fixed points

From previous discussion we have three fixed points (P_1, P_2 , and P_3) and two corresponding eigenvalue equations. First, we check the stability of P_1 .

Equation (12) implies that,

$$\begin{aligned}\lambda + \beta &= 0, \\ \lambda^2 + (1 + \sigma)\lambda - \sigma(\rho - 1) &= 0\end{aligned}$$

Above equations yield the following values of λ ,

$$\begin{aligned}\lambda_1 &= -\beta, \\ \lambda_2, \lambda_3 &= -\frac{(1 + \sigma)}{2} \pm \sqrt{\frac{(1 + \sigma)^2}{4} + \sigma(\rho - 1)}\end{aligned}$$

For simplicity, we assume that $\sigma, \beta > 0$ and vary ρ to find the condition for the point to be a stable fixed point. Then we see that $\lambda_1 < 0$ always.

Condition for stability: A fixed point is stable fixed point if *all* the corresponding eigenvalues are negative (or in case of complex values, the real part is negative).

Thus, in case of P_1 , it will be stable only if $\lambda_1, \lambda_2, \lambda_3 < 0$.

$$\begin{aligned}\text{if } \rho &= 1, & \lambda_2 &= 0, \lambda_3 < 0 \\ \text{if } 0 < \rho &< 1, & \lambda_2 &< 0, \lambda_3 < 0 \\ \text{if } \rho &> 1, & \lambda_2 &> 0, \lambda_3 < 0\end{aligned}\tag{14}$$

Thus, all the eigenvalues are negative, i.e. P_1 is a stable fixed point whenever $0 < \rho < 1$. And P_1 is unstable when $\rho > 1$.

Now, consider equation (13), for $P_2, P_3 \equiv (\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, \rho - 1)$

$$\begin{aligned}(\beta + \lambda)[\lambda^2 + (1 + \sigma)\lambda] + \beta(\rho - 1)\lambda + 2\sigma\beta(\rho - 1) &= 0 \quad \text{or} \\ \lambda^3 + (1 + \sigma + \beta)\lambda^2 + \beta(\sigma + \rho)\lambda + 2\sigma\beta(\rho - 1) &= 0\end{aligned}\tag{15}$$

For simplicity, we let,

$$\begin{aligned}A &= 1 + \sigma + \beta \\ B &= \beta(\sigma + \rho) \\ C &= 2\sigma\beta(\rho - 1)\end{aligned}$$

so that (15) reduces to the following,

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \quad (16)$$

We focus on the condition $\rho > 1$, since P_2, P_3 are complex-valued for $\rho \leq 1$. Rearranging (16) we get,

$$\lambda(\lambda^2 + B) = -(A\lambda^2 + C)$$

since the term on the RHS is always negative (for $\sigma, \beta > 0, \rho > 1$), it implies that,

$$\lambda(\lambda^2 + B) = -(A\lambda^2 + C) < 0$$

For LHS to satisfy above condition, one eigenvalue has to be negative. Therefore, we assume, $\lambda_1 < 0$. This means that either λ_2 and λ_3 are either both real or complex conjugates. But above condition also implies that $\lambda^2 < -B$ or equivalently, $\lambda^2 < -\frac{C}{A}$, which is only true for complex roots. Thus, we have, $\lambda_1 < 0$ and $\lambda_2, \lambda_3 \in \mathbb{C}$.

Condition for stability: All the eigenvalues corresponding to the fixed point have to be negative. In case of complex values, the real part should be negative.

To find whether $Re(\lambda_2)$ and $Re(\lambda_3)$, we need to find the value of ρ for which the mentioned condition holds. Let $\lambda_2, \lambda_3 = u \pm iv$. Then we can express (16) as,

$$\begin{aligned} (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) &= 0 \\ (\lambda - \lambda_1)(\lambda^2 - (\lambda_2 + \lambda_3)\lambda + \lambda_2\lambda_3) &= 0 \\ (\lambda - \lambda_1)(\lambda^2 - (u + iv + u - iv)\lambda + (u + iv)(u - iv)) &= 0 \\ (\lambda - \lambda_1)(\lambda^2 - 2u\lambda + u^2 + v^2) &= 0 \\ \lambda^3 - (2u + \lambda_1)\lambda^2 + (2u\lambda_1 + u^2 + v^2)\lambda - \lambda_1(u^2 + v^2) &= 0 \end{aligned} \quad (17)$$

Comparing equations (16) and (17), we get,

$$\begin{aligned} A = -(2u + \lambda_1) &= 1 + \sigma + \beta \\ B = 2u\lambda_1 + u^2 + v^2 &= \beta(\sigma + \rho) \\ C = -\lambda_1(u^2 + v^2) &= 2\sigma\beta(\rho - 1) \end{aligned}$$

where $u = Re(\lambda_2), Re(\lambda_3)$ and $v = Im(\lambda_2), Im(\lambda_3)$.

Consider the following equation,

$$C - AB = -\lambda_1(u^2 + v^2) + (2u + \lambda_1)(2u\lambda_1 + u^2 + v^2)$$

Simplifying,

$$\begin{aligned}
 C - AB &= -\lambda_1 u^2 - \lambda_1 v^2 + (4u^2 \lambda_1 2u^3 + 2uv^2 + 2u\lambda_1^2 + u^2 \lambda_1 + v^2 \lambda_1) \\
 &= 4u^2 \lambda_1 2u^3 + 2uv^2 + 2u\lambda_1^2 \\
 &= 2u(2u\lambda_1 + u^2 + v^2 + \lambda_1^2) \\
 C - AB &= 2u[(u + \lambda_1)^2 + v^2]
 \end{aligned}$$

In the above equation, the term inside the square brackets is always positive (real). This means that the sign of $C - AB$ is the same as that of u . Thus, for the eigenvalues $\lambda_2, \lambda_3 < 0$ we get the following condition,

$$Re(\lambda_2), Re(\lambda_3) < 0 \text{ whenever } C - AB < 0$$

Substituting values of A, B and C in the above condition we get,

$$\begin{aligned}
 C - AB &= 2\sigma\beta(\rho - 1) - (1 + \sigma + \beta)(\sigma + \beta)\beta < 0 \\
 2\sigma\rho - 2\sigma - (\sigma + \rho + \sigma^2 + \sigma\rho + \beta\sigma + \beta\rho) &< 0 \\
 (\sigma - \beta - 1)\rho - 3\sigma - \sigma^2 - \beta\sigma &< 0 \\
 (\sigma - \beta - 1)\rho &< 3\sigma + \sigma^2 + \beta\sigma
 \end{aligned}$$

Which gives us the following condition,

$$\rho < \frac{\sigma(3 + \sigma + \beta)}{(\sigma - \beta - 1)} = \rho_0 \quad (18)$$

Alternative method: We can find the value of ρ for which the real part of the eigenvalues transition from negative to positive, so that the value of ρ less than that value will have real part of eigenvalues negative. This transition happens when the real part of the eigenvalues is zero i.e. $u = 0$.

We substitute $\lambda = iv$ in equation (15), to get,

$$\begin{aligned}
 -iv^3 - (1 + \sigma + \beta)v^2 + \beta(\sigma + \rho)iv + 2\sigma\beta(\rho - 1) &= 0 \\
 \text{i.e.} \quad - (1 + \sigma + \beta)v^2 + 2\sigma\beta(\rho - 1) &= 0 \\
 \text{and} \quad -v^3 + \beta(\sigma + \rho)v &= 0
 \end{aligned} \quad (19)$$

From the second equation of (19), we get,

$$v = 0 \text{ or } v^2 = \sqrt{\beta(\sigma + \rho)}$$

Substituting the second value of v in the first equation of (19), we get,

$$\begin{aligned}\beta(\sigma + \rho)(1 + \sigma + \beta) &= 2\sigma\beta(\rho - 1) \\ \sigma + \rho + \sigma^2 + \sigma\rho + \sigma\beta + \rho\beta &= 2\sigma(\rho - 1) \\ \sigma^2 + 3\sigma - \sigma\rho + \sigma\beta + \rho\rho\beta &= 0 \\ \sigma^2 + 3\sigma + \sigma\beta &= \rho(\sigma - \beta - 1)\end{aligned}$$

Rearranging above equation,

$$\rho = \frac{\sigma(3 + \sigma + \beta)}{(\sigma - \beta - 1)}$$

which is same as equation (18).

Thus, the points all three eigenvalues corresponding to P_2, P_3 have negative real parts (and thus, P_2, P_3 are stable fixed points)whenever the following condition holds (otherwise they're unstable),

$$1 < \rho < \rho_0 \quad \text{where} \quad \rho_0 = \frac{\sigma(\sigma + \beta + 3)}{(\sigma - \beta - 1)} \quad (20)$$

Since, we have assumed $\sigma, \beta > 0$, the numerator of ρ_0 is always positive. But that's not the case for the denominator.

If $\sigma > \beta + 1$, $\rho_0 > 0$ and thus, $\rho < \rho_0$

If $\sigma < \beta + 1$, $\rho_0 < 0$ and $\rho > \rho_0$, since $\rho > 1$ for P_2 and P_3 .

Thus, we get an additional constraint for stability of P_2 and P_3 . We, therefore, revise (20) as follows,

$$1 < \rho < \rho_0 \quad \text{and} \quad \sigma > \beta + 1 \quad \text{where} \quad \rho_0 = \frac{\sigma(\sigma + \beta + 3)}{(\sigma - \beta - 1)} \quad (21)$$

1.5 Summary

Following results are true assuming $\sigma, \beta > 0$.

Fixed points:

$$\begin{aligned} P_1 &\equiv (0, 0, 0) \\ P_2 &\equiv (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1) \\ P_3 &\equiv (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1) \end{aligned}$$

Eigenvalue equations:

$$\begin{aligned} P_1 : \quad & (\beta + \lambda)[\lambda^2 + (1 + \sigma)\lambda - \sigma(\rho - 1)] = 0 \\ P_2, P_3 : \quad & \lambda^3 + (1 + \sigma + \beta)\lambda^2 + \beta(\sigma + \rho)\lambda + 2\sigma\beta(\rho - 1) = 0 \end{aligned}$$

Jacobian matrices:

$$\begin{aligned} J(P_1) &= \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \\ J(P_2) &= \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(\rho-1)} \\ \sqrt{\beta(\rho-1)} & \sqrt{\beta(\rho-1)} & -\beta \end{bmatrix} \\ J(P_3) &= \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{\beta(\rho-1)} \\ -\sqrt{\beta(\rho-1)} & -\sqrt{\beta(\rho-1)} & -\beta \end{bmatrix} \end{aligned}$$

Stability of P_1 :

$$\begin{aligned} &\text{stable fixed point if, } 0 < \rho < 1 \\ &\text{unstable fixed point if, } \rho > 1 \text{ or } \rho < 0 \end{aligned}$$

Stability of P_2 and P_3 :

$$\begin{aligned} &\text{stable fixed points if, } 1 < \rho < \rho_0 \text{ and } \sigma > \beta + 1 \\ &\text{unstable fixed points if, } \rho > \rho_0 \text{ or } \rho < 1 \text{ or } \sigma < \beta + 1 \end{aligned}$$

2 Numerical Analysis of Lorenz Equations

2.1 For a point near P_1 ($\rho < 1$)

Following are the plots when initial point is $I_0 \equiv (0.5, 0.5, 0.5)$ and $\sigma = 10, \beta = 2.667, \rho = 0.7$. Here, $P_1 \equiv (0, 0, 0)$

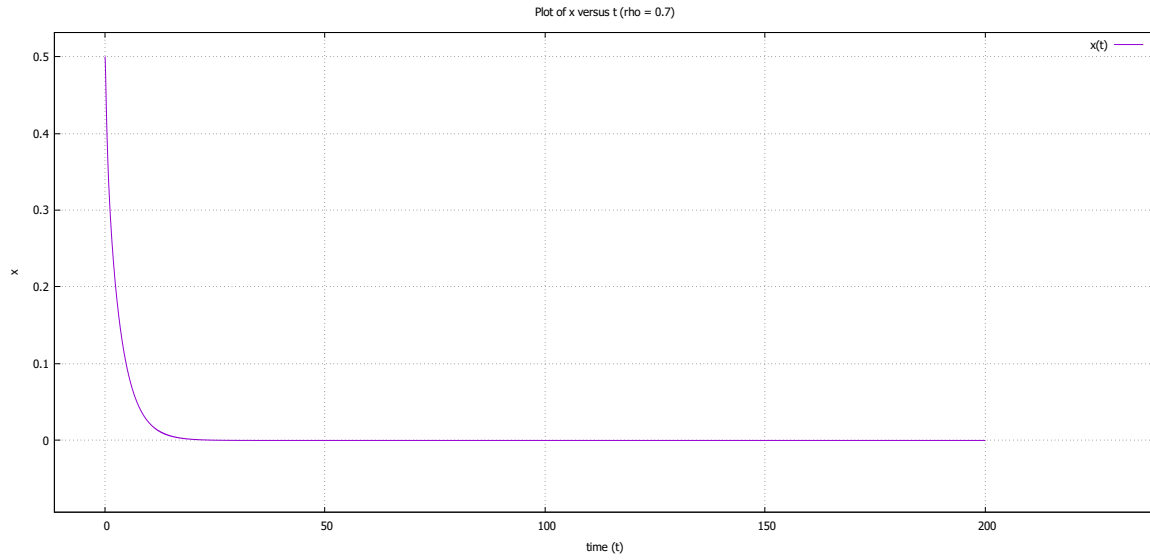


Figure 1: Plot of x vs t

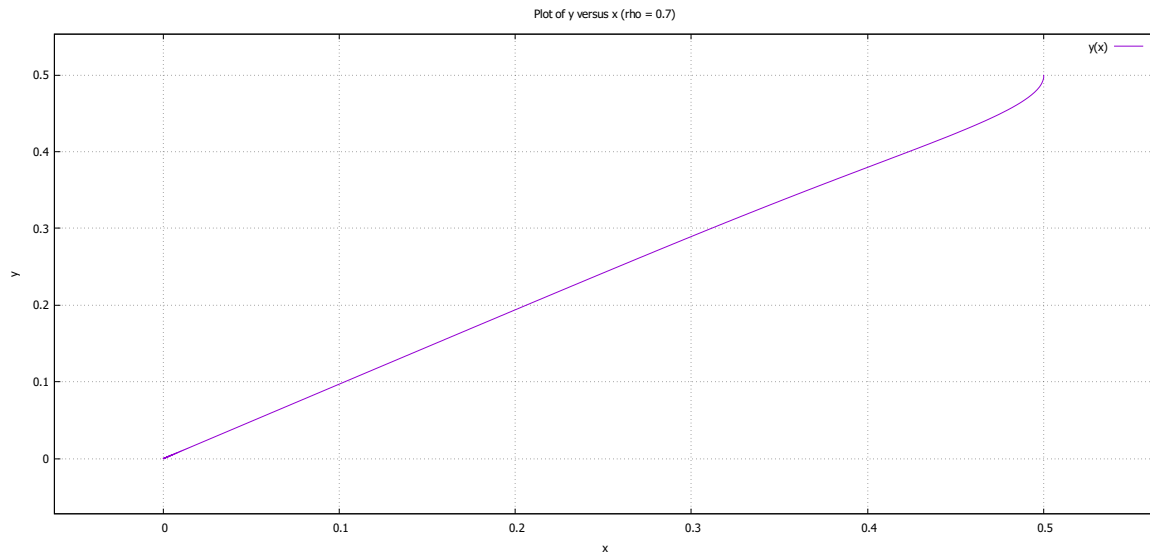


Figure 2: Plot of y vs x

2.2 For a point near P_1 ($\rho > 1$)

Following are the plots when initial point is $I_0 \equiv (0.5, 0.5, 0.5)$ and $\sigma = 10, \beta = 2.667, \rho = 1.5$. Here, $P_1 \equiv (0, 0, 0)$

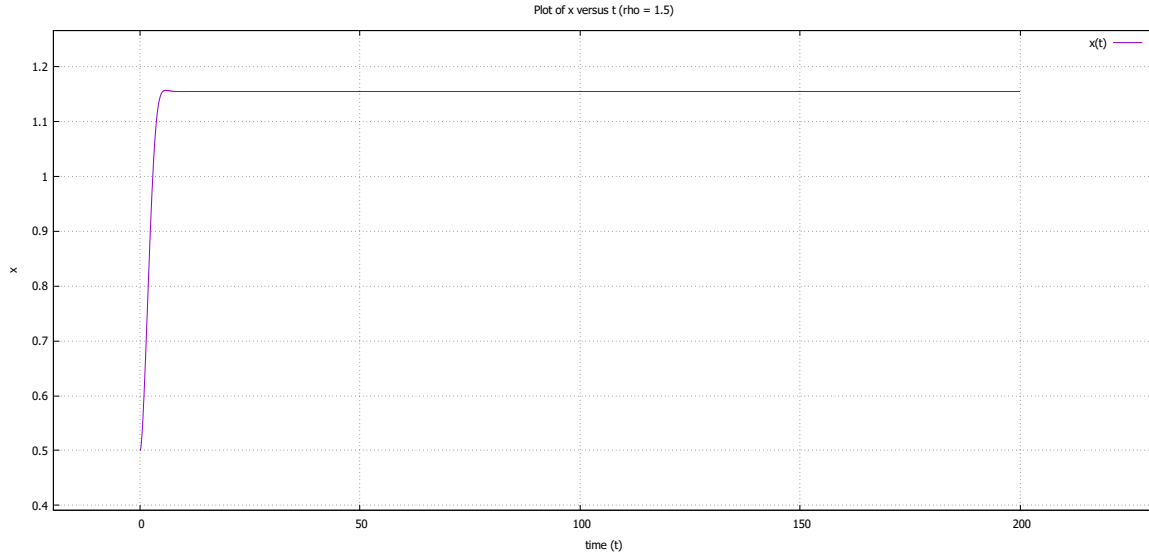


Figure 3: Plot of x vs t

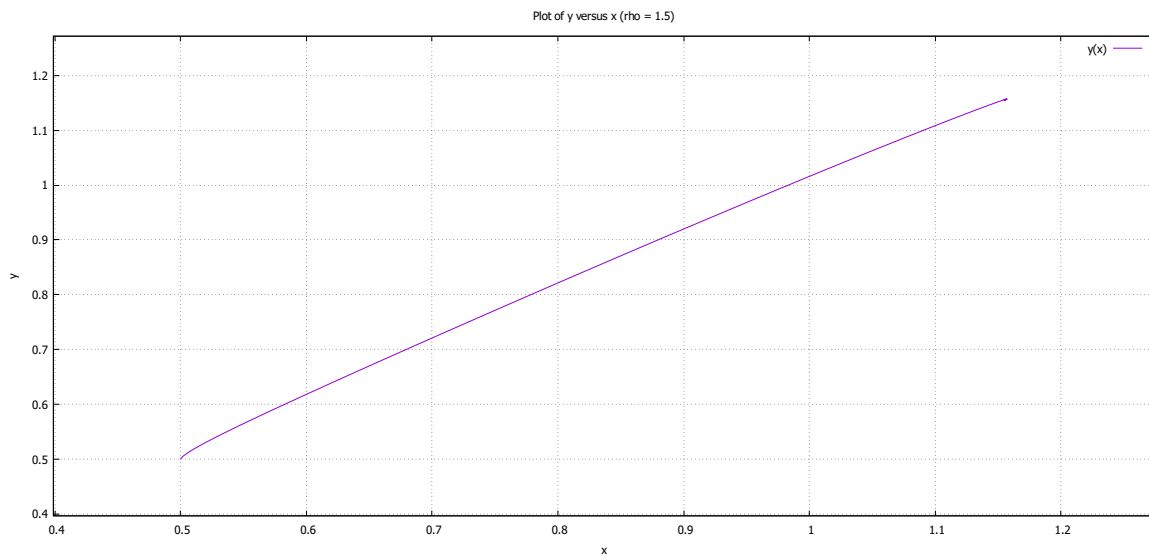


Figure 4: Plot of y vs x

2.3 For a point near P_1 ($\rho = 1$)

Following are the plots when initial point is $I_0 \equiv (0.5, 0.5, 0.5)$ and $\sigma = 10, \beta = 2.667, \rho = 1$. Here, $P_1 \equiv (0, 0, 0)$

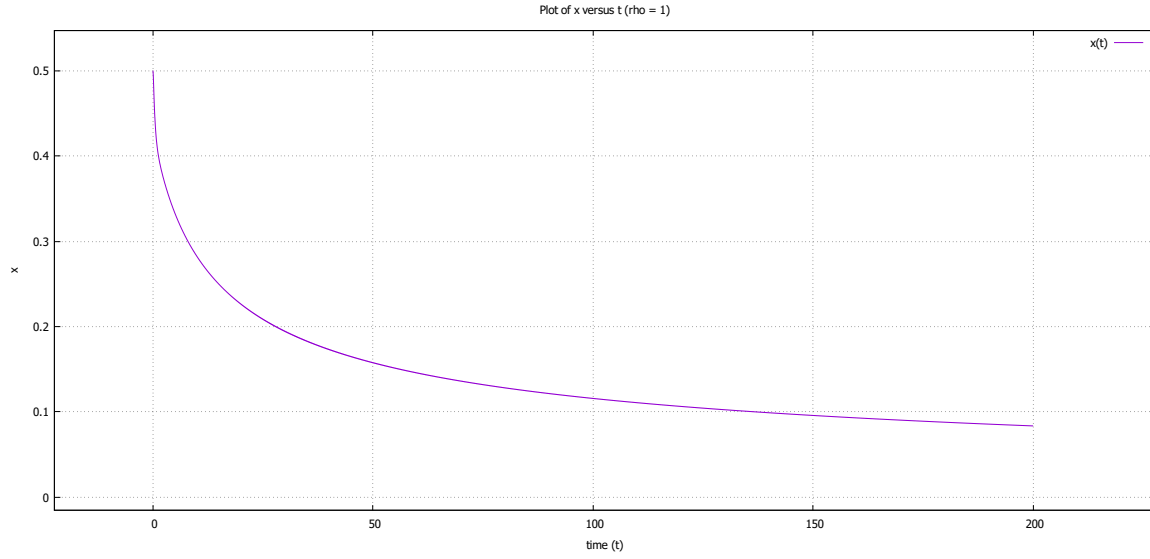


Figure 5: Plot of x vs t

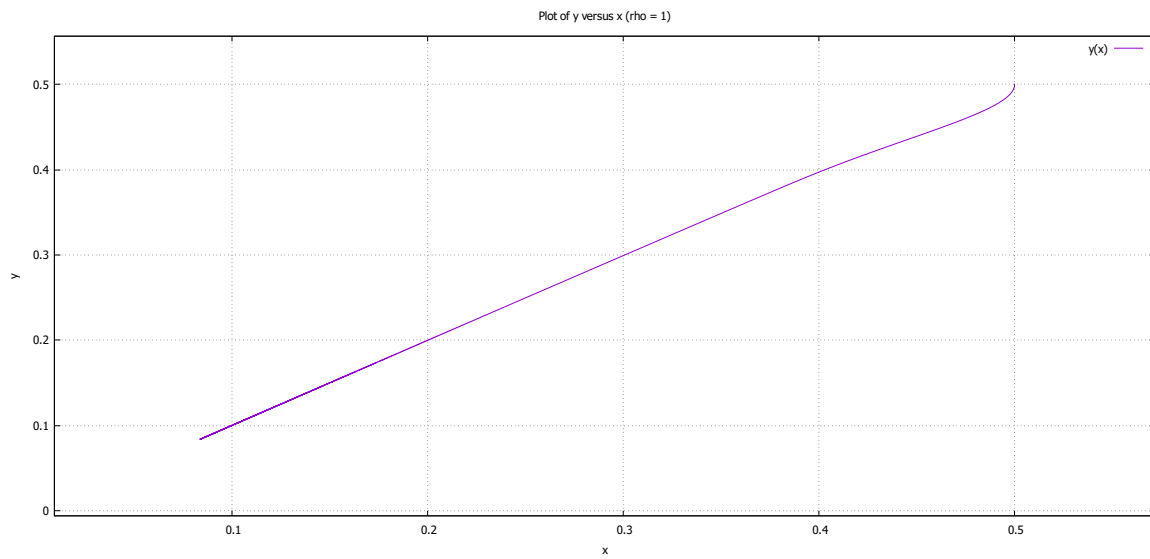


Figure 6: Plot of y vs x

2.4 For a point near P_2 ($\rho < \rho_0$)

Following are the plots when initial point is $I_0 \equiv (13, 13, 42)$ and $\sigma = 10, \beta = 5, \rho = 35$. Here $P_2 \equiv (14.83, 14.83, 44)$ and $\rho_0 = 45$

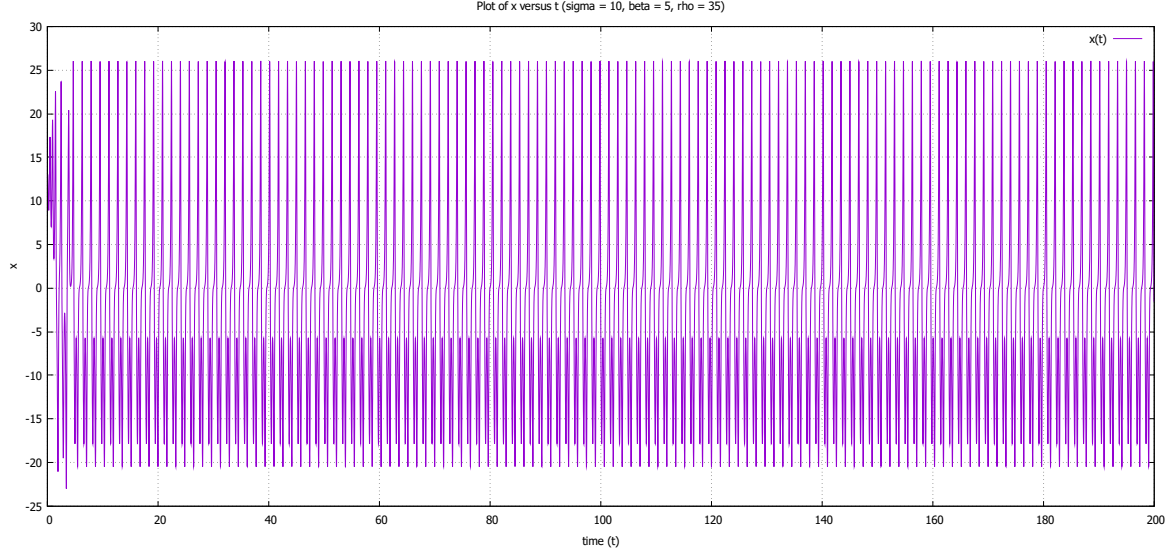


Figure 7: Plot of x vs t

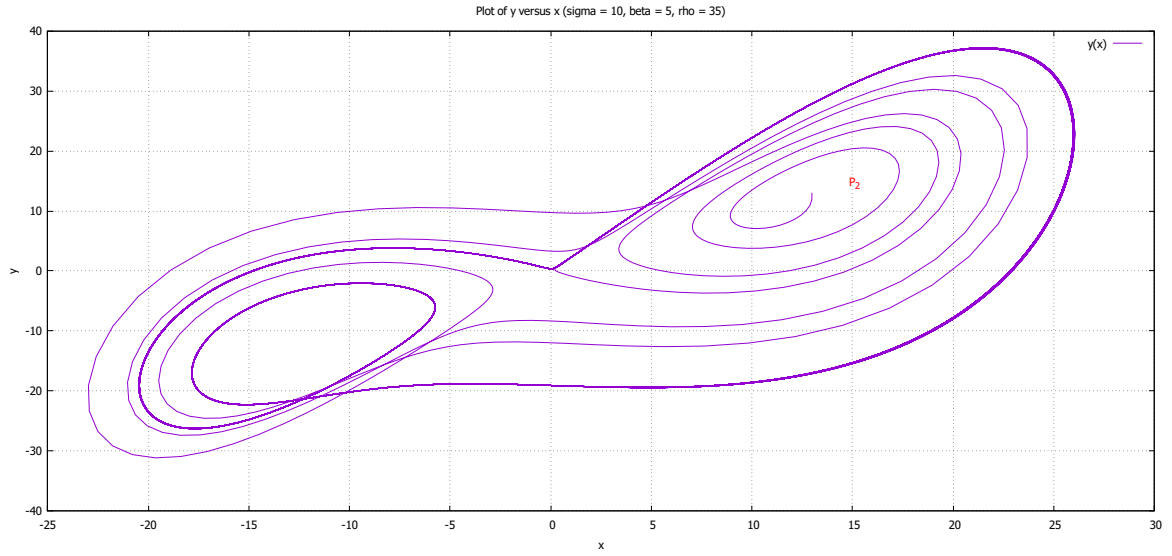


Figure 8: Plot of y vs x

2.5 For a point near P_2 ($\rho > \rho_0$)

Following are the plots when initial point is $I_0 \equiv (13, 13, 42)$ and $\sigma = 10, \beta = 5, \rho = 49$. Here $P_2 \equiv (14.83, 14.83, 44)$ and $\rho_0 = 45$

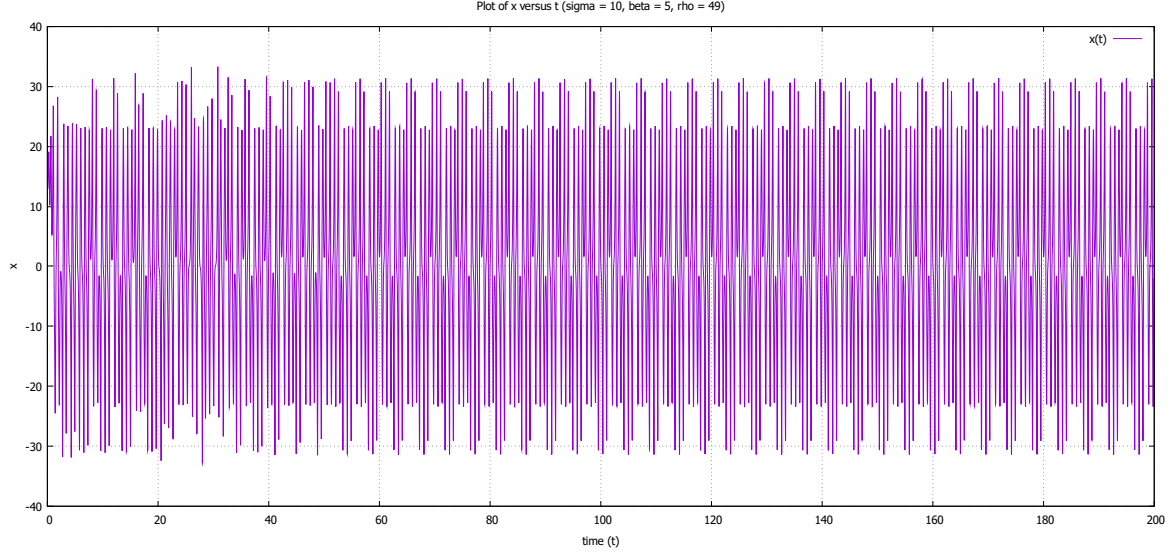


Figure 9: Plot of x vs t

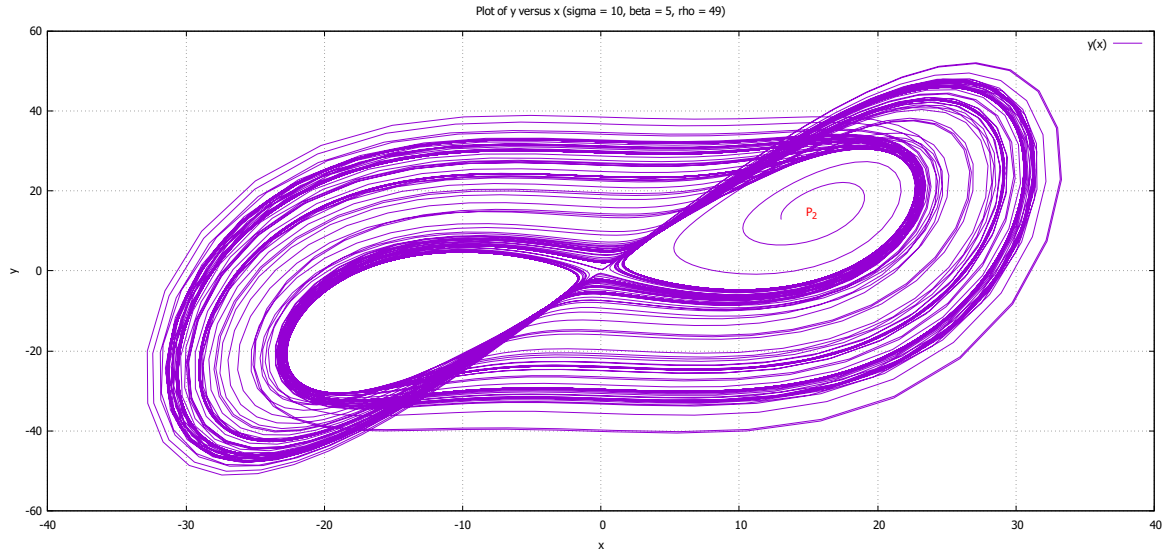


Figure 10: Plot of y vs x

2.6 For a point near P_2 ($\rho = \rho_0$)

Following are the plots when initial point is $I_0 \equiv (13, 13, 42)$ and $\sigma = 10, \beta = 5, \rho = 45$. Here $P_2 \equiv (14.83, 14.83, 44)$ and $\rho_0 = 45$

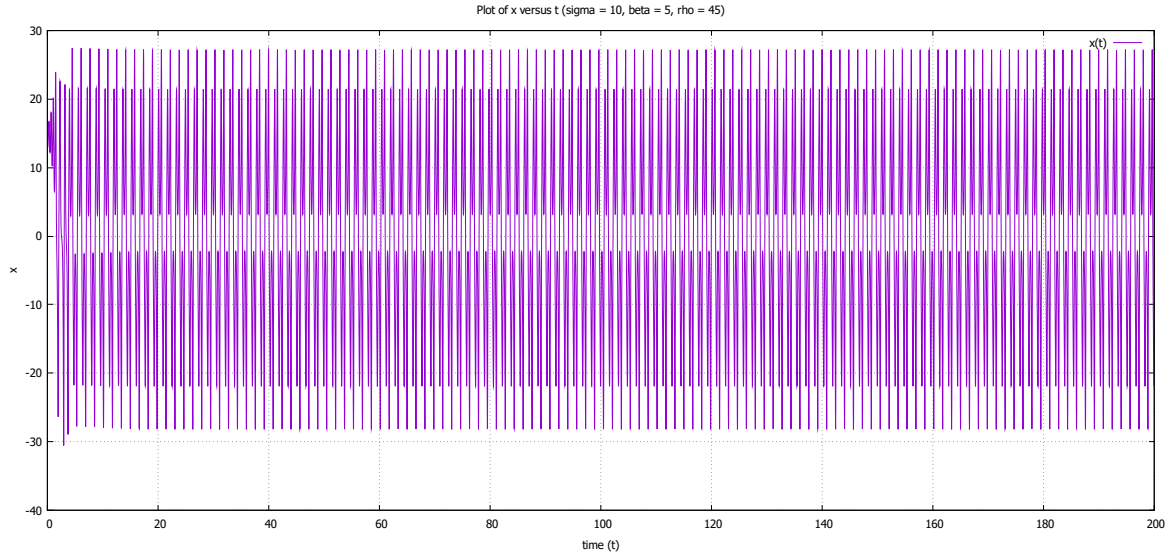


Figure 11: Plot of x vs t

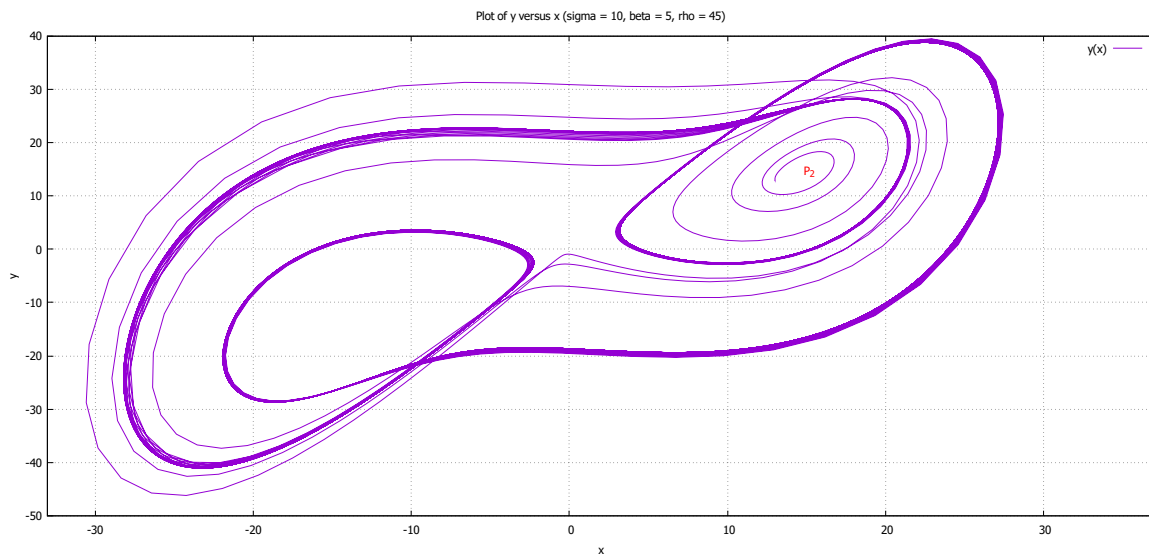


Figure 12: Plot of y vs x

2.7 For a point near P_3 ($\rho < \rho_0$)

Following are the plots when initial point is $I_0 \equiv (-15, -15, 45)$ and $\sigma = 10, \beta = 5, \rho = 32$. Here $P_3 \equiv (-14.83, -14.83, 44)$ and $\rho_0 = 45$

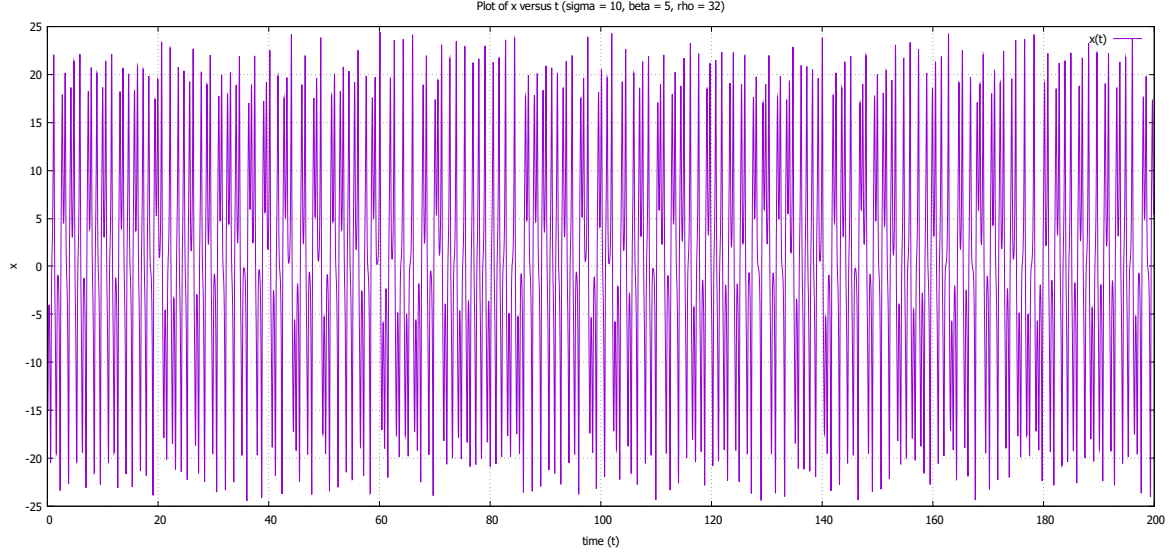


Figure 13: Plot of x vs t

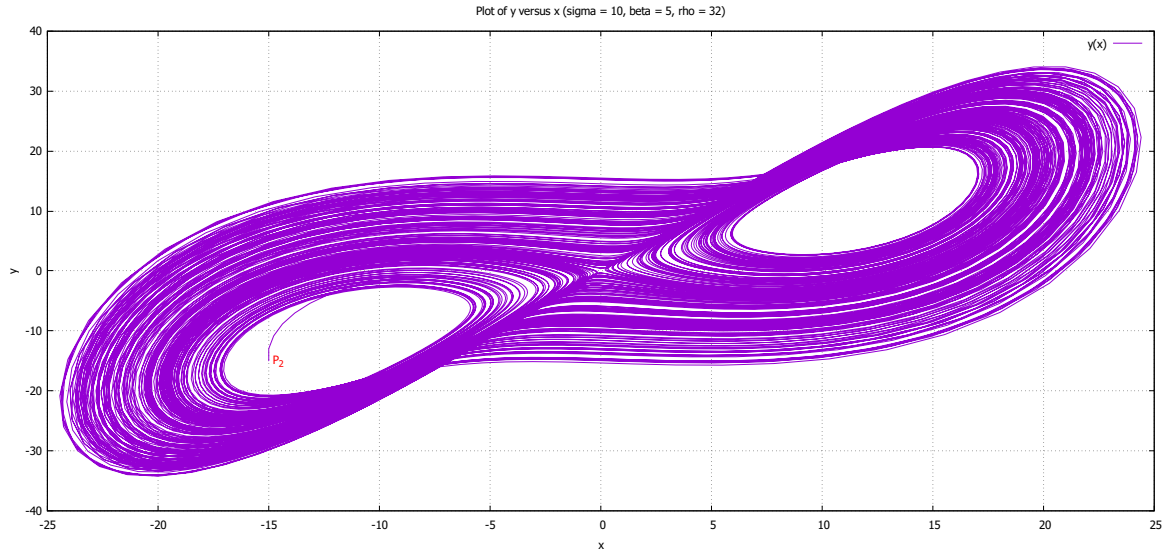


Figure 14: Plot of y vs x

2.8 For a point near P_3 ($\rho > \rho_0$)

Following are the plots when initial point is $I_0 \equiv (-15, -15, 45)$ and $\sigma = 10, \beta = 5, \rho = 47$. Here $P_2 \equiv (-14.83, -14.83, 44)$ and $\rho_0 = 45$

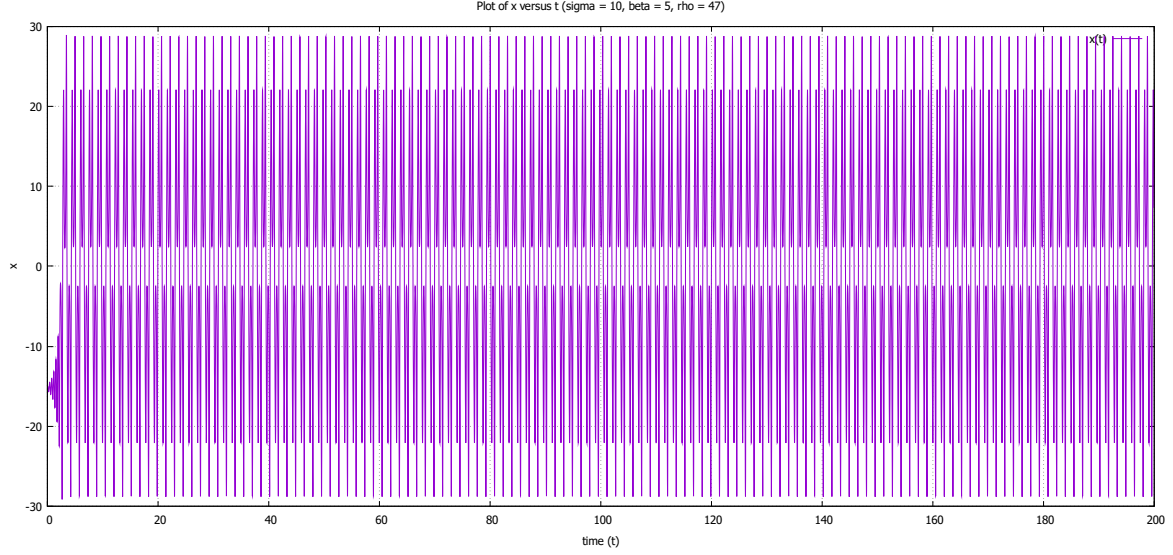


Figure 15: Plot of x vs t

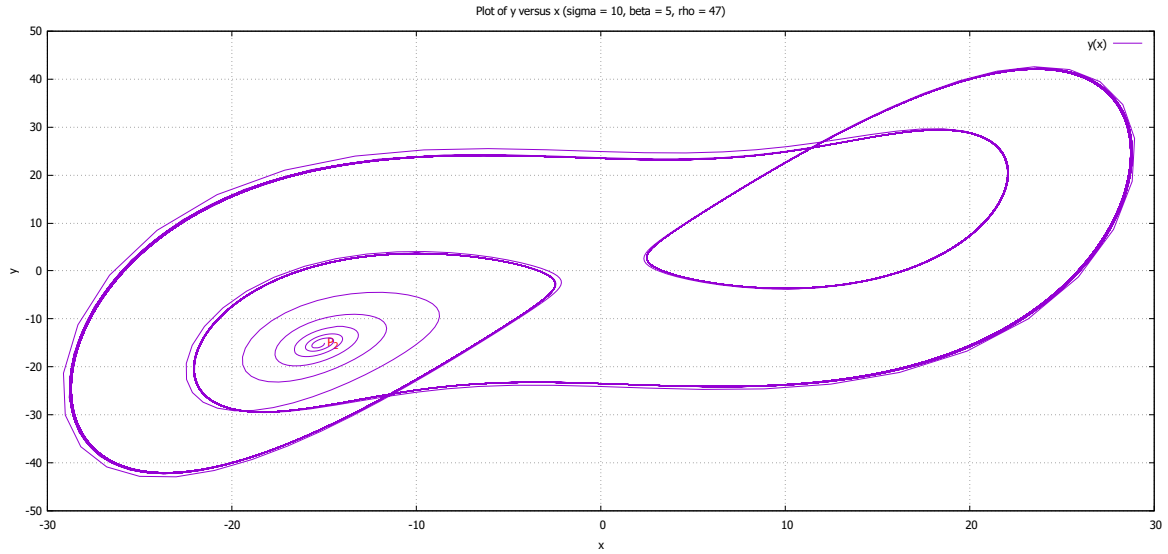


Figure 16: Plot of y vs x

2.9 For a point near P_3 ($\rho = \rho_0$)

Following are the plots when initial point is $I_0 \equiv (-15, -15, 45)$ and $\sigma = 10, \beta = 5, \rho = 45$. Here $P_3 \equiv (-14.83, -14.83, 44)$ and $\rho_0 = 45$

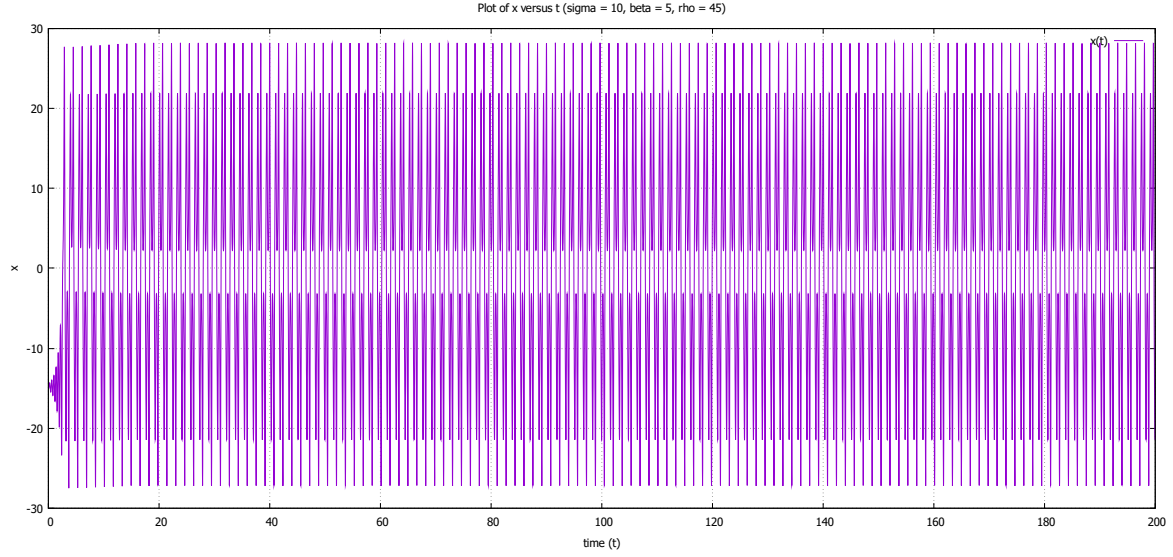


Figure 17: Plot of x vs t

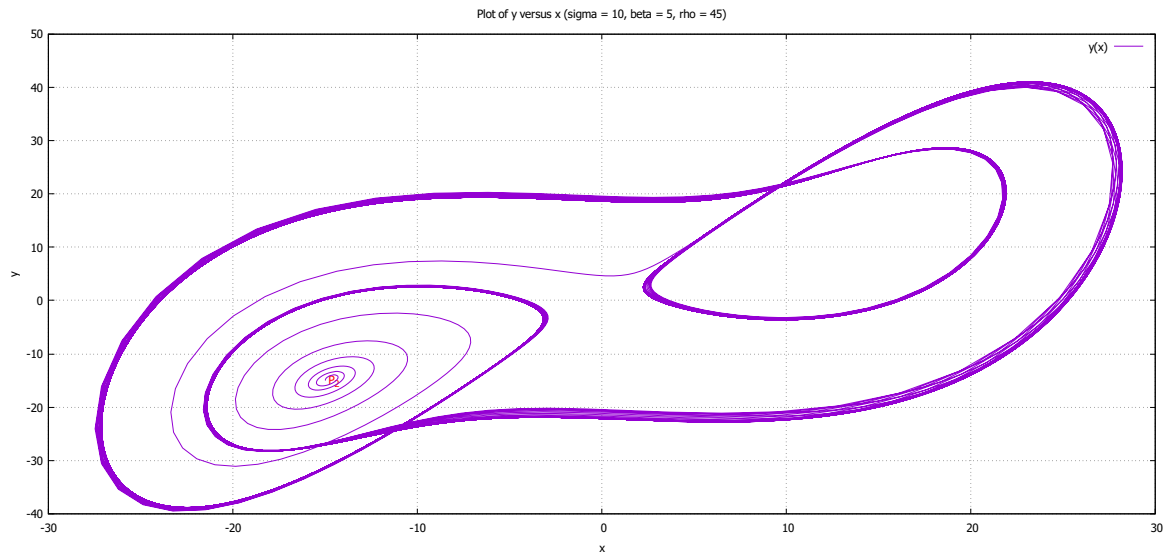
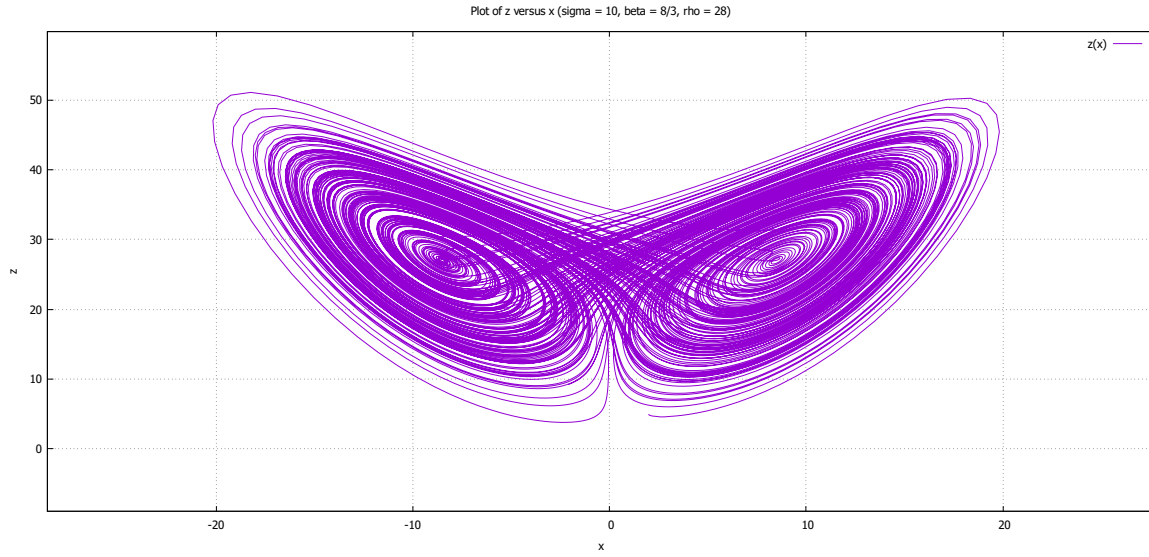
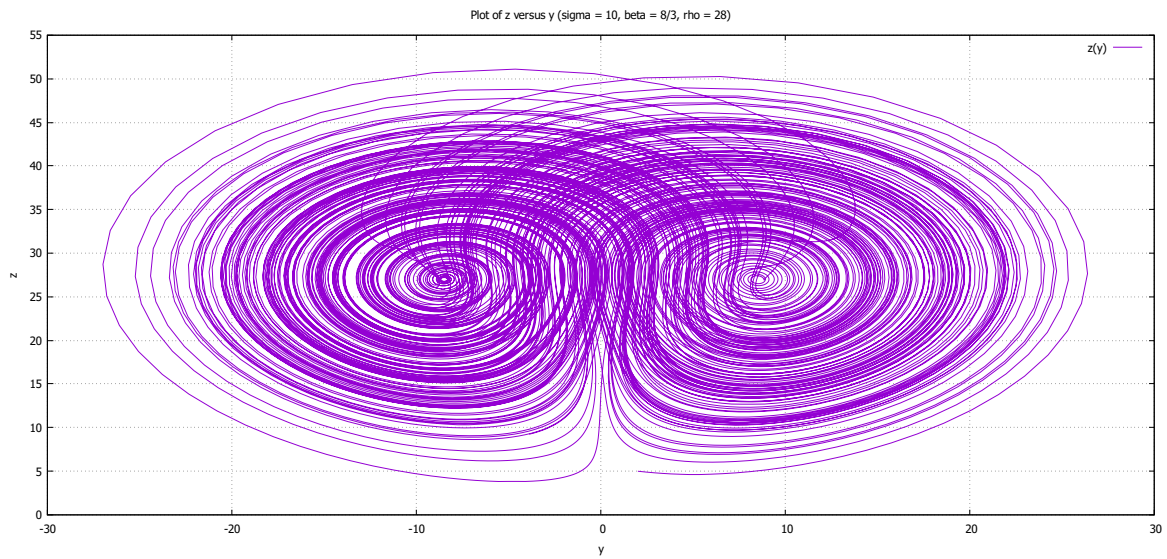


Figure 18: Plot of y vs x

2.10 For $\sigma = 10, \beta = 8/3, \rho = 28$

Following are the plots when initial point is $I_0 \equiv (2, 2, 5)$ and $\sigma = 10, \beta = 8/3, \rho = 28$.

**Figure 19:** Plot of z vs x **Figure 20:** Plot of z vs y

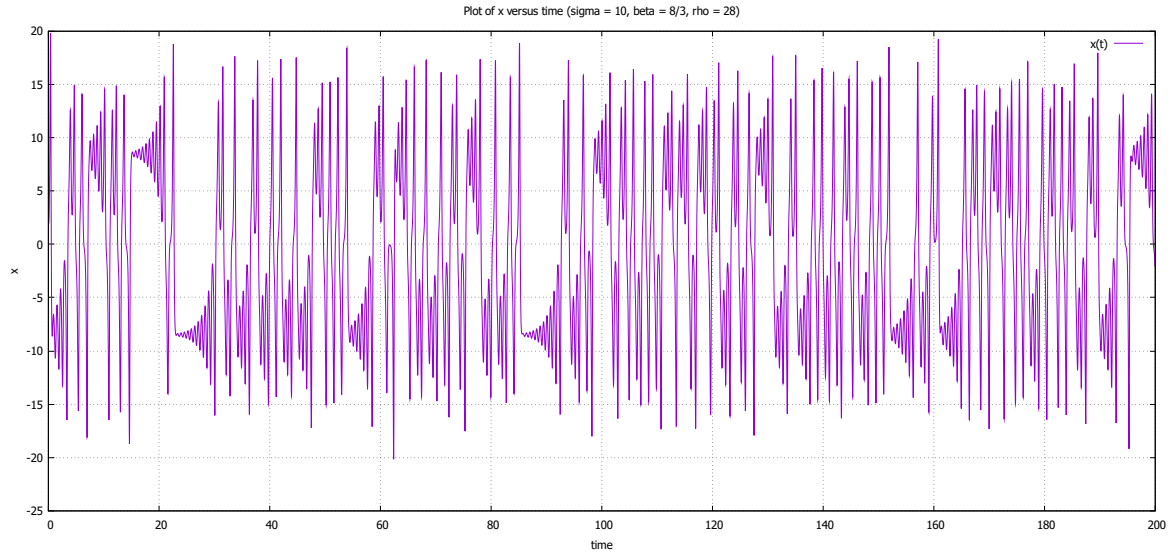


Figure 21: Plot of x vs t

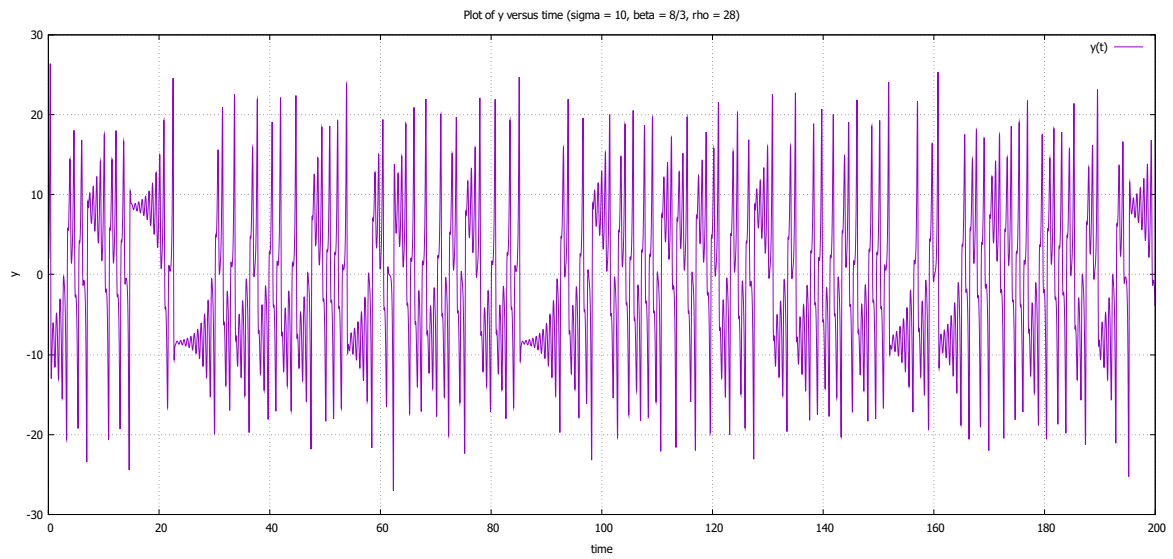


Figure 22: Plot of y vs t

3 Fortran code

Following pages contain the Fortran95 code used to generate above plots. The solution is generated using Euler's method with $h = 0.01$.

```
1 !Title: To find the solutions for Lorenz euqations using Euler's method
2 !PRN: 2019P042
3 !Name: Shreyes Madgaonkar
4
5 program LORENZ
6 implicit none
7
8 !declaration of variables and arrays
9 real, dimension(:), allocatable :: X, Y, Z, T
10 real :: FN, GN, HN
11 real :: H
12 integer :: I, J
13
14 !allocate arrays
15 J = 25000
16 allocate(X(J), Y(J), Z(J), T(J))
17
18 !step size
19 H = 0.01
20
21 !initial conditions
22 T(1) = 0
23 X(1) = 2.0
24 Y(1) = 2.0
25 Z(1) = 5.0
26
27 !Euler's method
28 do I = 1, 20000
29
30     X(I+1) = X(I) + H*FN(X(I), Y(I))
31     Y(I+1) = Y(I) + H*GN(X(I), Y(I), Z(I))
32     Z(I+1) = Z(I) + H*HN(X(I), Y(I), Z(I))
33
34     T(I+1) = T(I) + H
35
36 end do
37
38 !store data in a file
39 open(unit=21, file='plots\leqn1.dat')
40
41     do I = 1, 20000
42         write(21,*) T(I), X(I), Y(I), Z(I)
43     end do
44
45 close(21)
46
47 !deallocate arrays
48 deallocate(X, Y, Z, T)
49
50 end program LORENZ
51
52 !*****
53
54 real function FN(X, Y)
55 implicit none
```



```
56 real, intent(in) :: X, Y
57 real :: SIG
58 SIG = 10
59
60 FN = SIG*(Y - X)
61 return
62
63 end function FN
64
65 !*****
66
67 real function GN(X, Y, Z)
68 implicit none
69 real, intent(in) :: X, Y, Z
70 real :: R
71 R = 28
72
73 GN = X*(R - Z) - Y
74 return
75
76 end function GN
77
78 !*****
79
80 real function HN(X, Y, Z)
81 implicit none
82 real, intent(in) :: X, Y, Z
83 real :: B
84 B = 8.0/3.0
85
86 HN = X*Y - Z*B
87 return
88
89 end function HN
```