Analysis of Lorenz Equations

Shreyes Madgaonkar

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Contents

1 Stability Analysis of Lorenz Equations

1.1 Introduction

The dynamics of a Lorenz system is described by the following differential equations,

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned} \tag{1}$$

where $\sigma, \rho, \beta \;\epsilon\; \mathbb{R}$

Rewriting the above equations, for latter operations,

$$\begin{aligned} \dot{x} &= \sigma(y - x) &= f(x, y, x) \\ \dot{y} &= x(\rho - z) - y &= g(x, y, z) \\ \dot{z} &= xy - \beta z &= h(x, y, z) \end{aligned} \tag{2}$$

where $\sigma, \rho, \beta \;\epsilon\; \mathbb{R}$

1.2 Fixed points of Lorenz equations

To find the fixed points for (2) we set, $\dot{x} = \dot{y} = \dot{z} = 0$ to arrive at the following equations,

$$\begin{aligned} \sigma(y-x) &= f(x_0, y_0, z_0) &= 0 \\ x(\rho - z) - y &= g(x_0, y_0, z_0) &= 0 \\ xy - \beta z &= h(x_0, y_0, z_0) &= 0 \end{aligned} \tag{3}$$

where (x_0, y_0, z_0) represents fixed points. Above equations simplify to the following,

$$\begin{aligned} x_0 &= y_0 \\ x_0(\rho - z_0) - y_0 &= 0 \\ x_0 y_0 - \beta z_0 &= 0 \end{aligned}$$

These further simplify to,

$$\begin{aligned} x_0(\rho - z_0 - 1) &= 0 \\ x_0^2 &= \beta z_0 \end{aligned} \tag{4}$$

From the first equation of (4) we arrive at two conditions

$$x_0 = 0 \text{ and } z_0 = \rho - 1$$

The former condition implies that $y_0 = z_0 = 0$. Thus, the point $P_1 \equiv (0, 0, 0)$ is a fixed point.

From the condition, $z_0 = \rho - 1$ and second equation of (4), we get,

$$x_0 = y_0 = \pm\sqrt{\beta(\rho - 1)}$$

Following above argument, the other two fixed points are

$$P_2 \equiv (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1) \text{ and } P_3 \equiv (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho - 1)$$

Finally, we arrive at three fixed points for the system defined by (2)

$$\begin{aligned} P_1 &\equiv (0, 0, 0) \\ P_2 &\equiv (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1) \\ P_3 &\equiv (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho - 1) \end{aligned} \tag{5}$$

1.3 Jacobian and Eigenvalue equation

Let $\delta x = x - x_0$, $\delta y = y - y_0$, $\delta z = z - z_0$ be small perturbations around the fixed points represented by (x_0, y_0, z_0). Taylor expanding equations defined by (2) about fixed points we get,

$$\begin{aligned}
\delta \dot{x} &= f(x_0, y_0, z_0) + f'_x \delta x + f'_y \delta y + f'_y \delta z + \ldots \\
\delta \dot{y} &= g(x_0, y_0, z_0) + g'_x \delta x + g'_y \delta y + g'_y \delta z + \ldots \\
\delta \dot{z} &= h(x_0, y_0, z_0) + h'_x \delta x + h'_y \delta y + h'_y \delta z + \ldots
\end{aligned}$$

But from (3), $f(x_0, y_0, z_0) = g(x_0, y_0, z_0) = h(x_0, y_0, z_0) = 0$. Therefore, above equations reduce to the following linear approximations,

$$\begin{aligned}
\delta \dot{x} &= f'_x \delta x + f'_y \delta y + f'_z \delta z \\
\delta \dot{y} &= g'_x \delta x + g'_y \delta y + g'_z \delta z \\
\delta \dot{z} &= h'_x \delta x + h'_y \delta y + h'_z \delta z
\end{aligned} \tag{6}$$

Above equation can be written in matrix form as follows,

$$\frac{d}{dt}\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} f'_x & f'_y & f'_z \\ g'_x & g'_y & g'_z \\ h'_x & h'_y & h'_z \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} \tag{7}$$

Equation (5) can be written more compactly as,

$$\frac{d(X)}{dt} = JX \quad \text{where } J = \begin{bmatrix} f'_x & f'_y & f'_z \\ g'_x & g'_y & g'_z \\ h'_x & h'_y & h'_z \end{bmatrix} \tag{8}$$

From equation (2) and (8), we arrive at the following Jacobian matrix,

$$J = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z_0 & -1 & -x_0 \\ y_0 & x_0 & -\beta \end{bmatrix} \tag{9}$$

where (x_0, y_0, z_0) is a fixed point.
To find the eigenvalues for the Jacobian, consider the equation,

$$JX = \lambda X \quad \Rightarrow \quad Det(J - \lambda I) = |J - \lambda I| = 0$$

where λ represents eigenvalue and I is a 3×3 identity matrix.

$$\therefore\ |J - \lambda I| \ = \ 0 \ = \begin{vmatrix} -\sigma - \lambda & \sigma & 0 \\ \rho - z_0 & -1 - \lambda & -x_0 \\ y_0 & x_0 & -\beta - \lambda \end{vmatrix} \tag{10}$$

$$(\sigma + \lambda)[(1 + \lambda)(\beta + \lambda) + x_0^2] + \sigma[x_0 y_0 - (\beta + \lambda)(\rho - z_0)] = 0$$

$$(\sigma + \lambda)(1 + \lambda)(\beta + \lambda) + (\sigma + \lambda)x_0^2 + \sigma x_0 y_0 - \sigma(\beta + \lambda)(\rho - z_0)] = 0$$

$$(\beta + \lambda)[\lambda^2 + (1 + \sigma)\lambda - \sigma(\rho - z_0) + \sigma] + x_0^2\lambda + \sigma(x_0^2 + x_0 y_0) = 0 \tag{11}$$

Now, we determine the eigenvalue equations at fixed points P_1, P_2 and P_3 (see (5)). Substitutiong the values from (5) in (11), we get,

at P_1,

$$(\beta + \lambda)[\lambda^2 + (1 + \sigma)\lambda - \sigma(\rho - 1)] = 0 \tag{12}$$

and at P_2 and P_3,

$$(\beta + \lambda)[\lambda^2 + (1 + \sigma)\lambda] + \beta(\rho - 1)\lambda + 2\sigma\beta(\rho - 1) = 0 \tag{13}$$

Equations (12) and (13) are eigenvalue equations or characteristic polynomials for the eigenvalues represented by λ.

1.4 Stability of fixed points

From previous discussion we have three fixed points ($P_1, P_2,$ and P_3) and two corresponding eigenvalue equations. First, we check the stability of P_1.
Equation (12) implies that,

$$\begin{aligned} \lambda + \beta = 0, \\ \lambda^2 + (1+\sigma)\lambda - \sigma(\rho - 1) = 0 \end{aligned}$$

Above equations yield the following values of λ,

$$\begin{aligned} \lambda_1 &= -\beta, \\ \lambda_2, \lambda_2 &= -\frac{(1+\sigma)}{2} \pm \sqrt{\frac{(1+\sigma)^2}{4} + \sigma(\rho - 1)} \end{aligned}$$

For simplicity, we assume that $\sigma, \beta > 0$ and vary ρ to find the condition for the point to be a stable fixed point. Then we see that $\lambda_1 < 0$ always.

Condition for stability: A fixed point is stable fixed point if *all* the corresponding eigenvalues are negative (or in case of complex values, the real part is negative).

Thus, in case of P_1, it will be stable only if $\lambda_1, \lambda_2, \lambda_3 < 0$.

$$\begin{aligned} \text{if } \rho = 1, &\quad \lambda_2 = 0, \lambda_3 < 0 \\ \text{if } 0 < \rho < 1, &\quad \lambda_2 < 0, \lambda_3 < 0 \\ \text{if } \rho > 1, &\quad \lambda_2 > 0, \lambda_3 < 0 \end{aligned} \tag{14}$$

Thus, all the eigenvalues are negative, i.e. P_1 is a stable fixed point whenever $0 < \rho < 1$. And P_1 is unstable when $\rho > 1$.

Now, consider equation (13), for $P_2, P_3 \equiv (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho - 1)$

$$\begin{aligned} (\beta + \lambda)[\lambda^2 + (1+\sigma)\lambda] + \beta(\rho-1)\lambda + 2\sigma\beta(\rho-1) = 0 \quad \text{or} \\ \lambda^3 + (1+\sigma+\beta)\lambda^2 + \beta(\sigma+\rho)\lambda + 2\sigma\beta(\rho-1) = 0 \end{aligned} \tag{15}$$

For simplicity, we let,

$$\begin{aligned} A &= 1 + \sigma + \beta \\ B &= \beta(\sigma + \rho) \\ C &= 2\sigma\beta(\rho - 1) \end{aligned}$$

so that (15) reduces to the following,

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{16}$$

We focus on the condition $\rho > 1$, since P_2, P_3 are complex-valued for $\rho \leq 1$. Rearranging (16) we get,

$$\lambda(\lambda^2 + B) = -(A\lambda^2 + C)$$

since the term on the RHS is always negative (for $\sigma, \beta > 0, \rho > 1$), it implies that,

$$\lambda(\lambda^2 + B) = -(A\lambda^2 + C) < 0$$

For LHS to satisfy above condition, one eigenvalue has to be negative. Therefore, we assume, $\lambda_1 < 0$. This means that either λ_2 and λ_3 are either both real or complex conjugates. But above condition also implies that $\lambda^2 < -B$ or equivalently, $\lambda^2 < -\frac{C}{A}$, which is only true for complex roots. Thus, we have, $\lambda_1 < 0$ and $\lambda_2, \lambda_3 \ \epsilon\ \mathbb{C}$.

Condition for stability: All the eigenvalues corresponding to the fixed point have to be negative. In case of complex values, the real part should be negative.

To find whether $Re(\lambda_2)$ and $Re(\lambda_3)$, we need to find the value of ρ for which the mentioned condition holds. Let $\lambda_2, \lambda_3 = u \pm iv$. Then we can express (16) as,

$$\begin{aligned}
(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) &= 0 \\
(\lambda - \lambda_1)(\lambda^2 - (\lambda_2 + \lambda_3)\lambda + \lambda_2\lambda_3) &= 0 \\
(\lambda - \lambda_1)(\lambda^2 - (u + iv + u - iv)\lambda + (u + iv)(u - iv)) &= 0 \\
(\lambda - \lambda_1)(\lambda^2 - 2u\lambda + u^2 + v^2) &= 0
\end{aligned}$$

$$\lambda^3 - (2u + \lambda_1)\lambda^2 + (2u\lambda_1 + u^2 + v^2)\lambda - \lambda_1(u^2 + v^2) = 0 \tag{17}$$

Comparing equations (16) and (17), we get,

$$\begin{aligned}
A &= -(2u + \lambda_1) && = 1 + \sigma + \beta \\
B &= 2u\lambda_1 + u^2 + v^2 && = \beta(\sigma + \rho) \\
C &= -\lambda_1(u^2 + v^2) && = 2\sigma\beta(\rho - 1)
\end{aligned}$$

where $u = Re(\lambda_2), Re(\lambda_3)$ and $v = Im(\lambda_2), Im(\lambda_3)$.
Consider the following equation,

$$C - AB = -\lambda_1(u^2 + v^2) + (2u + \lambda_1)(2u\lambda_1 + u^2 + v^2)$$

Simplifying,

$$\begin{aligned}
C - AB &= -\lambda_1 u^2 - \lambda_1 v^2 + (4u^2\lambda_1 + 2u^3 + 2uv^2 + 2u\lambda_1^2 + u^2\lambda_1 + v^2\lambda_1) \\
&= 4u^2\lambda_1 + 2u^3 + 2uv^2 + 2u\lambda_1^2 \\
&= 2u(2u\lambda_1 + u^2 + v^2 + \lambda_1^2) \\
C - AB &= 2u[(u+\lambda_1)^2 + v^2]
\end{aligned}$$

In the above equation, the term inside the square brackets is always positive (real). This means that the sign of $C - AB$ is the same as that of u. Thus, for the eigenvalues $\lambda_2, \lambda_3 < 0$ we get the following condition,

$$Re(\lambda_2), Re(\lambda_3) < 0 \quad \text{whenever} \quad C - AB < 0$$

Substituting values of A, B and C in the above condition we get,

$$\begin{aligned}
C - AB = 2\sigma\beta(\rho - 1) - (1 + \sigma + \beta)(\sigma + \beta)\beta &< 0 \\
2\sigma\rho - 2\sigma - (\sigma + \rho + \sigma^2 + \sigma\rho + \beta\sigma + \beta\rho) &< 0 \\
(\sigma - \beta - 1)\rho - 3\sigma - \sigma^2 - \beta\sigma &< 0 \\
(\sigma - \beta - 1)\rho &< 3\sigma + \sigma^2 + \beta\sigma
\end{aligned}$$

Which gives us the following condition,

$$\rho < \frac{\sigma(3 + \sigma + \beta)}{(\sigma - \beta - 1)} = \rho_0 \tag{18}$$

Alternative method: We can find the value of ρ for which the real part of the eigenvalues transition from negative to positive, so that the value of ρ less than that value will have real part of eigenvalues negative. This transition happens when the real part of the eigenvalues is zero i.e. $u = 0$.
We substitute $\lambda = iv$ in equation (15), to get,

$$-iv^3 - (1 + \sigma + \beta)v^2 + \beta(\sigma + \rho)iv + 2\sigma\beta(\rho - 1) = 0$$

$$\begin{aligned}
\text{i.e.} \qquad & -(1 + \sigma + \beta)v^2 + 2\sigma\beta(\rho - 1) = 0 \\
\text{and} \quad & -v^3 + \beta(\sigma + \rho)v = 0
\end{aligned} \tag{19}$$

From the second equation of (19), we get,

$$v = 0 \text{ or } v^2 = \sqrt{\beta(\sigma + \rho)}$$

Substituting the second value of v in the first equation of (19), we get,

$$\beta(\sigma+\rho)(1+\sigma+\beta) = 2\sigma\beta(\rho-1)$$
$$\sigma+\rho+\sigma^2+\sigma\rho+\sigma\beta+\rho\beta = 2\sigma(\rho-1)$$
$$\sigma^2+3\sigma-\sigma\rho+\sigma\beta+\rho\rho\beta = 0$$
$$\sigma^2+3\sigma+\sigma\beta = \rho(\sigma-\beta-1)$$

Rearranging above equation,

$$\rho = \frac{\sigma(3+\sigma+\beta)}{(\sigma-\beta-1)}$$

which is same as equation (18).

Thus, the points all three eigenvalues corresponding to P_2, P_3 have negative real parts (and thus, P_2, P_3 are stable fixed points)whenever the following condition holds (otherwise they're unstable),

$$1<\rho<\rho_0 \quad \text{where} \quad \rho_0 = \frac{\sigma(\sigma+\beta+3)}{(\sigma-\beta-1)} \tag{20}$$

Since, we have assumed $\sigma, \beta > 0$, the numerator of ρ_0 is always positive. But that's not the case for the denominator.
If $\sigma > \beta+1$, $\rho_0 > 0$ and thus, $\rho < \rho_0$
If $\sigma < \beta+1$, $\rho_0 < 0$ and $\rho > \rho_0$, since $\rho > 1$ for P_2 and P_3.
Thus, we get an additional constraint for stability of P_2 and P_3. We, therefore, revise (20) as follows,

$$1<\rho<\rho_0 \quad \text{and} \quad \sigma>\beta+1 \quad \text{where} \quad \rho_0 = \frac{\sigma(\sigma+\beta+3)}{(\sigma-\beta-1)} \tag{21}$$

1.5 Summary

Following results are true assuming $\sigma, \beta > 0$.

Fixed points:

$$\begin{aligned}
P_1 &\equiv (0,0,0) \\
P_2 &\equiv (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1) \\
P_3 &\equiv (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1)
\end{aligned}$$

Eigenvalue equations:

$$\begin{aligned}
P_1 : &\qquad (\beta+\lambda)[\lambda^2+(1+\sigma)\lambda-\sigma(\rho-1)] = 0 \\
P_2, P_3 : &\qquad \lambda^3+(1+\sigma+\beta)\lambda^2+\beta(\sigma+\rho)\lambda+2\sigma\beta(\rho-1) \quad = 0
\end{aligned}$$

Jacobian matrices:

$$J(P_1) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

$$J(P_2) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(\rho-1)} \\ \sqrt{\beta(\rho-1)} & \sqrt{\beta(\rho-1)} & -\beta \end{bmatrix}$$

$$J(P_3) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{\beta(\rho-1)} \\ -\sqrt{\beta(\rho-1)} & -\sqrt{\beta(\rho-1)} & -\beta \end{bmatrix}$$

Stability of P_1:

$$\begin{aligned}
\text{stable fixed point if,} &\quad 0 < \rho < 1 \\
\text{unstable fixed point if,} &\quad \rho > 1 \ \text{ or } \ \rho < 0
\end{aligned}$$

Stability of P_2 and P_3:

$$\begin{aligned}
\text{stable fixed points if,} &\quad 1 < \rho < \rho_0 \ \text{ and } \ \sigma > \beta+1 \\
\text{unstable fixed points if,} &\quad \rho > \rho_0 \text{ or } \ \rho < 1 \text{ or } \ \sigma < \beta+1
\end{aligned}$$

2 Numerical Solutions of Lorenz Equations

2.1 For a point near P_1 ($\rho < 1$)

Following are the plots when initial point is $I_0 \equiv (0.5, 0.5, 0.5)$ and $\sigma = 10, \beta = 2.667, \rho = 0.7$. Here, $P_1 \equiv (0, 0, 0)$

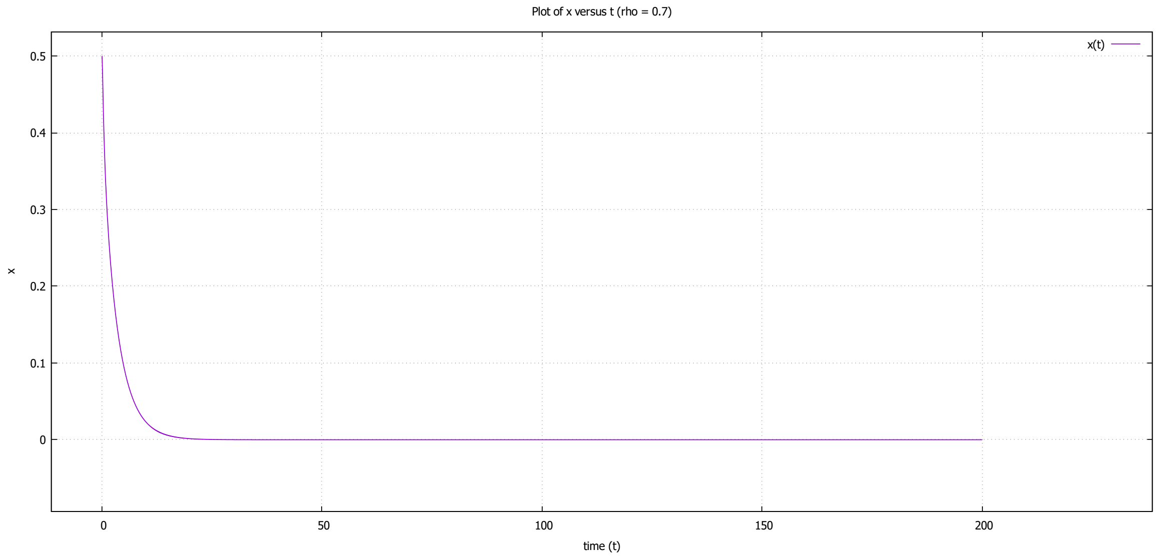

Figure 1: Plot of x vs t

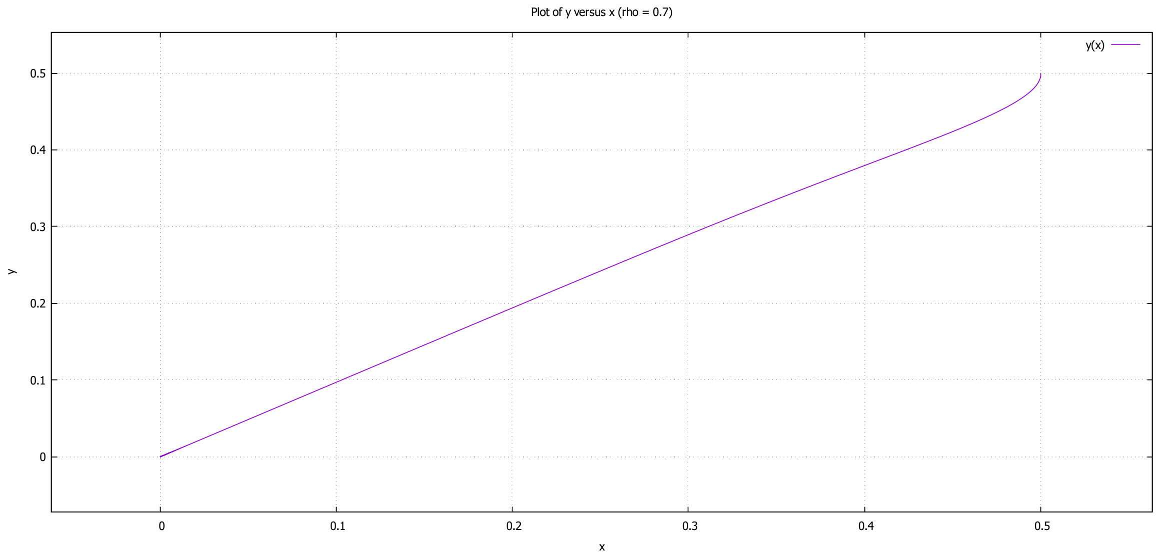

Figure 2: Plot of y vs x

2.2 For a point near P_1 ($\rho > 1$)

Following are the plots when initial point is $I_0 \equiv (0.5, 0.5, 0.5)$ and $\sigma = 10, \beta = 2.667, \rho = 1.5$. Here, $P_1 \equiv (0, 0, 0)$

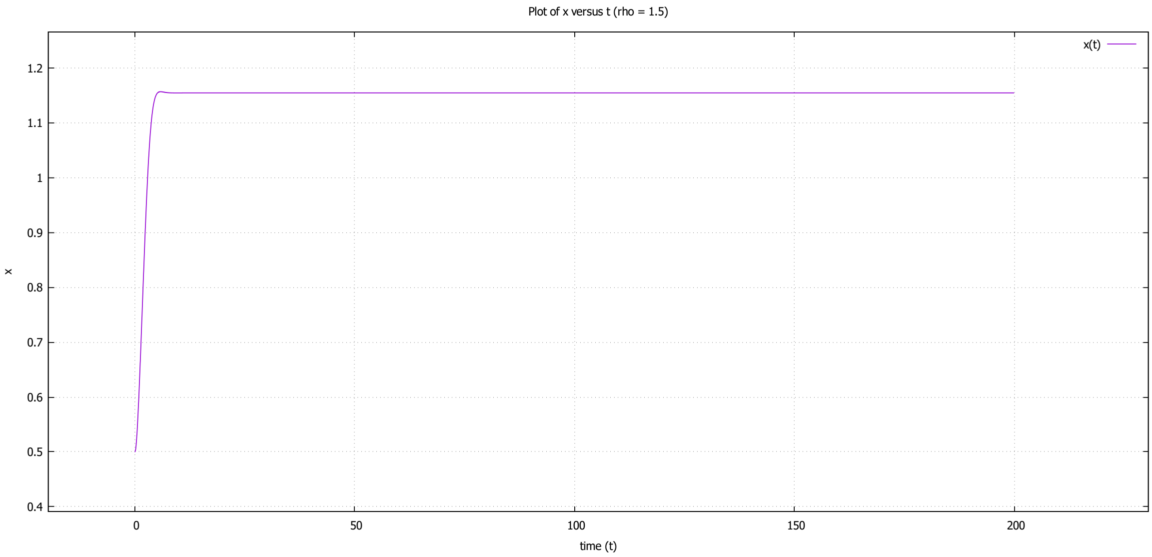

Figure 3: Plot of x vs t

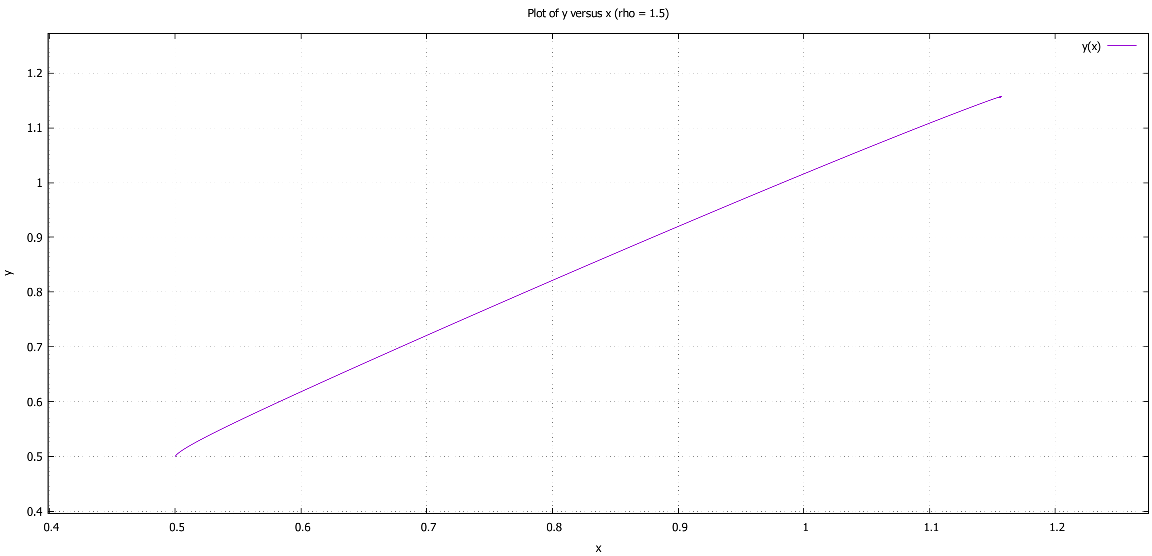

Figure 4: Plot of y vs x

2.3 For a point near P_1 ($\rho = 1$)

Following are the plots when initial point is $I_0 \equiv (0.5, 0.5, 0.5)$ and $\sigma = 10, \beta = 2.667, \rho = 1$. Here, $P_1 \equiv (0, 0, 0)$

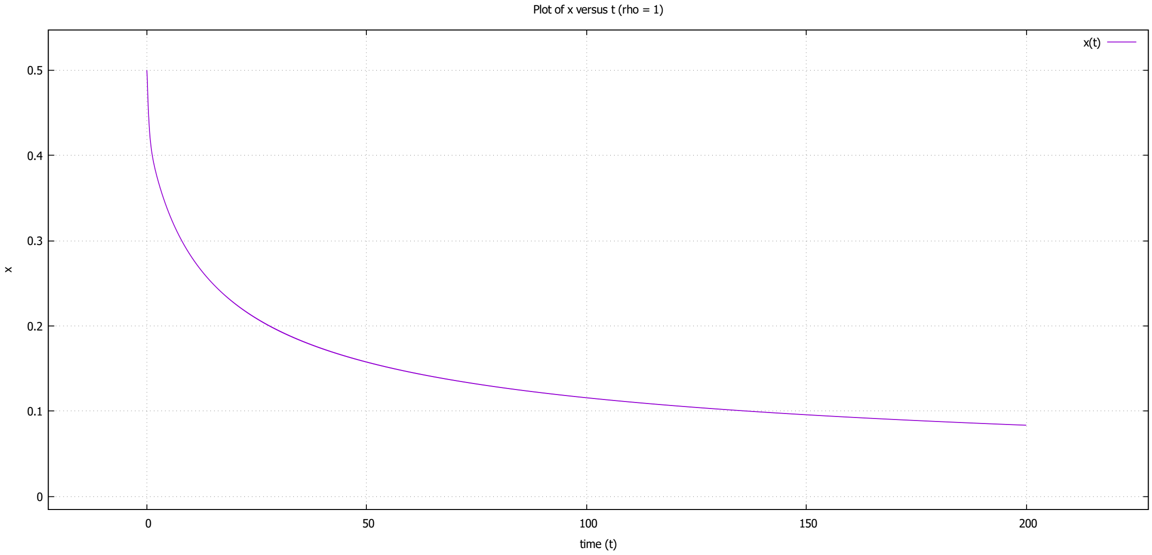

Figure 5: Plot of x vs t

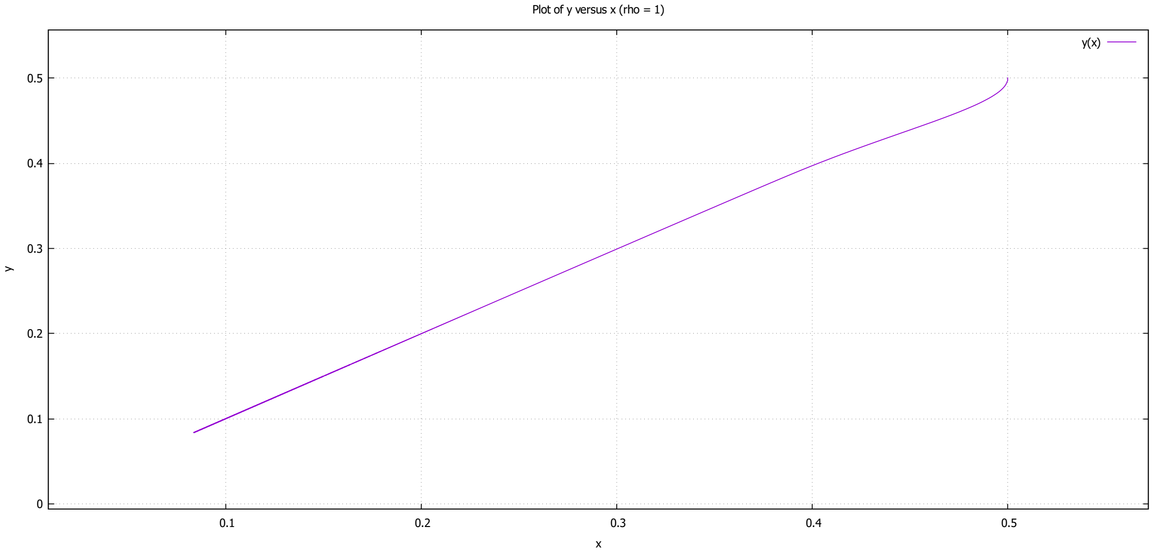

Figure 6: Plot of y vs x

2.4 For a point near P_2 ($\rho < \rho_0$)

Following are the plots when initial point is $I_0 \equiv (6, 6, 20)$ and $\sigma = 10, \beta = 2, \rho = 10$. Here $P_2 \equiv (4.24, 4.24, 9)$ and $\rho_0 = 21.42$

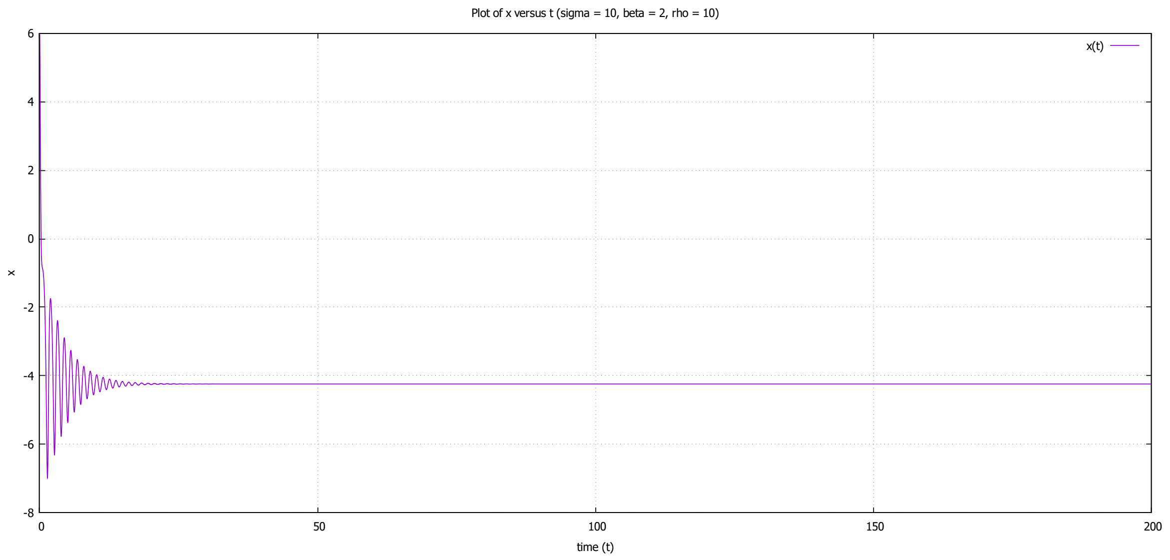

Figure 7: Plot of x vs t

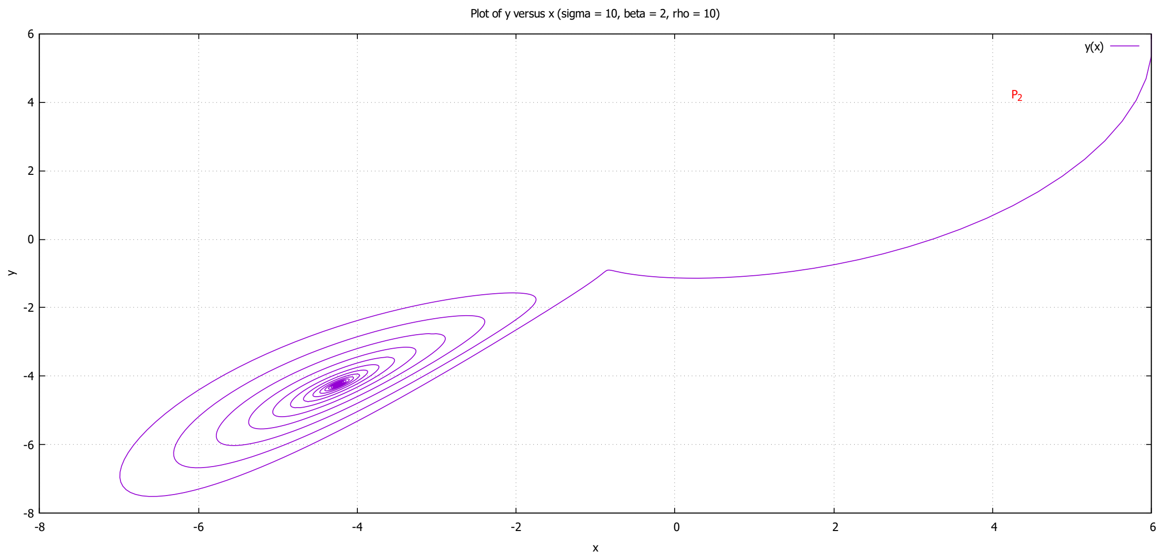

Figure 8: Plot of y vs x

2.5 For a point near P_2 ($\rho > \rho_0$)

Following are the plots when initial point is $I_0 \equiv (6, 6, 20)$ and $\sigma = 10, \beta = 2, \rho = 25$. Here $P_2 \equiv (6.92, 6.92, 24)$ and $\rho_0 = 21.42$

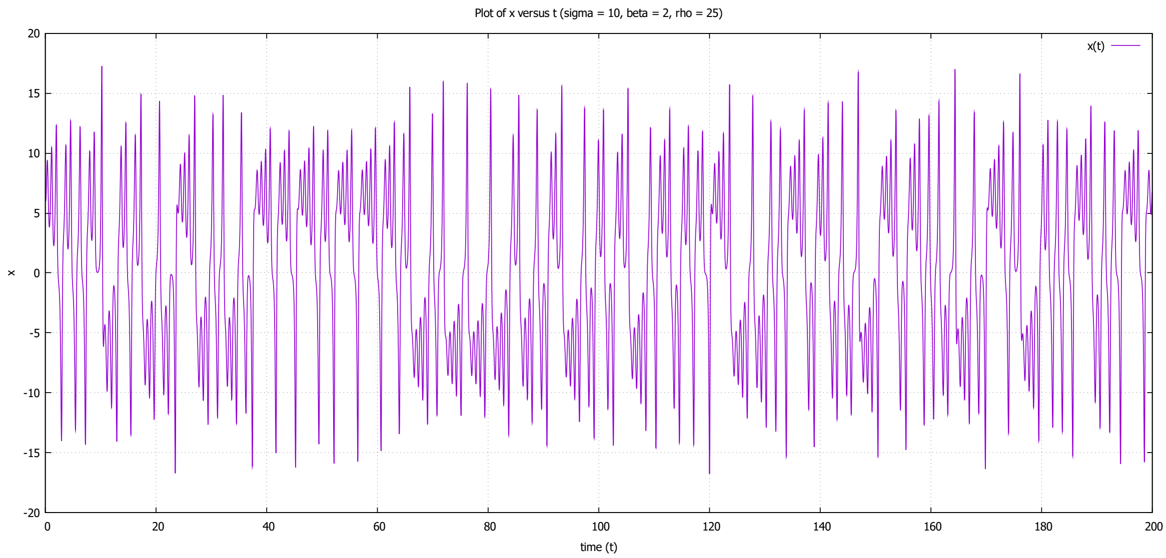

Figure 9: Plot of x vs t

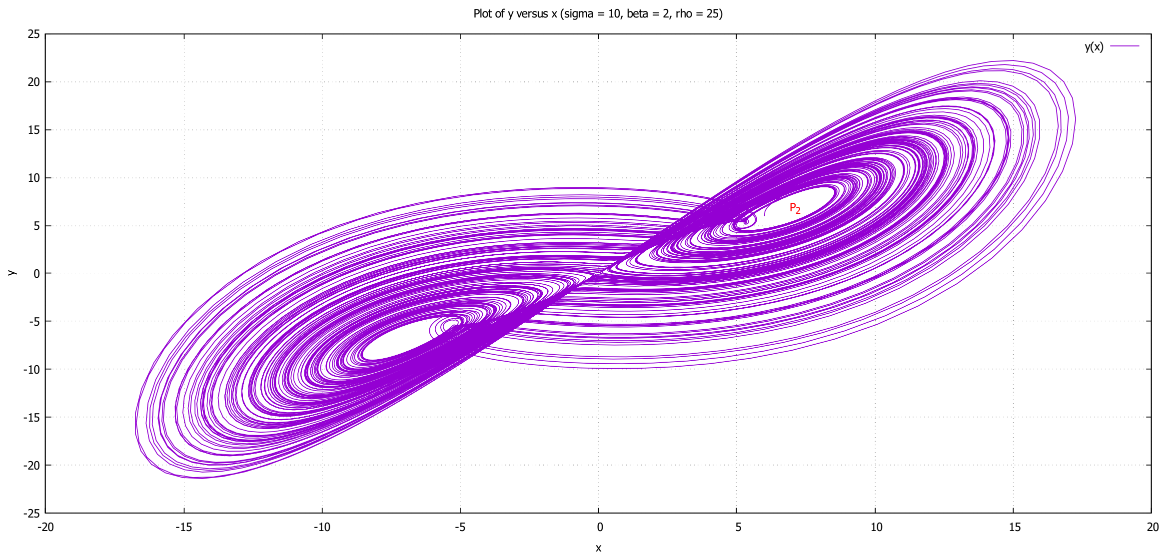

Figure 10: Plot of y vs x

2.6 For a point near P_2 ($\rho = \rho_0$)

Following are the plots when initial point is $I_0 \equiv (6, 6, 20)$ and $\sigma = 10, \beta = 2, \rho = 21.42$. Here $P_2 \equiv (6.39, 6.39, 20.42)$ and $\rho_0 = 21.42$

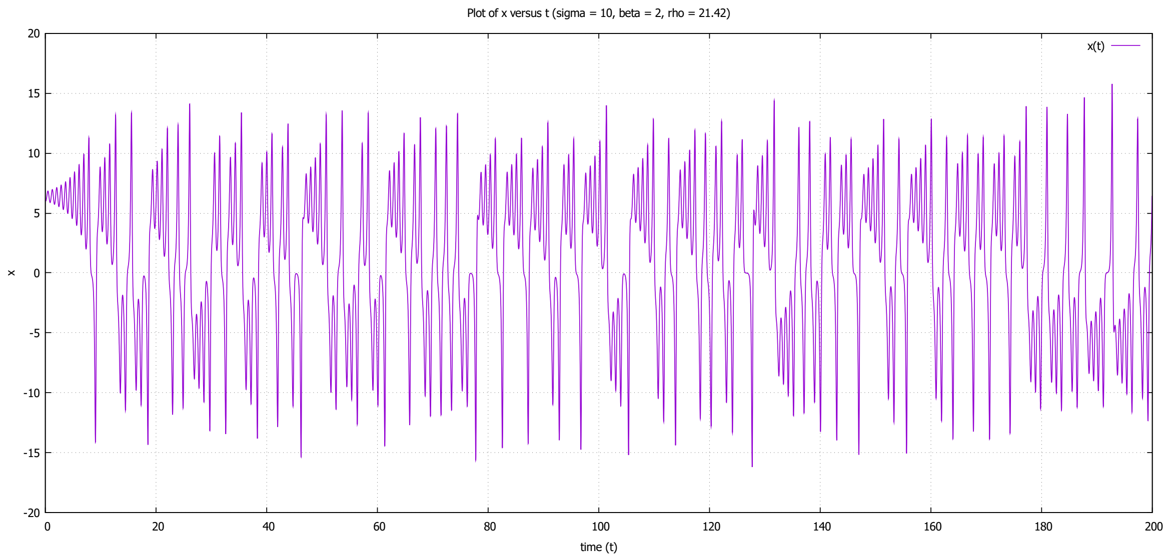

Figure 11: Plot of x vs t

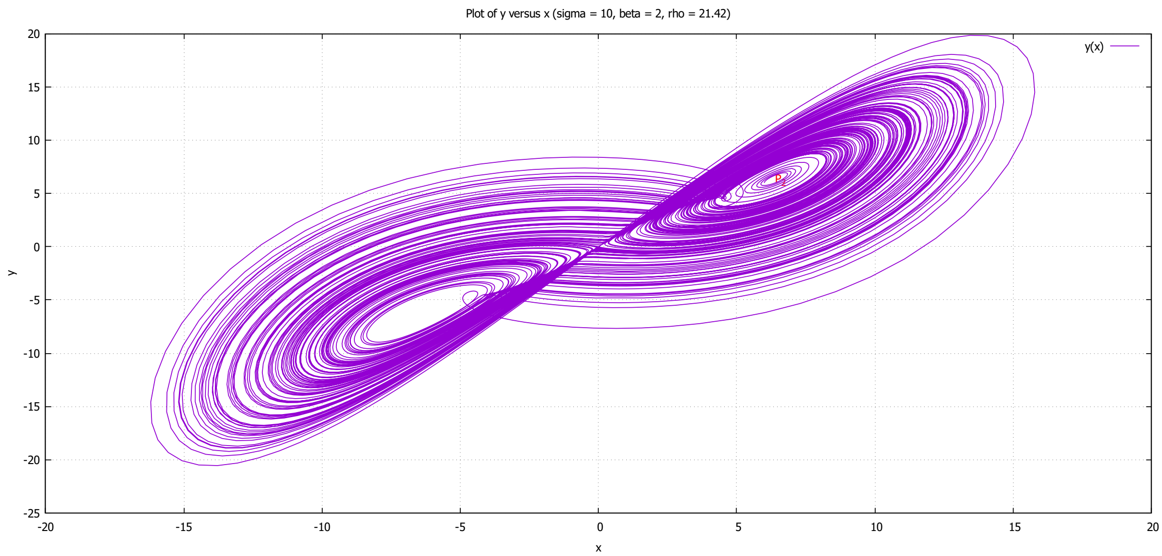

Figure 12: Plot of y vs x

2.7 For a point near P_3 ($\rho < \rho_0$)

Following are the plots when initial point is $I_0 \equiv (-6, -6, 20)$ and $\sigma = 10, \beta = 2, \rho = 10$. Here $P_3 \equiv (-4.24, -4.24, 9)$ and $\rho_0 = 21.42$

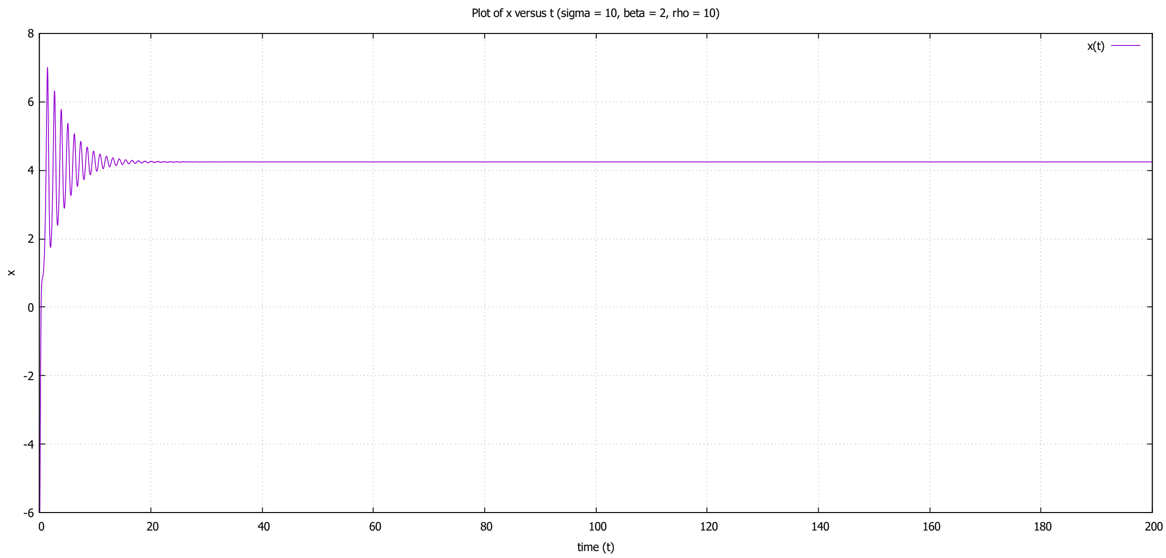

Figure 13: Plot of x vs t

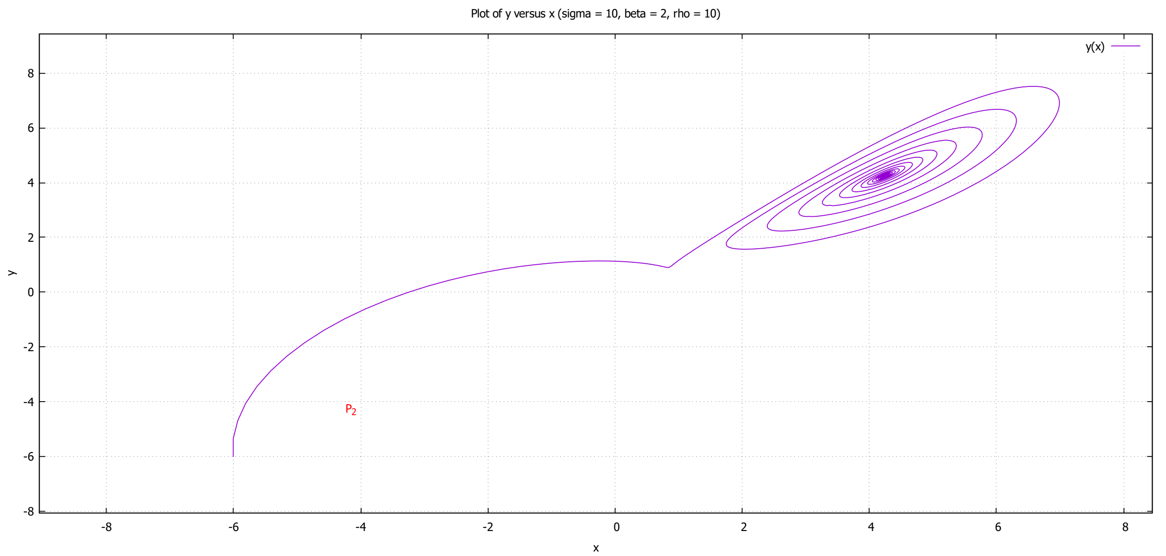

Figure 14: Plot of y vs x

2.8 For a point near P_3 $(\rho > \rho_0)$

Following are the plots when initial point is $I_0 \equiv (-6, -6, 20)$ and $\sigma = 10, \beta = 2, \rho = 25$. Here $P_3 \equiv (-6.92, -6.92, 24)$ and $\rho_0 = 21.42$

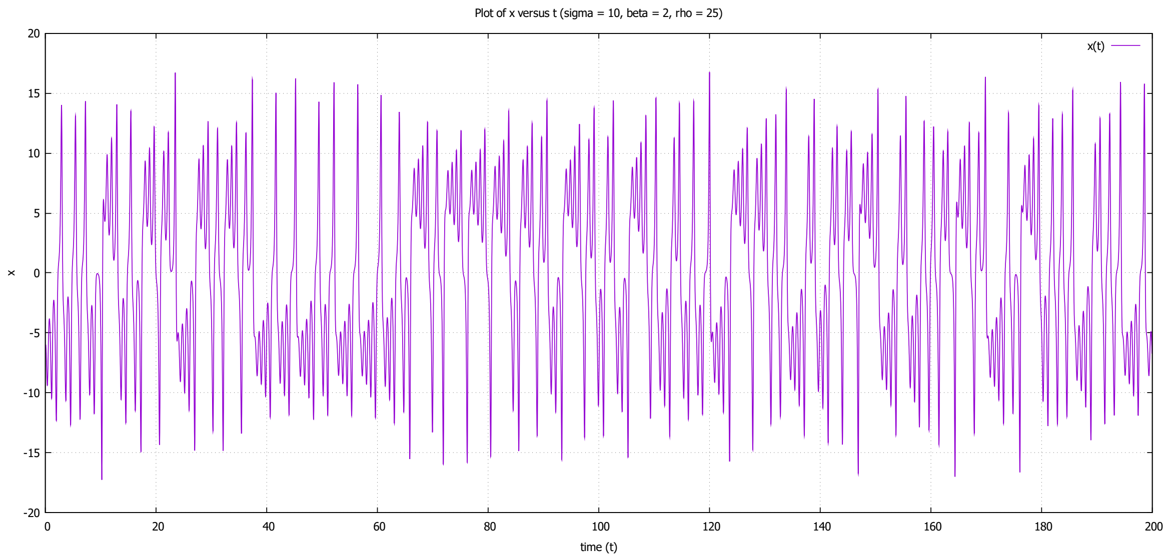

Figure 15: Plot of x vs t

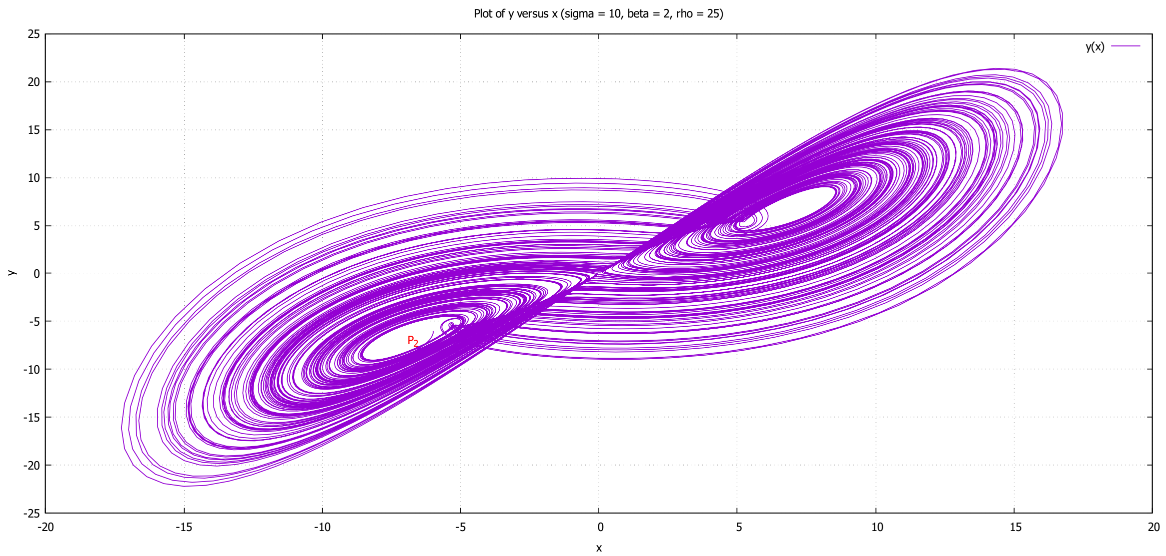

Figure 16: Plot of y vs x

2.9 For a point near P_3 ($\rho = \rho_0$)

Following are the plots when initial point is $I_0 \equiv (-6, -6, 20)$ and $\sigma = 10, \beta = 2, \rho = 21.42$. Here $P_3 \equiv (-6.39, -6.39, 20.42)$ and $\rho_0 = 21.42$

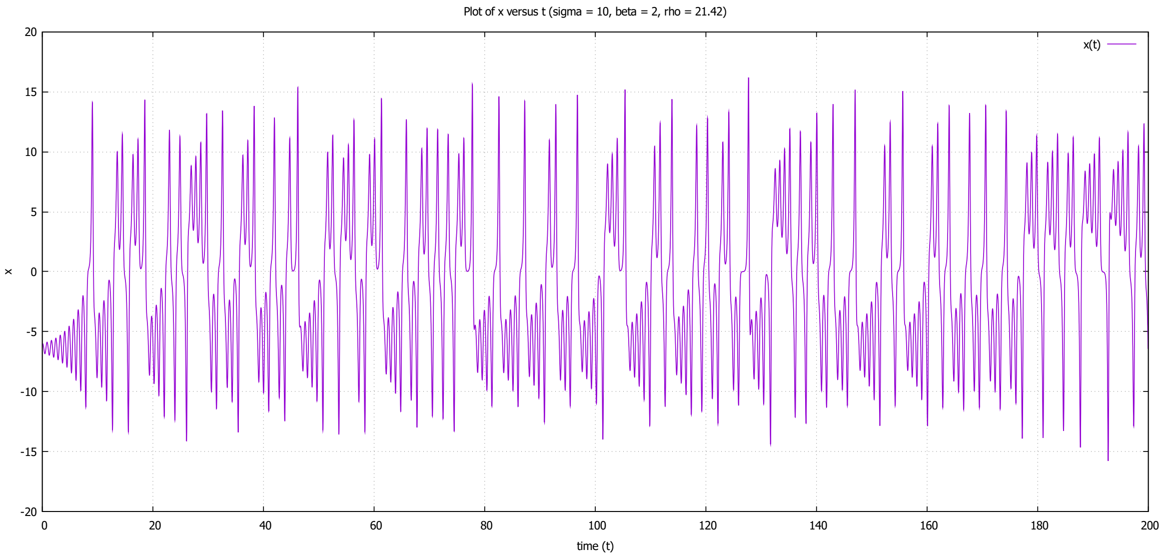

Figure 17: Plot of x vs t

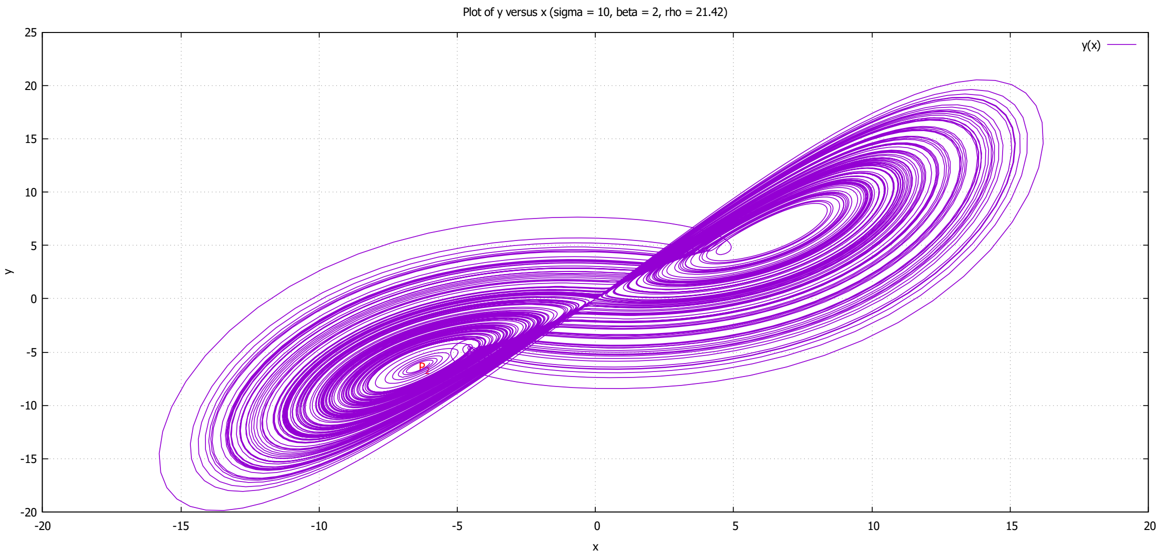

Figure 18: Plot of y vs x

2.10 For $\sigma = 10, \beta = 8/3, \rho = 28$

Following are the plots when initial point is $I_0 \equiv (2, 2, 5)$ and $\sigma = 10, \beta = 8/3, \rho = 28$.

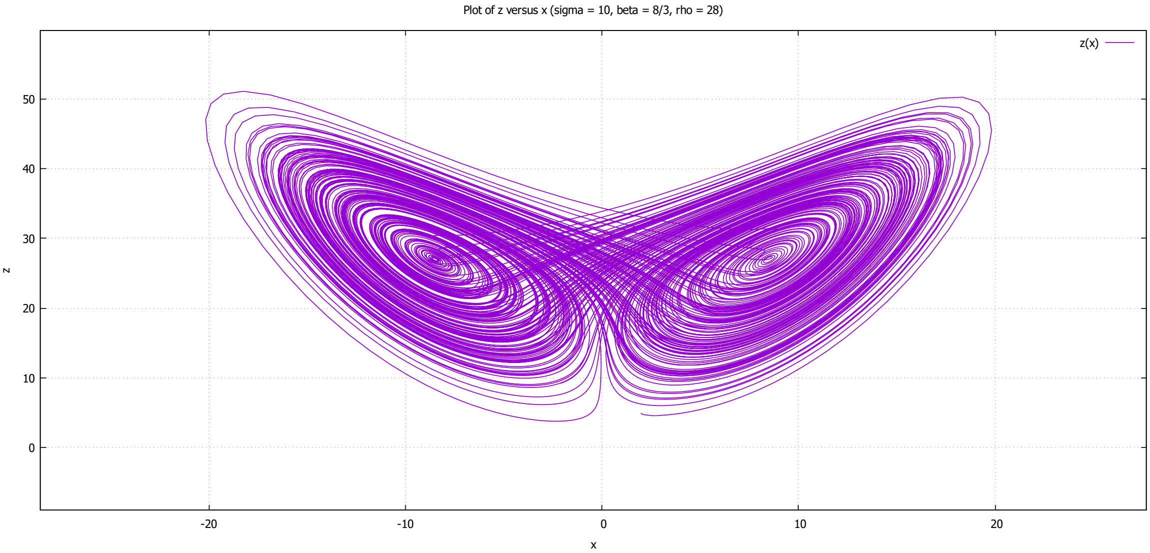

Figure 19: Plot of z vs x

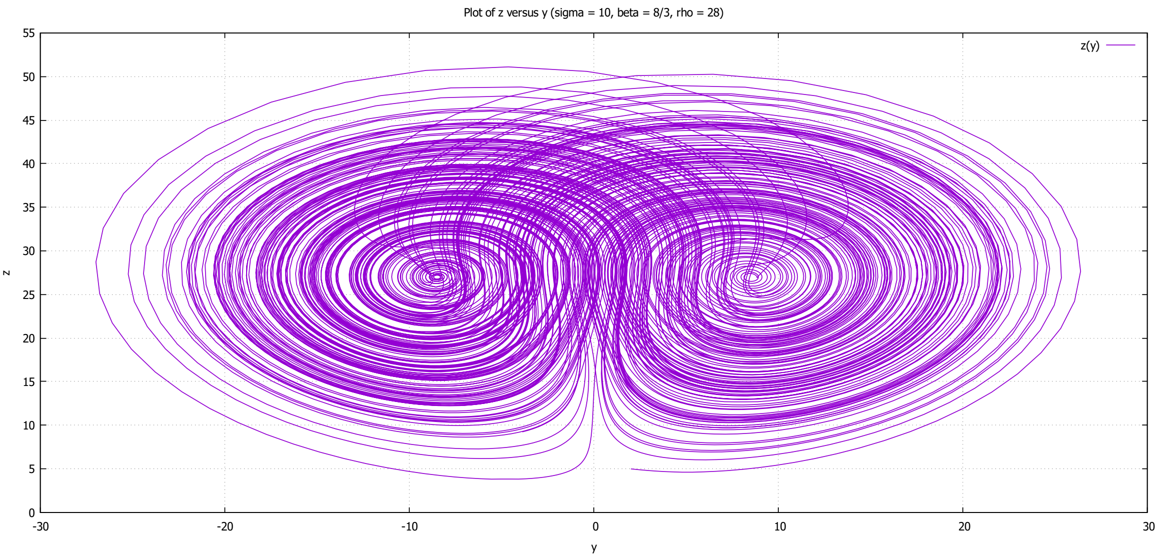

Figure 20: Plot of z vs y

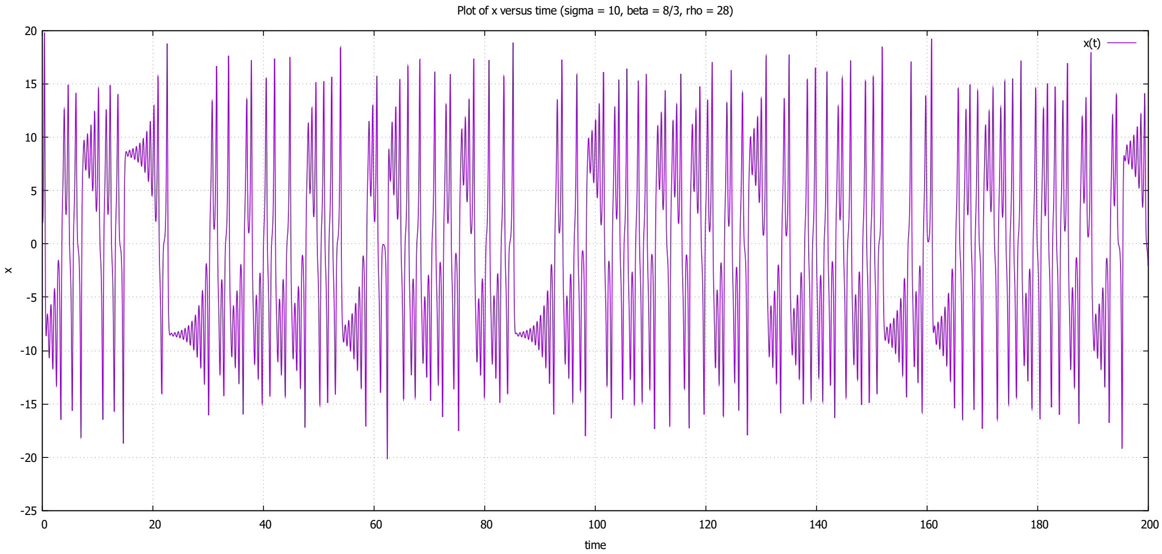

Figure 21: Plot of x vs t

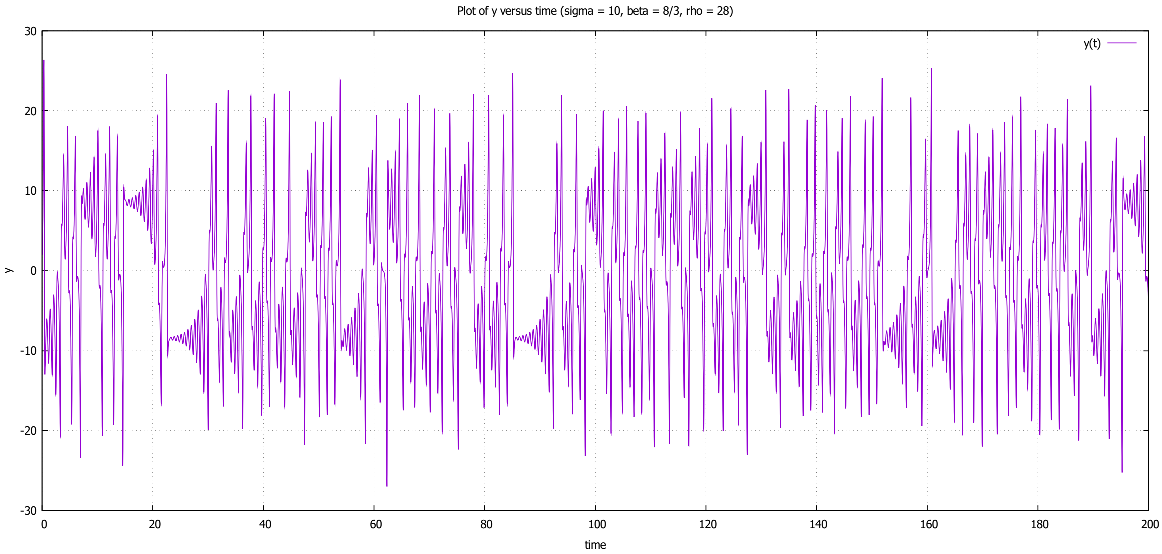

Figure 22: Plot of y vs t

3 Fortran code

Following pages contain the Fortran95 code used to generate above plots. The solution is generated using Euler's method with $h = 0.01$.

Following repo contains all the gnu scripts, plots and Fortran95 files: `https://github.com/mshreyes/Computational-Physics/tree/master/LSA`

```fortran
!Title: To find the solutions for Lorenz equations using Euler's method
!PRN: 2019P042
!Name: Shreyes Madgaonkar

program LORENZ
implicit none

!declaration of variables and arrays
real, dimension(:), allocatable :: X, Y, Z, T
real :: FN, GN, HN
real :: H
integer :: I, J

    !allocate arrays
    J = 25000
    allocate(X(J), Y(J), Z(J), T(J))

    !step size
    H = 0.01

    !initial conditions
    T(1) = 0
    X(1) = 2.0
    Y(1) = 2.0
    Z(1) = 5.0

    !Euler's method
    do I = 1, 20000

        X(I+1) = X(I) + H*FN(X(I), Y(I))
        Y(I+1) = Y(I) + H*GN(X(I), Y(I), Z(I))
        Z(I+1) = Z(I) + H*HN(X(I), Y(I), Z(I))

        T(I+1) = T(I) + H

    end do

    !store data in a file
    open(unit=21, file='plots\leqn1.dat')

        do I = 1, 20000
            write(21,*) T(I), X(I), Y(I), Z(I)
        end do

    close(21)

    !deallocate arrays
    deallocate(X, Y, Z, T)

end program LORENZ

!***********************************************************

real function FN(X, Y)
implicit none
```

```fortran
real, intent(in) :: X, Y
real :: SIG
SIG = 10

FN = SIG*(Y - X)
return

end function FN

!************************************************************

real function GN(X, Y, Z)
implicit none
real, intent(in) :: X, Y, Z
real :: R
R = 28

GN = X*(R - Z) - Y
return

end function GN

!************************************************************

real function HN(X, Y, Z)
implicit none
real, intent(in) :: X, Y, Z
real :: B
B = 8.0/3.0

HN = X*Y - Z*B
return

end function HN
```