Diff TraX

Differential Calculus

→ Lines and curves have slopes
→ You can use those slopes to find solves in real problems
→ There are a set of rules you need to follow to define those slopes
→ There is some underlying theory that helps to underpin how to do this

~~y = m~~

Straight line and slopes

$$y = mx + c$$

↑ slope

$$\text{slope} = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Find the derivatives of the function $f(x) = x^2 - 3x + 12$ by first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^2 - 3x - 3h + 12$$
$$= x^2 + 2xh + h^2 - 3x - 3h + 12$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 12 - x^2 + 3x - 12}{h}$$

$$= \lim_{h \to 0} \frac{(2xh + h^2 - 3h)}{h} = \lim_{h \to 0} 2x + h - 3 = 2x - 3$$

Tangent versus Derivative

Derivative is a way to show rate of change, the amount by which a function is changing at a given point.

→ When we find the derivative of a function we say that we "differentiate" the function and that the function is "differentiable"

$$y = 4$$

$$\frac{dy}{dx} = 4x^0 = 0 \times 4x^{0-1} = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Differentiating polynomial:

$$ax^n$$

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

Tangent line

tangent line of a functio $f(x)$ at a point $(a, f(a))$ is:

$$\boxed{y = f'(a)(x-a) + f(a)}$$

$f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2}$

$y = f'(a)(x-a) + f(a)$

$f(x) = 4x^6 - 6x^5 + 7x^4 + 8x^3 + 10x^2 - 7x - 1$

$f'(x) = 24x^5 - 30x^4 + 28x^3 + 24x^2 + 20x - 7$

$y = f'(3)(x-3) + f(3)$

Tangent $y = 4427(x-3) + 2129$

Tangents and Normal of a graph:

two lines are ~~norma~~ perpendicular exactly when the product of the slopes $m_1 m_2 = -1$

If ~~a~~ the point of the graph is $(a, f(a))$, and the derivative $f'(a)$ is not equal to zero at this point then the equation ~~the~~ of the line normal to the graph at this point is

$$\boxed{y = -\frac{(x-a)}{f'(a)} + f(a)}$$

Now $f(x) = 12x - 2$

$f'(x) = 12$

equation of line perpendicular to $f(x)$ at $x = -4$?

$$y = -\frac{(x+4)}{f'(-4)} + f(-4)$$

$$= -\frac{x+4}{12} \cancel{-48} - 50$$

~~Now, $f(x) = x^2$~~

Features of function:

Determine if a function is increasing or decreasing

$f(x) = \cancel{8}x^3 - 3x^2 + 4$

$f'(x) = 3x^2 - 6x.$

let $f'(x) = 0$

$3x(x-2) = 0$

$\Rightarrow x = 0$ or $x = 2$

so, $f(x)$ is increasing when $x < 0$, decreasing when $0 < x < 2$ and ~~it~~ is ~~creasi~~ increasing again when $x > 2$

Now

$\cancel{f(x) = -3x^6 - x}$

Stationary points:

$\frac{df}{dx} = 0$ → find x → plug it to $f(x)$.

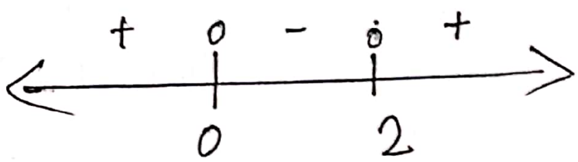
$\cancel{f(x)} = f(0) = 4$

$f(2) = 0$

Categorising Stationary points:

- Local maxima
- Local minima

Check using sign diagram:



$f'(-1) = +ve$

$f'(1) = -ve$

$f'(3) = +ve$

Now

$f(x) = 5x - 7 + 8x^{-1}$

$f'(x) = 5 - \frac{8}{x^2}$

Now, if $f'(x) = 0$

$\frac{5x^2 - 8}{x^2} = 0$

$x = 0, x = \sqrt{\frac{8}{5}}$

$f(0) = -7$

$f\left(\sqrt{\frac{8}{5}}\right) = -7 + 4\sqrt{10}$

For this point we get maxima

Concavity

Concave up → look like a cup

Concave down ⇒ look like a cap

$f(x) = x^3 - 3x^2 + 4$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

if $f''(x) > 0$ on a region then the graph of $f(x)$ is concave up on that region

if $f''(x) < 0$ on a region then the graph of $f(x)$ is concave down on that region

However if the value of $f''(x)$ is zero at a number c and the sign $f''(x)$ changes from positive to negative at c then the function $f(x)$ has a point of inflection at $(c, f(c))$

Point of inflection: A point where the graph is concave up on one side and concave down on the other.

	increasing	decreasing
Concave up		
Concave down		

<u>Now</u>

$\cancel{f(x) = 4x^4 - 4x^3 + 2x^2 - 9x + 9}$

$f(x) = 4x^4 - 4x^3 + 2x^2 - 9x + 16$

$f'(x) = 16x^3 - 12x^2 + 4x - 9$

$f''(x) = 48x^2 - 24x + 4$

<u>To sum up everything</u>

- First derivative test for increasing/ decreasing

  $f'(x) > 0$ on interval $\rightarrow$ $f(x)$ increasing on interval

  $f'(x) < 0$ on interval $\rightarrow$ $f(x)$ decreasing on interval

- Second derivative ~~d~~ test for concavity

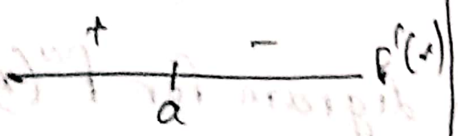
  $f''(x) > 0$ on interval $\rightarrow$ $f(x)$ concave up on interval

  $f''(x) < 0$ on interval $\rightarrow$ $f(x)$ concave down on interval

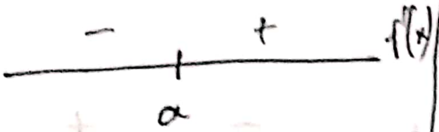
- First derivative test for stationary points

$f'(x) = 0$ at $x = a$ $\rightarrow$ $f(x)$ has stationary point at $x=a$

- Sign diagram for first derivative at stationary points



f(x) has local maximum at $x=a$



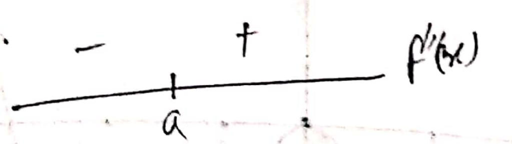
f(x) has local minimum at $x=a$

- Second derivative test for local maximum/minimum

$f''(a) > 0$, $x=a$ stationary point $\rightarrow$ $f(x)$ has local min at $x=a$

$f''(a) < 0$, $x=a$ stationary point $\rightarrow$ $f(x)$ has local max at $x=a$

- Sign diagram for second derivative at an inflection point



f(x) has a point at $x=a$

f(x) has a point at $x=a$

Graphing whole function:

$f(x) = x^4 - 24x^2$

$f'(x) = 4x^3 - 48x$

$f''(x) = 12x^2 - 48$

x intercepts:

if $f(x) = 0$

$x^4 - 24x^2 = 0$

$\Rightarrow x^2(x^2 - 24) = 0$

$x = 0;\ x = \pm 2\sqrt{6}$

turning points

if $f'(x) = 0$

$4x^3 - 48x$

$\Rightarrow 4x(x^2 - 12) = 0$

$\Rightarrow x = 0 \qquad x = \pm 2\sqrt{3}$

inflection points:

if $f''(x) = 0$

$12x^2 - 48$

$\Rightarrow x = \pm 2$

Sign digram for $f'(x)$

$-$		$+$		$-$		$+$	
	$-2\sqrt{3}$		0		$2\sqrt{3}$		x

Sign digram for $f''(x)$

$+$		$-$		$+$	
	-2		2		x

sketch