Introduction to Martingales with Applications in Finance

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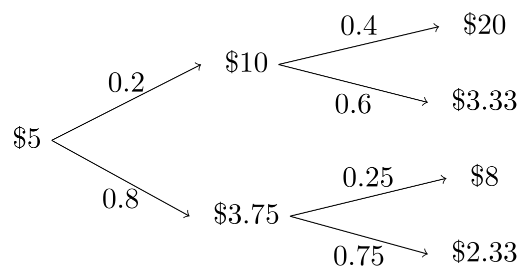
Recall the Markov Property, which states that "the probability of a future event depends on only the present state." We can create a stochastic process, Markov chains, based off this property. Now, we will introduce another type of stochastic process that is based off of a different assumption, called a **martingale**[1]. The martingale assumption states that the expected value of future states, given all past and present information, is equal to the value of the current state. In discrete time:

$$\mathrm{E}[X_{n+1} \mid X_n, X_{n-1}, ..., X_1, X_0] = X_n, \text{ for all } n$$

Exercise 1 (Two Period Binomial Pricing Model) The binomial pricing model is a simple model used to predict how the price of a stock will move in discrete time intervals. Given a stock is currently selling for X dollars, assume it has a probability to sell for $X + s$ at the next time with probability p, and that it has the probability to sell for $X - t$ at next time with probability $1 - p$. This can be extended through multiple time periods, resulting in a tree with many "branches" similar to coin flip models used in earlier probability classes.

Suppose you have a stock that is currently selling for \$5. Assuming discrete time intervals, the stock is expected to have a 20% chance to sell for \$10 tomorrow, and a 80% chance to sell for \$3.75 tomorrow. If the stock sells for \$10 tomorrow, the stock is expected to have a 40% chance to sell for \$20 the next day, as well as a 60% chance to sell for \$3.3333 the next day. If the stock sells for \$3.75 tomorrow, the stock is expected to have a 25% chance to sell for \$8 the next day, as well as a 75% chance to sell for \$2.3333 the next day. Prove that this simple scenario is, in fact, a martingale.

[1] The name "martingale" comes from the martingale betting strategy. Begin by betting \$1. The idea is that whenever you lose money, you bet twice the amount lost in the previous round on the next round of gambling, so that on the next win you are guaranteed to earn \$1 in profit, provided you do not run out of money before then.



Answer: Need to show expected value of next state given previous information = value of current state, for any time period. Use Law of Total Probability:

$$E[X_1 \mid X_0 = 5] = 0.2(10) + 0.8(3.75) = 5$$
$$E[X_2 \mid X_1 = 10] = 0.4(20) + 0.6(3.3333) = 10$$
$$E[X_2 \mid X_1 = 3.75] = 0.25(8) + 0.75(2.3333) = 3.75$$

Thus, this process is a martingale since the expected value of the next state is equal to the current state for all periods.

The collection of present information that we condition on, $X_n, ..., X_0$, is called the **filtration**, denoted $\mathcal{F}$.

Exercise 2 (Fair Coin Flip Filtration) Suppose we flip a fair coin three times. Denote heads = 1 and tails = 0. Let the sample space, or all possible outcomes, be denoted by Ω.

a) What is the sample space Ω? In other words, what are all the possible outcome sequences of this process?

   Answer: {000, 001, 010, 011, 100, 101, 110, 111}

b) Suppose we know the outcome of the first coin flip. What are all the possible events that could result from the first coin flip?

   Answer: H = First coin flip is heads, T = First coin flip is tails

c) How would we represent these events as sets of unique outcomes from our sample space?

   Answer: $H = \{100, 101, 110, 111\}$, $T = \{000, 001, 010, 011\}$

d) How would we represent the complements of these events as sets of unique outcomes from our sample space?

   Answer: $H^C = \{000, 001, 010, 011\}$, $T^C = \{100, 101, 110, 111\}$

e) How would we represent all possible unions of these events (and complements of events) as sets of unique outcomes from our sample space?

Answer: $H \cup T = \{000, 001, 010, 011, 100, 101, 110, 111\}$

f) How would we represent all possible intersections of these events as sets of unique outcomes from our sample space?

Answer: $H \cap T = \emptyset$

g) Suppose we know that the first coin flip resulted in heads. Which of the above events, complements, unions, and intersections have taken place? Include the sample space Ω, which represents the event of any possible outcome happening, and the empty set $\emptyset$, which represents a null event.

Answer: $\emptyset,\ H,\ T^C,\ H \cup T,\ \Omega$

h) What **unique** sets of outcomes are described by the events that occur when the first coin flip results in heads? What events can we pick that correspond to these sets of outcomes so that each unique set of outcomes appears only once?

Answer: $\emptyset$, $\{100, 101, 110, 111\}$, $\{000, 001, 010, 011, 100, 101, 110, 111\}$. We can pick $\{\emptyset,\ H,\ \Omega\}$ to describe these unique sets.

The **filtration** can be loosely defined as the accumulation of **all** present information we have in the form of all possible events that have taken place up until the present in a stochastic process.

- This includes not just simple events like "heads/tails on the first flip," but all of their unique complements, unions and intersections.
- However, we only want to describe each unique set of outcomes once.

In Exercise 1, the following events are **sufficient** to describe the filtration event space after the first coin flip out of three:

$$\mathscr{F}_1 = \{\emptyset,\ T = \{000, 001, 010, 011\},\ H = \{100, 101, 110, 111\},\ \Omega\}$$

Where $\emptyset$ is the null event and Ω is the sample space, or the event that any outcome happens.

In the discrete case with a finite set of outcome sequences, such as in Exercise 1, this is easier (but still very painstaking) to enumerate, as the variables $X_1, ..., X_n$ are sufficient to describe the accumulation of all information up until the present.

- However, this becomes much more complicated in the continuous-time case or in a stochastic process with infinite outcomes (such as when we have an infinitely long stochastic process).

- As a result, we can simplify our notation to denote the expected value of a future state conditioned on present information as

$$E[X_{n+1} \mid \mathcal{F}_n]$$

where $\mathcal{F}_n$ represents the filtration, or a measure of the accumulation of all the present events/information we have.

The **accumulation** of information is an important part of the definition of filtration as well. How does filtration "accumulate" information?

Exercise 3 (Fair Coin Flip Filtration) Suppose we flip a fair coin three times. Denote heads = 1 and tails = 0. Let the sample space, or all possible outcomes, be denoted by Ω.

a) Suppose we know the outcome of both the first and second coin flips. What are the possible events that could result from the first and second coin flips?

   Answer: HH, HT, TT, TH

b) How would we represent these events as sets of unique outcomes from our sample space?

   Answer: HH = {110, 111}, HT = {100, 101}, TT = {000, 001}, TH = {010, 011}

c) What are all the possible complements, unions, and intersections of these sets of unique outcomes?

   Answer: $\emptyset$, {110, 111}, {100, 101}, {000, 001}, {010, 011},

   {110, 111, 100, 101}, {110, 111, 000, 001}, {110, 111, 010, 011},

   {100, 101, 000, 001}, {100, 101, 010, 011}, {000, 001, 010, 011},

   {110, 111, 100, 101, 000, 001}, {110, 111, 100, 101, 010, 011},

   {110, 111, 000, 001, 010, 011}, {100, 101, 000, 001, 010, 011},

   {000, 001, 010, 011, 100, 101, 110, 111}

d) Compare the sets of outcomes that are possible once we know the first coin flip only ($\mathcal{F}_1$) to these possible sets of outcomes from knowing the first two coin flips ($\mathcal{F}_2$) in part (c). What do you notice? Are all the outcomes in $\mathcal{F}_1$ present in $\mathcal{F}_2$?

   Answer: Yes, $\mathcal{F}_1$ is completely contained in $\mathcal{F}_2$. Specifically, we see that $\emptyset$, {000, 001, 010, 011}, {100, 101, 110, 111} and Ω from $\mathcal{F}_1$ are found in $\mathcal{F}_2$

The definition of filtration as the **accumulation** of information to the present means that given a time s and a time t where $s < t$,

$$\mathcal{F}_s \subset \mathcal{F}_t$$

In other words, all the information at time s is contained in the present information at time t. In general, in the discrete case,

$$\emptyset \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$$

In the continuous case, for all $s < t$,

$$\emptyset \subset \mathcal{F}_s \subset \mathcal{F}_t$$

Therefore, when finding the expected value of a future state X_{n+1} conditional on present information, we only need the filtration at the current time, $\mathcal{F}_n$.

We can incorporate this concept of filtration as a measure of present information into our definition of martingales. We can rewrite the definition of a martingale as a stochastic process where:

$$\mathrm{E}[X_{n+1} \mid \mathcal{F}_n] = X_n, \text{ for all } n$$

If the present information defined in a martingale stochastic process can be completely described by a filtration $\mathcal{F}_n$ ($\mathcal{F}_n$-measurable), then the stochastic process is **adapted** to the filtration $\mathcal{F}_n$.

A **submartingale** is a different type of martingale in which the expected value of the next state is *greater* than the value of the current state, given all present information.

$$\mathrm{E}[X_{n+1} \mid \mathcal{F}_n] \geq X_n$$

- That is, over time, we expect a submartingale to have an upwards trend.

Similarly, a **supermartingale** is a different type of martingale in which the expected value of the next state is *less* than the value of the current state, given information on the present and all previous states.

$$\mathrm{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n$$

- That is, over time, we expect a submartingale to have a downwards trend.

Exercise 4 (Call Options) A call option on a stock is when an investor pays a premium to be able to buy a stock at a specified price (called the strike price) up until a specified expiration date. They are not obligated to exercise their right to purchase the stock at the strike price (since they paid a premium), but doing so at the correct time can net them a profit.

a) In what direction would an investor who bought a call option on a stock want the stock price to move and why?

   Answer: The investor would want the stock price to move up. That way, they could buy the stock at the strike price (which has been set and is unmoving upon purchase of the call option), and sell the stock at the higher price immediately for a profit.

b) If you were to simulate the investor's ideal direction of the stock price using a type of martingale, which would you use and why?

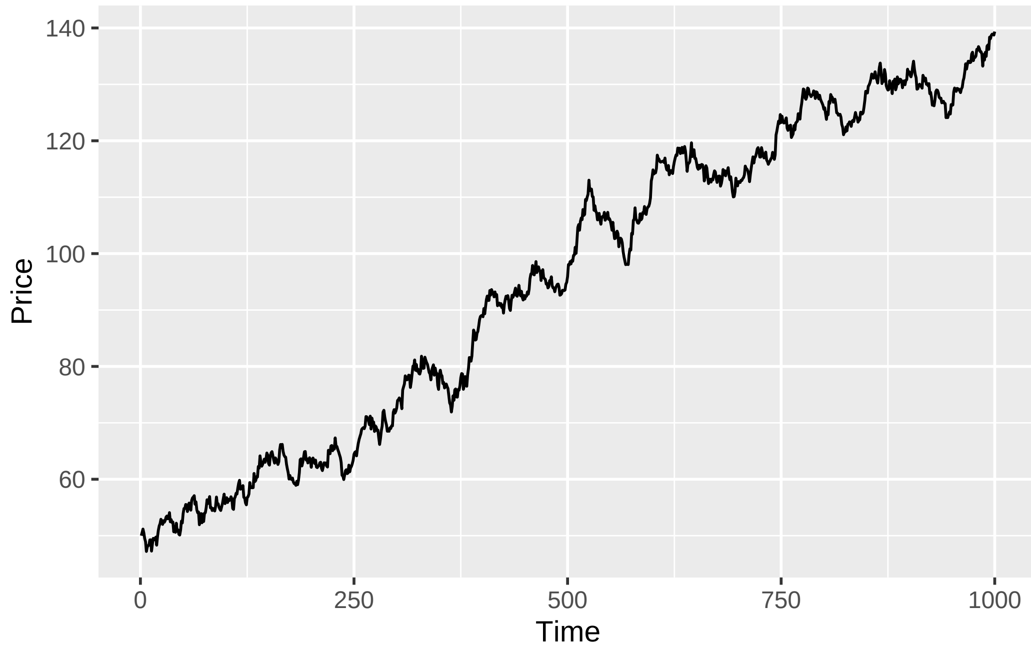
   Answer: We would use a submartingale, because then the expected price of the stock at the next time period would be expected to be higher than the current price of the stock. This would result in an upward trend.

c) Carry out the simulation of the investor's ideal direction of the stock price, with a starting stock price of \$50. Make sure you implement your answer to part (b) in the coding process somehow, and to include a visual as well.

   Answer: See example code and plot below.

```r
subsupermartingale <- function(current_price, nreps, mean, sd){
  curr = current_price
  currvec = c(curr)
  for (rep in 2:nreps){
    curr = curr + rnorm(1, mean = mean, sd = sd)
    currvec <- append(currvec, curr)
  }
  df <- data.frame(Time = rep(1:nreps),
                   Price = currvec)
  head(df)

  ggplot(data = df, aes(x=Time, y=Price)) +
    geom_line()
}
```

```r
subsupermartingale(50, 1000, 0.1, 1)
```



Exercise 5 (Put Options) A put option on a stock can be thought of as the opposite of a call option. It is when the investor pays a premium to SELL the stock at a specified strike price up until a specified date.

a) In what direction would an investor who bought a put option want the stock price to move and why?

   Answer: The investor would want the stock price to move down. If the stock price moves down and the investor has the ability to sell the stock at the strike price for more than its current market value, the investor will net a profit.

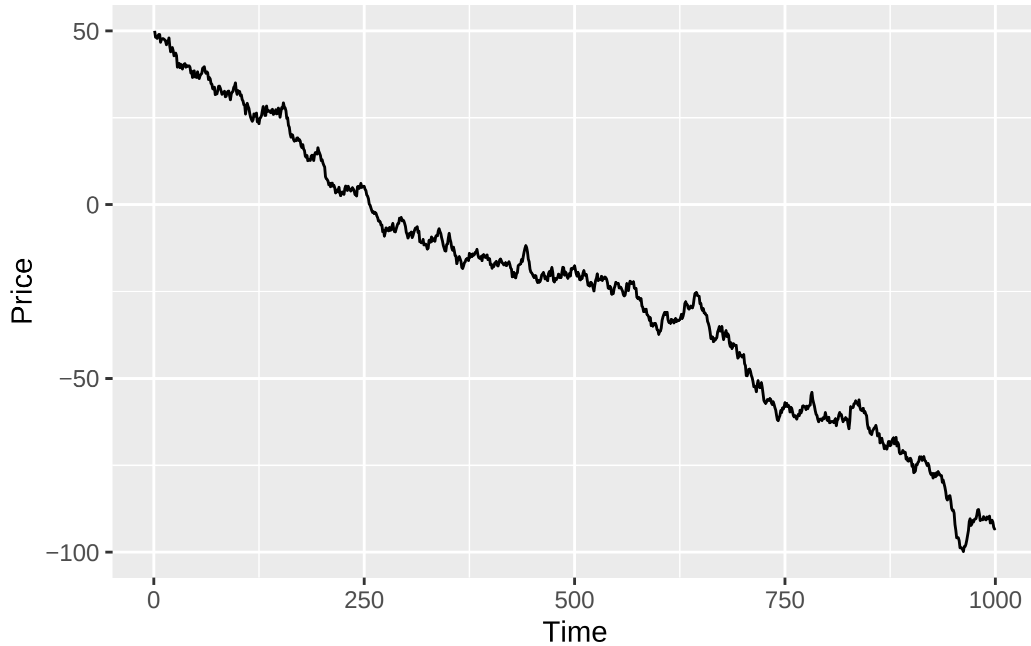
b) If you were to simulate the investor's ideal direction of the stock price using a type of martingale, which would you use and why?

   Answer: We would use a supermartingale, because then the expected price of the stock at the next time period would be expected to be LOWER than the current price of the stock. This would result in a downward trend.

c) Carry out the simulation of the investor's ideal direction of the stock price, with a starting stock price of $50. Include a visual as well. Note: your code should not be very different from your code in the call option example.

   Answer: See example code and plot below.

```r
subsupermartingale(50, 1000, -0.1, 1)
```



d) What are some clear issues with modeling "ideal scenarios" for stock prices with supermartingales or submartingales? Hint: it might be easier to tell with your put graph than your call graph.

Answer: Submartingales and supermartingales do not typically accurately reflect a stock's trend over time. For example, a stock will not trend downwards until it hits a negative price, and a stock will (typically) not keep skyrocketing upwards in price.

Exercise 6 (NVDA Martingale Inference) You are given recent stock data on NVIDIA and want to see if you can classify it as a martingale, submartingale, or supermartingale using the open prices for each day.

a) Propose a method for classifying the stock data as a type of martingale. Why do you think this method is valid?

Answer: Take the mean of the changes between opening stock prices. If the mean is positive, the stock behaves similarly to a submartingale, since an average positive difference means the stock typically moves upward. If the mean is negative, the stock behaves similarly to a supermartingale, since an average negative difference means the stock typically moves downward.

b) Use code to run through your method. What are your results?

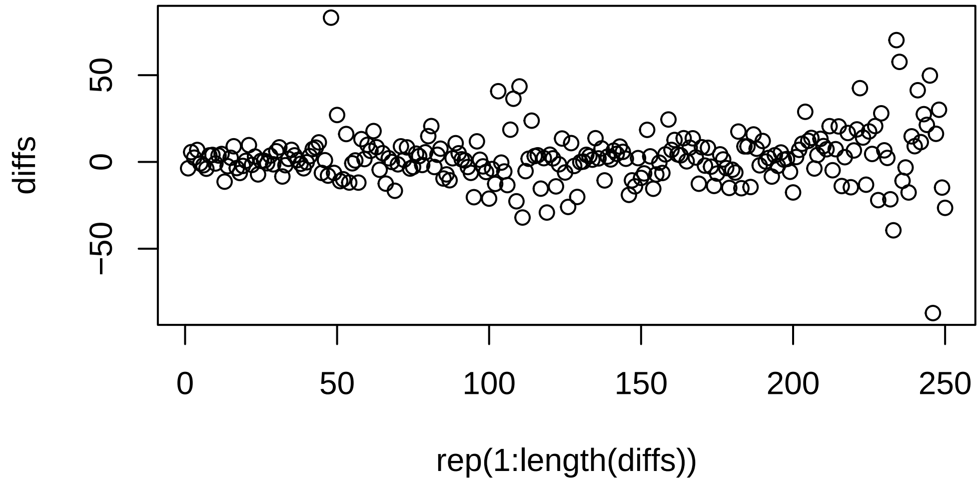
Answer: See example code below.

```
nvidia <- read.csv("https://query1.finance.yahoo.com/v7/finance/download/NVDA?period1=1679

nvidia$prevopen <- lag(nvidia$Open)

nvidia$opendiff <- nvidia$Open - nvidia$prevopen

diffs <- nvidia$opendiff[!is.na(nvidia$opendiff)]

mean_diffs <- mean(diffs)
mean_diffs
```

```
[1] 2.43792
```

Graph of differences:

```
plot(rep(1:length(diffs)), diffs)
```



The mean of the differences in opening price is about 2.438. Thus, we can classify this process as more likely to be a submartingale than either a martingale or a supermartingale.

The Markov and martingale properties are not mutually exclusive. Stochastic processes can exhibit both assumptions.

Exercise 7 (Brownian Motion) Suppose we have a standard Brownian motion $\{W(t)\}$.

a) Is $\{W(t)\}$ a Markov process? Explain.

Answer: Yes, because it is a continuous analog of a simple random walk. A random walk is a Markov process because the next state of the random walk only depends on the current state of the random walk.

b) Is $\{W(t)\}$ a martingale? Support your explanation with mathematical derivations based on the definition of a standard Brownian motion.

Answer: Yes. See the derivation below:

For all $s < t$,

$$E[W(t) \mid \mathcal{F}_s] = E[W(t) - W(s) + W(s) \mid \mathcal{F}_s]$$

By the Linearity of Expectation,

$$= E[W(t) - W(s) \mid \mathcal{F}_s] + E[W(s) \mid \mathcal{F}_s]$$
$$= E[W(t) - W(s) \mid \mathcal{F}_s] + W(s)$$

By the independent increments property of Brownian Motion, $(W(t) - W(s)) \perp\!\!\!\perp \mathcal{F}_s$. Therefore,

$$= E[W(t) - W(s)] + W(s)$$
$$= E[W(t)] - E[W(s)] + W(s)$$

By the definition of a standard Brownian motion, $W(t) \sim Normal(0, \sqrt{t})$. Therefore,

$$= 0 - 0 + W(s) = W(s)$$
$$\therefore E[W(t)|\mathcal{F}_s] = W(s)$$

Therefore a standard Brownian motion is a martingale.

A standard Brownian motion is an example of of a martingale stochastic process that also follows the Markov Property. Likewise, a Brownian motion with drift is an example of a sub- or supermartingale that also follows the Markov property.

What if we want to stop a martingale when we see certain states? For example, what if we want to sell a stock when we see a certain price pattern? A **stopping time** a random variable, T, representing the first time we see a certain event occur in our present information, $\mathcal{F}_n$.

Exercise 8 (Discrete Time and Continuous Time Stopping Time) Suppose we have the following discrete sequence of stock prices in our filtration $\mathcal{F}_6$:

t	0	1	2	3	4	5	6
X(t)	75	25	50	100	50	200	25

Define the stopping time T as $T =$ The first time we see a price of \$50 or \$100.

a) What is the event the stopping time is based on?

   Answer: The event that we see a price of \$50 or \$100, for the first time.

b) What is the value of the stopping time T in this example?

   Answer: $T = 2$

c) Suppose instead we have a continuously evolving stock price over time. In our filtration $\mathcal{F}_6$, we only know that $X(0) = 75$, $X(5) = 200$, and $X(6) = 25$. What do we know about T?

   Answer: We know T is a random time between time 0 and 5. This is because based off our current information, the value \$100 must have occurred at some point between $X(0)$ and $X(5)$

d) What is the expected value of the stock price at time T? That is, what is $\mathrm{E}[X(T)]$ in this example?

   Answer: Be careful, $E[X(T)] \neq 50$, 100 or any metric dependent on the two! Logically, by the definition of martingales, the expected value of $X(t)$ at any time is going to be constant and equal to the initial value, $X(0) = E[X_0]$. Therefore, $E[X(T)] = 75$.

The previous question is a demonstration of **Doob's Optional-Stopping Theorem**, which states that given a stopping time T,

$$\mathrm{E}[X_T] = E[X_0]$$

In other words, the expected value of the martingale at the stopping time is equal to the expected value of the initial value.

- This makes sense logically; if the expected value at the next time given all present information is equal to the present time, the expected value should remain constant over time, even at the stopping time.
- There are certain conditions under which this must be true. Most importantly, the event that T is based on must be bounded. For example, Exercise 8 is an example of a stopping time based on an event bounded below by 50 and above by 100.

We can use this property to solve important questions in finance.

Exercise 9 (Stop-Limit Order) Suppose an investor owns shares of a stock that is currently priced at $50. For simplicity, assume that this stock price follows a Brownian motion. The investor wants to sell these shares when the stock price hits $70 to net a profit, but he also wants to buy more shares when the price hits $40. What is the probability of the stock price hitting $70 before the stock price hits $40, as well as the probability the stock price hits $40 before hitting $70?

> *Answer: First note that our stopping time scenarios are bounded, meaning there is an upper bound ($70) and a lower bound ($40), which means that we can apply the optional stopping theorem. Then we can reason as follows:*
>
> $W_t = 70$ *or* $W_t = 40$ *with respective probabilities* p *and* $1 - p$. *Note:* W_t *is the value of the stock at stopping time t.*
>
> *Apply the optional stopping theorem:* $E[W_t] = E[W_0] = 50$
>
> *Now,* $E[W_t] = 70p + 40(1 - p) = 50$
>
> *Solve for* p *to yield* $p = 1/3$
>
> *Therefore there is a 1/3 probability that the stock price hits $70 before $40, and a 2/3 probability that the stock price hits $40 before $70.*

In general, martingales are an important part of modeling stock prices in finance. The infamous Black-Scholes formula for example, which is used to accurately price options, is based on martingale concepts. Understanding martingales can lead to the ability to make good decisions in the stock market.

References

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