Final Examination: Mathematical Thinking – Section 3

(Dec 01, 2021): 1730-17:30 (Dec 04, 2023)

72 hours (Marks 360/2 = 180 Marks)

1. The Exam is open textbook only if you have bought one
2. Internet and any help using Internet (like ChatGPT, Google, etc) is not allowed and would be considered as Plagiarism
3. You will assemble in Capstone project groups and will discuss and solve Sections 1 and 2 This is Question 1 Upload your solution on DevOps
4. Question 2 is Reading the chapter 2 "Encryption" and understand the mathematical language of encryption You are tasked with understanding the chapter and solve 2 18 exercise problem and help Election Commission of Pakistan by developing in Python a Secure Voting Machine based on the idea of 2 18 Submit your solution and Python code on DevOps
5. Any person at Random will be called for an oral examination on October 26, 2021, to explain solutions and whatever marks he gets will be the marks of the whole group (please make sure that all group members understand the solutions)
6. In all proofs, show your complete rough work to illustrate your thought process and then at the end neatly convert them into a literary masterpiece of mathematical poems and prose
7. Remember, this course rewards the entire process of thinking and not a particular endpoint and result The thinking journey and discovery of different paths is more important than reaching the destination itself (that is also important in real life though)
8. BEST OF LUCK on your thinking endeavor

	Novice (0 points)	Apprentice (2 points)	Practitioner (4 points)
Logical Correctness	The answer given is fundamentally wrong.	The approach is generally correct, but there is at least one significant error.	Other than perhaps a minor slip, the proof is complete and correct.
Clarity	Overall, the argument is hard or impossible to follow	Can follow it with some effort. Some parts may be clearer than others	Clear and easy to follow throughout
Opening	No opening statement of what is being proved. No mention of use of standard method, where relevant (e.g. induction).	There is a statement of what is being proved (inc. mention of a standard method, if relevant), but it comes later and/or is incomplete.	Clear, correct opening statement of what is being proved, with statement of method if a standard method is used.
Stating the conclusion	Argument ends abruptly, without stating or acknowledging a conclusion	Argument ends with some form of concluding statement, but it is not clear and definitive	Argument concludes with a clear and concise statement indicating that the desired result has been established
Reasons	Significant steps presented without justification.	Some significant steps are justified, but at least one is not.	Reasons are given for all significant steps.
Overall valuation	Overall, this is not a good answer	The answer is fairly good, but there is room for improvement	Discounting small, minor slips, this is a good answer.

Question 1: Prove that if x, y □ R. Then |x + y| ≤ |x| + |y|.

Thinking Process Evidence:

1- I'll start by analyzing the possible values of x and y.

2- we can see if both x and y are positive or 0, then equality holds as $x+y = x+y$

3. Another case is if one of x or y is a negative number, the on the right side of inequality, the mod operator will make that number positive and we will get $x+y$.

4. Now, on the left side we will either have $|-x+y|$ or $|x-y|$.

5- On left side we have difference of two numbers and on right side of inequality we have sum of same two numbers

6- The difference of two numbers is always less then the sum.

Final and Neat Proof (follow Rubrics)

Proof:

We want to prove that $|x+y| \leq |x|+|y|$

We'll have 3 cases:

Case 1:

let's suppose $x=0$, $y=0$

then

$$|0+0| \leq |0|+|0|$$

$$0 \leq 0 \quad \text{True}$$

Case 2:

let's suppose x, y are positive numbers

say $x=m$, $y=n$

$$|m+n| \leq |m|+|n|$$

$$m+n \leq m+n \quad \text{True}$$

Case 3:

let's suppose ~~x~~ one of x, y is -ive

say $x=m$, $y=-n$

$$|m+(-n)| = |m|+|-n|$$

$$|m-n| \leq m+n$$

$|m-n|$ is difference of m and n
$m+n$ is sum of m and n and
difference is always less then sum.
so using arguments in mentioned 3 cases we have proved that

$|x+y| \leq |x|+|y|$ ∎

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Question 2: Prove are infinitely many Fibonacci numbers and they grow exponentially
[*Hint: Remember Fibonacci numbers* $f_n = f_{n-1} + f_{n-2}$ and induction helps in recursive reasoning]

Thinking Process Evidence:

1- We Know that first 3 numbers in fabonacci series are 0, 1, 2

2: Our base case is f (3), f(4)...

3: We Know $f_n = f_{n-1} + f_{n+2}$

4. Also $f_{n+1} = f_n + f_{n-1}$

5- And as n increases the ratio between two consecutive numbers in fabonacci series is ϕ (golden ratio)

6. We will assume that for f_K this holds and then prove it also holds for f_{K+1}

Final and Neat Proof (follow Rubrics)

Prof. We have to prove that fabonacci series is infinite and increases exponentially

we know that in fabonacci series

$$f_n = f_{n-1} + f_{n-2}$$

and $f(0) = 0$

$f_1 = 1$

$f_2 = 2$

$f_3 = 2+1 = 3$

we know, for some number n in fabonacci series $\frac{f_n}{f_{n-1}} \approx \phi$

Assume there are K numbers in series we have to prove that $(K+1)^{th}$ term exists

if $n = K$ $\quad \frac{f(K)}{f(K+1)} \approx \phi$

if $n = K+1$ $\quad \frac{f(K+2)}{f(K+1)} = \frac{f(K+1) + f(K)}{f_{K+1}} = 1 + \frac{f(K)}{f_{K+1}}$

if $\frac{f_{K+1}}{f_K} \approx \phi$ then $\frac{f_K}{f_{K+1}} \approx 1/\phi$

so $\frac{f_{K+2}}{f_{K+1}} \approx 1 + \frac{1}{\phi} \approx \phi$

so by mathematical induction we proved that there are infinite numbers in fabonacci series As their growth tends towards golden ratio and the increase is exponential with growth factor of ϕ. ∎

Question 3: Prove If n ∈ N and θ ∈ R, then [cos(θ) + i sin(θ)]ⁿ = cos(nθ) + i sin(nθ). *[Hint: "i" shows the complex number and try PMI for doing the proof]*

Thinking Process Evidence:

1- we will start with the base case $n=1$

2- The hypothesis is true for base cas

3- we will assume that hypothesis is true for some n and will try to prove for n+1

4- if we simplify $(\cos\theta + i\sin\theta)^{n+1}$ we will get $(\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta)$

5- we know $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
ie ou Assumption in step 3

6- we know $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

7- After step 3,4,5, ~~over~~ we are left with
~~cos(nθ)~~ $[\cos(n\theta) + i\sin(n\theta)][\cos\theta + i\sin\theta]$

$= \underline{\cos\theta \cdot \cos(n\theta)} + i\cos(n\theta)\sin\theta + i\cos\theta\sin(n\theta)$
$\underline{- \sin\theta\sin(n\theta)}$

8- The underlined terms are the identity in step 6.

9- if we take i common from other two terms
we get $i(\cos(n\theta)\sin\theta + \cos\theta\sin(n\theta))$
$= \sin(n\theta+\theta)$ using
$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$

10 so we are left with
$= \cos(n\theta+\theta) + i\sin(n\theta+\theta)$
$= \cos((n+1)\cdot\theta) + i\sin((n+1)\cdot\theta)$

11- we proved for n+1, so hypothesis is true.

Final and Neat Proof (follow Rubrics):

Proof: we have to prove

$[\cos(\theta) + i \sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta)$

Base case: $n=1$

$[\cos(\theta) + i\sin(\theta)]^1 = \cos(\theta) + i\sin(\theta)$

$(\cos(1\cdot\theta) + i\sin(1\cdot\theta)) = \cos\theta' + i\sin\theta$

Assume this holds for some $n=K$

$(\cos\theta + i\sin\theta)^K = \cos(K\theta) + i\sin(K\theta)$ — ①

We have to prove that this holds for $n=K+1$

$(\cos(\theta) + i\sin\theta)^{K+1} = (\cos\theta + i\sin\theta)^K(\cos\theta + i\sin\theta)$

we Know the value of $(\cos\theta + i\sin\theta)^K$ from ①

$= (\cos(K\theta) + i\sin K\theta)(\cos\theta + i\sin\theta)$

$= \cos(K\theta)\cos\theta + i\sin(K\theta)\cos\theta + i\sin\theta\cos(K\theta) - \sin\theta\,\cancel{K\theta}\,\sin(K\theta)$

$= \cos(\theta)\cos(K\theta) - \sin\theta\cdot\sin K\theta + i(\sin(K\theta)\cos\theta + \cos(K\theta)\sin\theta)$

using identities $\cos(\alpha+\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

and $\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$

we get

$= \cos(K\theta+\theta) + i(\sin(K\theta+\theta))$

$= \cos((K+1)\cdot\theta) + i(\sin[(K+1)\cdot\theta])$

we proved

$(\cos\theta + i\sin\theta)^{K+1} = \cos((K+1)\cdot\theta) + i\sin((K+1)\cdot\theta)$

By priciple of Mathematical Induction

we proved

$[\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta)$ ∎

Question 4: Prove that only prime triplet (i.e three primes, each at a difference of 2 from the next) is 3,5,7.

Thinking Process Evidence:

1- we know only even prime is 2.

2. if we take 3 consecutive odd numbers they have a difference of 2.

3- we will try to prove that in any 3 consecutive odd numbers, one will give remainder 0 after divided by 3.

4- if number is ~~divide~~ divisible by 3 it cannot be prime.

5- so no consecutive prime triplets exist.

Final and Neat Proof (follow Rubrics):

Proof:

We have to prove that there are no prime triplets except 3, 5, 7 with difference of 2 from next.

We know that only even number that is prime is 2. So our triplets must be odd.

Let 3 odd numbers with difference of 2

$2n+1, \quad 2n+3, \quad 2n+5$

Let's divide each by 3

- $2n+1$ leaves remainder 1
- $2n+3$ leaves remainder 0
- $2n+5$ leaves remainder 2

Now remainders are 1, 0, 2 consecutively. and when we will move through consecutive odd number. these are possible remainders and will repeat.

So if we pick any 3 consecutive odd no. one will leave remainder 0. So there cannot be any 3 consecutive prime numbers. each with difference of 2 from next. ■

Question 5: If a data structure of type tree has *n* vertices, then it has *n*−1 edges.

Thinking Process Evidence:

1- we will use induction to prove that tree with n ~~edges~~ vertice has n-1 edge.

2- let suppose n=1 , edges=0
n=2 , edge = 1
n=3 , edges=2

3- then will assume for som k it holds and will prove for k+1

4- if we remove a node from tree of (k+1) we will have Tree (k) with k-1 edges.

5- if we add one vertice to a tree , it only adds one edge so Tree(k+1) has k edges

6- This way our hypothesis is proved

Final and Neat Proof (follow Rubrics):

Proof.

We need to prove that ~~the~~ any tree with n ~~edges~~ vertices has (n-1) ~~ve~~ edges.

Base case: n=1 n-1 = 1-1 = 0 edges

Assume for any tree of k vertices, it has k-1 edges.

Now we will prove for k+1

we know if we remove a leaf node we will have tree with k vertices and will have k-1 edges.

To attach the vertice again we only need one edge so k-1 edges will be k-1+1 = k edges

using PMI we proved that tree with n vertices has n-1 edges. ∎

Question 6: Prove that any odd prime number p can be represented as sum of squares of two integers if and only if $p \equiv 1 \bmod 4$ where mod operator is modulo operator.

Thinking Process Evidence

let prime be p and integers be a, b.

$p = a^2 + b^2$

also $p = 4q + 1$ (p mod 4 = 1)

we know p is odd, so one of a^2 or b^2 should be ~~od~~ square of odd or in a, b one should be odd. so

suppos $a = 2n+1$
$b = 2m$

$(2m)^2 + (2n+1)^2 = 4q+1$

$4m^2 + 4n^2 + 2n + \cancel{1} = 4q + \cancel{1}$

$\cancel{4m^2 + 2n + 4q = 0}$

$\cancel{4n^2 + n + 2q}$

$4m^2 + 4n^2 + 4n + 1 = 4q + 1$

take left side and take mod 4

$0 + 0 + 0 + 1 \bmod 4 = 0 + 1$

$= 1 \bmod 4 = 01$

so $p \bmod 4 = 1$

Now In backward

$p \bmod 4 = 1 \equiv p = 4k + 1$

for backward part we know we have a^2+b^2 and prove it is equal to $1+4k$

so

$(2n)^2 + (2n+1)^2$

$= 4n^2 + 4n^2 + 4n + 1$

$= 1 + 4(n^2 + n^2 + n)$

$= 1+4(2n^2+n)$

$2n^2+n$ is to k

$= 1+4k$

Final and Neat Proof (follow Rubrics):

Proof:

we have to prove that any odd prime p can be represented as sum of squares of two integers if $p \bmod 4 = 1$

if p is odd and is sum of squares let's ~~sy~~ say a^2+b^2 then .
either a is odd or b is odd

$p = a^2+b^2$

$a = 2n$

$b = 2m+1$

then

$p = 4n^2 + 4m^2 + 4m + 1$ — ①

and $p = 4q+1$ ∴ using remainder theorem

$4q+1 = 4n^2 + 4m^2 + 4m + 1$

$0 + 1 = 0 + 0 + 0 + 1 \pmod 4$

$1 \equiv 1$

if we divide p by 4 we get 1 and if we divide a^2+b^2 by 4 we get 1

So all primes ~~of~~ in form a^2+b^2 will have remainder 1 when divided by 4

Now we have to prove if a prime is sum of squares then it will leave remainder 1
then we have to prove $p = 4k+1$

taking equation 1

$p = 4n^2 + 4m^2 + 4m + 1$

$p = 4(n^2 + m^2 + m) + 1$

where n, m are two number
let's suppose $n^2 + m^2 + m = k$

say $p = 4k+1$

Hence proved that any odd number p can be represented as sum of squares of two integer if and only if $p \equiv 1 \pmod 4$ ■