# ETHEREAL MISSION DESIGN TASK

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## Contents

# 1 Problem Description

Assume a launch vehicle with the specifications provided in Table 1 is launched from SDSC, SHAR to a Low Earth Orbit. With sufficient assumptions, calculate the following:

(*a*) The $\Delta V$ required including all applicable losses and launch azimuth to reach a prograde, circular orbit at 400 km above the Earth's surface and 22 degrees above the equator.

(*b*) For the same altitude and tilt angle as specified in (1a), calculate the additional $\Delta V$ required for a retrograde orbit and the subsequent launch azimuth.

(*c*) Calculate the propellant and structural masses of both the stages and payload mass that can be carried to reach the orbit described in (1a) assuming direct ascent.

(*d*) Calculate the feasible payload mass assuming a Hohmann transfer with a parking orbit at 250 km and compare with (1c).

(*e*) Calculate the time, altitude, velocity, and downrange of the stages at burnout for the trajectory calculated in (1d).

(*f*) Calculate the propellant and liftoff mass for the same rocket to carry 1.4 times the payload as calculated in (1c).

## 1.1 Table 1: Launch Vehicle Specifications

| **Parameter** | **Stage 1** | **Stage 2** |
|---|---|---|
| **Liftoff Mass (kg)** | 672,470 | |
| **Max Rocket Diameter (m)** | 6.5 | |
| **Thrust (kN)** | 8.2 MN | 1.2 MN |
| **No. of Engines** | 1 | 4 |
| **Isp (s)** | 278 | 302 |
| **Structural Coefficient** | 5.13% | 4.04% |

Table 1: Launch Vehicle Specifications

## 1.2 Part (a)

## 1.3 Launch Azimuth Calculation

The launch azimuth is given by the angle of the trajectory with respect to the north, and it is a function of the orbital inclination and latitude.

The launch azimuth $A$ is given by the following equation:

$$\cos(i) = \cos(\phi) \times \sin(A)$$

Where:

- $i = 22^\circ$ is the orbital inclination,

- $\phi = 13.7^\circ$ is the launch site latitude,
- $A$ is the launch azimuth.

Rearranging the equation to solve for $A$:

$$A = \arcsin\left(\frac{\cos(i)}{\cos(\phi)}\right)$$

Substituting the given values:

$$A = \arcsin\left(\frac{\cos(22^\circ)}{\cos(13.7^\circ)}\right)$$

Approximating the values:

$$A \approx \arcsin\left(\frac{0.927}{0.974}\right) \approx \arcsin(0.951) \approx 72^\circ$$

Thus, the launch azimuth $A$ is approximately $72^\circ$.

Now we have to calculate the total $\Delta V_{\text{required}}$ which is given by the sum of various components:

$$\Delta V_{\text{required}} = \Delta V_{\text{orbital velocity}} + \Delta V_{\text{total losses}} - \Delta V_{\text{gain}}$$

Orbital Velocity $\Delta V_{\text{orbital velocity}}$ which is given by:

$$\Delta V_{\text{orbital velocity}} = \frac{r}{\mu} \text{ where } \mu = 398600\,\text{km}^3/\text{s}^2 \text{ and } r = R_0 + h = 6378\,\text{km} + 400\,\text{km} = 6778\,\text{km}$$

Substituting the values:

$$\Delta V_{\text{orbital velocity}} = \frac{6778}{398600} = 7.67\,\text{km/s}$$

## 1.4 Calculation of Equatorial Velocity

The total angular velocity $w$ of the Earth combines its daily rotation and orbital motion around the Sun:

$$w = 2\pi + \frac{2\pi}{365.26}$$

Now, to convert this to radians per second, we divide by the number of seconds in a day:

$$w_{\text{total}} = \frac{2\pi + \frac{2\pi}{365.26}}{24 \times 60 \times 60}$$

Next, the equatorial velocity is given by:

$$V_{\text{eq}} = w_{\text{total}} \times R_{\text{Earth}}$$

Where $R_{\text{Earth}} = 6378\,\text{km}$.

Thus, the equatorial velocity is:

$$V_{\text{eq}} = \left( \frac{2\pi + \frac{2\pi}{365.26}}{24 \times 60 \times 60} \right) \times 6378\,\text{km}$$

After calculating, we find:

$$V_{\text{eq}} \approx 0.4651\,\text{km/s}$$

The velocity due to the Earth's rotation at the launch site (13.7 degrees latitude) is given by:

$$\Delta V_{\text{launch site}} = V_{\text{eq}} \times \cos(\phi)$$

Where $\phi = 13.7°$:

$$\Delta V_{\text{launch site}} = 0.4651\,\text{km/s} \times \cos(13.7°) = 0.4518\,\text{km/s}$$

Effective contribution of launch Azimuth

$$\Delta V_{\text{gain}} = 0.4518\,\text{km/s} \times \sin(72.7°) = 0.4312\,\text{km/s}$$

Now for losses The orbit speed required is given by:

$$\Delta V_{\text{orbit}} = \ln\left(\frac{m_0}{m_f}\right) - \int_0^{t_f} \frac{2T}{m} \sin^2\left(\frac{\delta + \alpha}{2}\right) dt - \int_0^{t_f} \frac{D}{m} dt - \int_0^{t_f} g \sin\gamma\, dt \tag{1}$$

The terms of Equations can be identified as follows:

- The first term represents the **ideal** $\Delta V_{\text{ideal}}$.
- The second term represents the **steering loss** $\Delta V_{\text{steering loss}}$.
- The third term represents the **drag loss** $\Delta V_{\text{drag loss}}$.
- The fourth term represents the **gravity loss** $\Delta V_{\text{gravity loss}}$.

Thus, the final representation is:

$$\Delta V_{\text{ideal}} = \Delta V_{\text{orbit}} + \Delta V_{\text{steering loss}} + \Delta V_{\text{drag loss}} + \Delta V_{\text{gravity loss}} + \Delta V_{\text{thrust loss}} \tag{2}$$

Here, the term $\Delta V_{\text{ideal}}$ represents the $\Delta V$ that the rocket must deliver—as calculated using the rocket equation—in order to end up with the orbital speed $\Delta V_{\text{orbit}}$. The steering loss $\Delta V_{\text{steering loss}}$ represents the loss of energy that occurs because the vehicle's engine(s) thrust line is not parallel to the velocity direction. The terms $\Delta V_{\text{drag loss}}$ and $\Delta V_{\text{gravity loss}}$ provide for the $\Delta V$ that the rocket loses due to aerodynamic drag and the component of the vehicle's weight that occurs along the velocity direction.

It is easy to see that because of these losses, it is necessary to design considerably more performance capability into the launch vehicle compared to that required to simply accelerate to orbital speed in a vacuum. It can also be seen that to minimize aerodynamic drag, the launch vehicle should climb vertically out of the atmosphere as quickly as possible, while to minimize gravity loss, the vehicle should turn horizontally ($\gamma \approx 0$) as quickly as possible—two conflicting actions.

Finally, one can see that the actual calculation of these losses and the necessary ideal speed requires detailed knowledge of the time histories of the vehicle's thrust, angle of attack, thrust vector control (TVC) angle, drag, weight, and flight path angle. **Without such knowledge, we can only estimate them using physics and statistical data from past launches. Procedures to estimate these losses will be covered in the following sections.**

## 1.5 Gravity losses

The first method uses conservation of energy. We wish to find the speed $\Delta v_{\text{grav loss}}$ that is required to coast upwards from a starting radius $r_0$ to zero speed at a final radius $r_f$ in a vacuum.

The total energy (kinetic + potential) must be conserved:

$$K_0 + U_0 = K_f + U_f$$

where:

- $K = \frac{1}{2}v^2$ is the kinetic energy per unit mass,
- $U = -g_0 \frac{R_e^2}{r}$ is the gravitational potential energy at a distance $r$ from the center of the Earth,
- $g_0$ is the gravitational acceleration at the surface,
- $R_e$ is the Earth's radius.

Thus, applying energy conservation:

$$\frac{\Delta v_{\text{grav loss}}^2}{2} - g_0 \frac{R_e^2}{r_0} = 0 - g_0 \frac{R_e^2}{r_f} \tag{3}$$

Solving Equation (3.19) for $\Delta v_{\text{grav loss}}$, we find:

$$\Delta v_{\text{grav loss}} = \sqrt{2 g_0 R_e^2 \left( \frac{1}{r_0} - \frac{1}{r_f} \right)}$$

(4)

If the vehicle launches from the Earth's surface at sea level (SL), then $r_0 = R_e$, and $h = r_f - R_e$ (altitude above Earth's surface). Substituting these:

$$r_f = R_e + h$$

reduces to:

$$\Delta v_{\text{grav loss–SL}} = \sqrt{2g_0R_e^2\left(\frac{1}{R_e} - \frac{1}{R_e+h}\right)} \tag{5}$$

$$= \sqrt{\frac{2g_0R_eh}{R_e+h}} \tag{6}$$

$$= \sqrt{\frac{2g_0h}{1+\frac{h}{R_e}}} \tag{7}$$

Substituting the known values:

$$g_0 = 9.80665\,\text{m/s}^2, \quad h = 400 \times 10^3\,\text{m}, \quad R_e = 6,378,137\,\text{m}$$

$$\Delta v_{\text{grav loss-SL}} = \sqrt{\frac{2 \times 9.80665 \times 400 \times 10^3}{1 + \frac{400\times10^3}{6,378,137}}}$$

From conservation of energy, the theoretical gravity loss $\Delta v_{\text{grav, theoretical}}$ is given by:

$$\Delta v_{\text{grav, theoretical}} = 2.717046567\,\text{km/s}$$

However, to account for real-world effects such as:

- Propellant consumption (reducing required thrust over time),
- Thrust not being perfectly antiparallel to the gravity vector (due to gravity turn),

it is recommended to multiply the theoretical value by a correction factor of 0.8 to obtain a more realistic estimate:

$$\Delta v_{\text{grav, actual}} = 0.8 \times \Delta v_{\text{grav, theoretical}}$$

Substituting the given value:

$$\Delta v_{\text{grav, actual}} = 0.8 \times 2.717046567$$

$$\Delta v_{\text{grav, actual}} = 2.1736372536\,\text{km/s}$$

Thus, the actual estimated gravity loss is:

$$\boxed{\Delta v_{\text{grav, actual}} = 2.1736\,\text{km/s}}$$

## Drag Loss Estimation

As was the case with gravity losses, drag loss can be calculated as an integral:

$$\Delta v_{\text{drag loss}} = \int_{t_0}^{t_{\text{final}}} \frac{D}{m}\,dt \tag{8}$$

Expanding the drag force:

$$\Delta v_{\text{drag loss}} = \int_{t_0}^{t_{\text{final}}} \frac{C_D(M,\alpha)S_{\text{ref}}}{2}\frac{\rho(h)v^2(t)}{m(t)}\,dt$$

(9)

It is evident from Eq. that to exactly calculate a vehicle's $\Delta v_{\text{drag loss}}$, both the drag force $D$ and the vehicle's mass $m$ as functions of time must be known. This, in turn, requires the following vehicle and trajectory information.

Aerodynamic drag is the resistance offered by air to a body moving through it. The drag force $D$ acts in the opposite direction of the velocity vector and is given by:

$$D = C_D \frac{1}{2} \rho v^2 S_{\text{ref}} \tag{3.24}$$

where:

- $C_D$ is the drag coefficient,
- $\rho$ is the air density,
- $v$ is the speed of the launch vehicle,
- $S_{\text{ref}}$ is the reference area (typically the maximum cross-sectional area normal to the flow).

The drag coefficient $C_D$ is usually taken as a percentage of the maximum drag coefficient $C_{D_{\max}}$, which varies with Mach number. From Fig. 3.13, $C_{D_{\max}}$ is approximately 0.42.

To accurately estimate the energy required to overcome drag during ascent, the deceleration caused by drag per unit mass must be integrated through the powered ascent phase:

$$\Delta v_{\text{drag loss}} = \int_0^{t_{\text{burn}}} \frac{D(t)}{m(t)}\, dt \tag{3.25}$$

where $t_{\text{burn}}$ is the total burn time.

Since velocity and altitude information as functions of time are generally unknown at this stage, explicit calculation is impossible. Therefore, an approximate method must be used, as presented in [?]. The estimation curve used is shown in Fig.

Using Fig. to obtain the parameter $K_D$, the velocity loss due to drag can be estimated as:

$$\Delta v_{\text{drag loss}} = \frac{K_D C_D S_{\text{ref}}}{W_0} \tag{3.26}$$

where:

- $K_D$ = Drag estimation factor
- $C_D$ = Drag coefficient (assumed)
- $S_{\text{ref}}$ = Reference area (maximum cross-sectional area)
- $W_0$ = Initial vehicle weight

Other points to note:

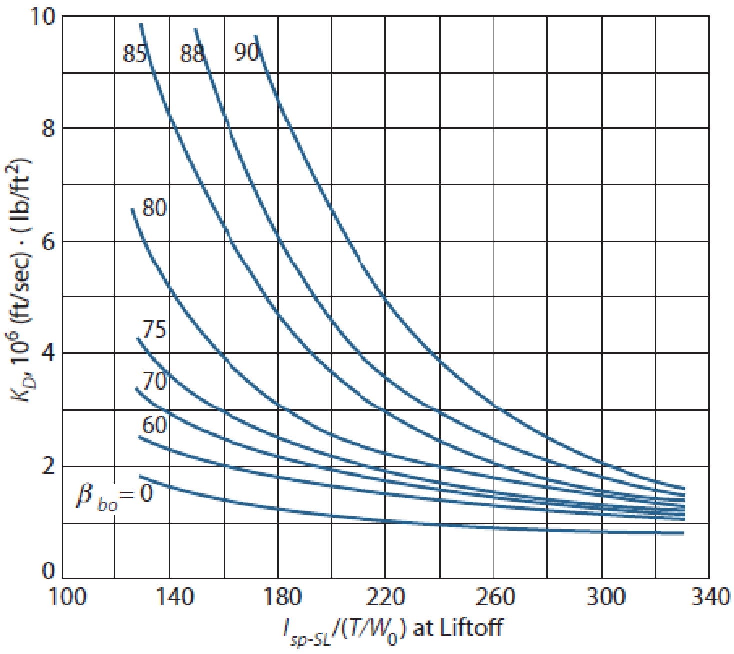


Figure 1: Drag loss estimation curves..

- Typically, $C_D \approx 0.42$ is used as a guideline, although it is significantly higher in the transonic region.
- Drag losses are usually smaller compared to gravity losses, typically in the range of 20–200 m/s (0.02–0.20 km/s).

The given values are:

$$K_D = 13{,}132{,}800$$
$$C_D = 0.42$$
$$S_{\text{ref}} = 33.1830724\,\text{m}^2$$
$$W_0 = 6{,}594{,}677.926\,\text{N}$$

Substituting the values into the formula:

$$\begin{aligned}
\Delta v_{\text{drag loss}} &= \frac{13{,}132{,}800 \times 0.42 \times 33.1830724}{6{,}594{,}677.926} \\
&= \frac{183{,}041{,}607.3}{6{,}594{,}677.926} \\
&\approx 27.754\,\text{m/s}
\end{aligned}$$

Converting to km/s:

$$\Delta v_{\text{drag loss}} = 0.027754\,\text{km/s}$$

Thus, the estimated drag loss is:

$$\boxed{\Delta v_{\text{drag loss}} = 0.027754\,\text{km/s}}$$

## Propulsion Losses and Thrust Loss Calculation

Propulsion losses occur due to non-optimal nozzle expansion when the nozzle exit pressure is not equal to the ambient pressure. This happens because most rocket engines are optimized for a single exhaust pressure. The total thrust $T$ is the sum of momentum thrust and pressure thrust during boost, given by:

$$T = \dot{m}v_e + (P_e - P_1)A_e$$

Where: - $\dot{m}$ is the mass flow rate, - $v_e$ is the exhaust velocity, - $P_e$ is the nozzle exit pressure, - $P_1$ is the ambient pressure, - $A_e$ is the exhaust area.

The variation in thrust due to changes in altitude is shown in Figure **??**.

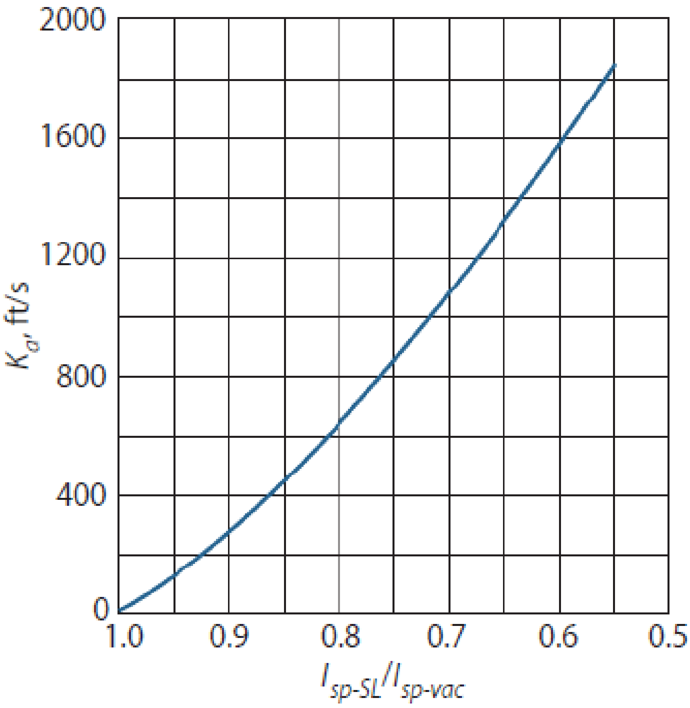


Figure 2: Thrust loss approximation curve.

Given: - Sea-level specific impulse $I_{sp-SL} = 278\,\text{s}$, - Vacuum specific impulse $I_{sp-vac} = 302\,\text{s}$, - Thrust-to-weight ratio $T/W_0 = 1.183$, - Propulsion drag $\Delta v_{\text{drag}} = 0.0728472\,\text{km/s}$, - PD Estimation Factor $K_a = 239$.

The ratio of specific impulses is calculated as:

$$\frac{I_{sp-SL}}{I_{sp-vac}} = \frac{278}{302} = 0.9205$$

From Figure, we can estimate the thrust loss parameter $K_a$. Fo convert from feet per second (ft/s) to meters per second (m/s), we use the following conversion factor $1\,\text{ft} = 0.3048\,\text{m}$. Thus, for $K_a = 239\,\text{ft/s}$, the conversion is:

$$K_a = 239\,\text{ft/s} \times 0.3048\,\frac{\text{m}}{\text{ft}} = 72.8472\,\text{m/s}$$

The thrust loss can be calculated as:

$$\Delta v_{\text{thrust loss}} = 0.07284\,\text{km/s}$$

## 1.6 Steering Losses

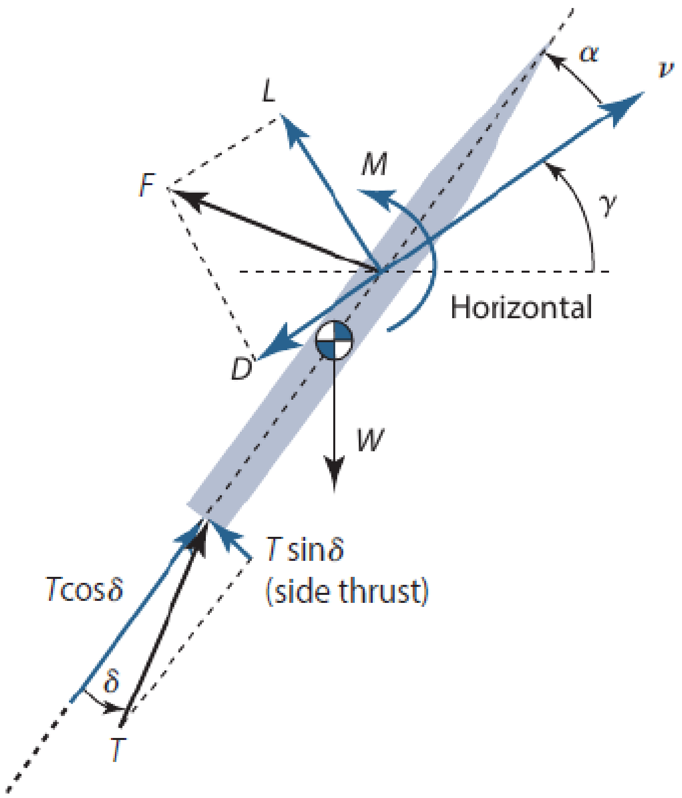


Figure 3: Forces on a launch vehicle in flight.

Steering losses occur when a vehicle's thrust vector is not parallel to its velocity vector. This can be understood by considering the definition of delivered mechanical power $P$, which is the scalar or "dot" product of the thrust vector $T$ and the velocity vector $v$, i.e.,

$$P = \mathbf{T} \cdot \mathbf{v}$$

Thus, it can be seen that any angular separation between thrust and velocity directions will lead to a loss of thrust power delivered to the vehicle. The angular separation may be seen in Figure **??**, where the sum of the angle of attack $\alpha$ and any thrust vector control angle $\delta$ is shown. Symbolically, the steering loss is given by:

$$\Delta v_{T \neq v} = \int_0^{t_b} T_v \left[1 - \cos(\delta + \alpha)\right] \, dt$$

Where: - $T_v$ is the thrust vector, - $\delta$ is the thrust vector control angle, - $\alpha$ is the angle of attack, - $t_b$ is the burn time.

In general, the steering losses are on the order of 30–400 m/s. The actual value usually depends greatly on the design of the upper step, specifically its thrust-to-weight ratio $c$ (whether the upper step is 'overpowered' or not). Upper stages with lower $c$-values need longer burn times to accelerate to orbital speed, and tend to have to spend more time "lofted," or intentionally sent to a higher-than-preferred altitude. During this period, they are flying at an angle of attack with their thrust vectors above the velocity direction to partially offset gravity, leading to higher steering losses.

The calculation of steering losses requires knowledge of the vehicle's angle of attack and thrust vector angles as a function of time; this information is usually not known at early stages of the design and is difficult to estimate. Therefore, we often use values from vehicles with similar expected performance.

### Steering Drag Estimation

Given the steering drag of 0.243 km/s, we can convert this value to SI units (m/s):

$$\Delta v_{\text{steering drag}} = 0.243\,\text{km/s} = 243\,\text{m/s}$$

This value represents the drag loss associated with the steering maneuver during ascent.

## Velocity Gain and Loss Calculation for 400 km Altitude

To calculate the total velocity required for achieving a 400 km altitude, we need to account for various velocity losses due to propulsion, gravity, drag, and steering losses. We will first sum up the velocity contributions and then compute the final required $\Delta v$.

- **Velocity Gain:** $\Delta v_{\text{gain}} = 0.43123421\,\text{km/s}$
- **Propulsion Drag:** $\Delta v_{\text{propulsion drag}} = 0.0728472\,\text{km/s}$
- **Actual Gravity Losses:** $\Delta v_{\text{gravity}} = 2.173637253\,\text{km/s}$
- **Drag Losses:** $\Delta v_{\text{drag}} = 0.027754258\,\text{km/s}$
- **Steering Drag:** $\Delta v_{\text{steering drag}} = 0.243\,\text{km/s}$
- **Total Velocity Losses:** $\Delta v_{\text{losses}} = 2.517238712\,\text{km/s}$
- **Total $\Delta v$ required:** $\Delta v_{\text{required}} = 9.754562677\,\text{km/s}$

### Summing the Velocities

To calculate the total $\Delta v_{\text{required}}$ for reaching a 400 km orbit, we need to sum the velocity gain and the velocity losses as follows:

$$\Delta v_{\text{total}} = \Delta v_{\text{propulsion drag}} + \Delta v_{\text{gravity}} + \Delta v_{\text{drag}} + \Delta v_{\text{steering drag}} - \Delta v_{\text{gain}}$$

Substituting the given values:

$$\Delta v_{\text{total}} = 0.0728472\,\text{km/s} + 2.173637253\,\text{km/s} + 0.027754258\,\text{km/s} + 0.243\,\text{km/s} - 0.43123421\,\text{km/s}$$

Thus, the total $\Delta v$ required to reach a 400 km orbit is:

$$\boxed{9.7546\,\text{km/s}}$$

## 1.7 Part (b): Calculation of $\Delta V$ for Retrograde Orbit

For a retrograde orbit, the $\Delta V$ is higher due to the opposite direction of the velocity vector relative to the prograde orbit. The required $\Delta V$ can be calculated similarly, but the launch azimuth will be different to adjust the launch trajectory.

#### 1. Launch Azimuth for Retrograde Orbit

The launch azimuth for the retrograde orbit is = 180 -72.7 = 107.3°.

#### 2. Orbital Velocity for Retrograde Orbit

The orbital velocity for a retrograde orbit at 400 km altitude is the same magnitude as the prograde orbit but in the opposite direction:

$$v_{\text{retrograde}} = 7.784\,\text{km/s}$$

#### 3. Earth's Rotational Speed at Launch Site

The rotational speed of Earth at the launch site (SDSC, SHAR) is 0.451791531 m/s.

#### 4. Total $\Delta v$ Required for Retrograde Orbit

The total $\Delta v$ required to reach the retrograde orbit, including all applicable losses, is calculated as:

$$\Delta v_{\text{retro}} = \Delta v_{\text{prograde}} + 2v_{\text{rot}}$$

$$\Delta v_{\text{retrograde}} = 10.613\,\text{km/s}$$

#### 5. Additional $\Delta v$ Required for Retrograde Orbit

The additional $\Delta v$ required to reach a retrograde orbit compared to the prograde orbit is:

$$\Delta v_{\text{additional}} = 0.86246842\,\text{km/s}$$

Thus, the launch azimuth for the retrograde orbit is 107.3°, and the additional $\Delta v$ required for the retrograde orbit is 0.86246842 km/s.

## 1.8 Part (c): Mass Calculation for Direct Ascent

The change in velocity $\Delta V$ is given by the Tsiolkovsky rocket equation:

$$\Delta V = I_{sp} \cdot g \cdot \ln\left(\frac{m_i}{m_f}\right)$$

Where:

- $\Delta V$ is the change in velocity,
- $I_{sp}$ is the specific impulse (s),
- $g$ is the standard acceleration due to gravity ($9.81\,\mathrm{m/s^2}$),
- $m_i$ is the initial mass,
- $m_f$ is the final mass.

Using the Tsiolkovsky rocket equation for each stage, we have:

$$\Delta v_1 = v_{e1} \ln\left(\frac{m_{01}}{m_{f1}}\right)$$

$$\Delta v_2 = v_{e2} \ln\left(\frac{m_{02}}{m_{f2}}\right)$$

We want to maximize the total velocity increment:

$$\Delta v_{\text{total}} = \Delta v_1 + \Delta v_2$$

Where:

- $m_{01}$ is the initial mass of the first stage (including the second stage and payload),
- $m_{f1}$ is the final mass of the first stage (structural mass $m_{s1}$ plus the initial mass of the second stage $m_{02}$),
- $m_{02}$ is the initial mass of the second stage (including the payload $m_{\text{pl}}$),
- $m_{f2}$ is the final mass of the second stage (structural mass $m_{s2}$ plus the payload $m_{\text{pl}}$).

The step mass for each stage $i$ is:

$$m_i = m_{s_i} + m_{\text{pl}}$$

The structural coefficient $\varepsilon$ is defined as:

$$\varepsilon = \frac{m_{s_i}}{m_{s_i} + m_{\text{pl}}}$$

The mass ratios for each stage are:

$$n_1 = \frac{m_{01}}{m_{f1}} = \frac{m_1 + m_2 + m_{\text{pl}}}{\varepsilon_1 m_1 + m_2 + m_{\text{pl}}}$$

$$n_2 = \frac{m_{02}}{m_{f2}} = \frac{m_2 + m_{\text{pl}}}{\varepsilon_2 m_2 + m_{\text{pl}}}$$

Solving for $m_0/m_{\text{pl}}$, we get:

$$\frac{m_0}{m_{\text{pl}}} = \frac{(1-\varepsilon_1)(1-\varepsilon_2)n_1 n_2}{(1-\varepsilon_1 n_1)(1-\varepsilon_2 n_2)}$$

Applying ln and considering this equation as $f(n_1, n_2)$ subjected to $g(n_1, n_2)$, we have:

$$\Delta v_{\text{total}} - v_{e1}\ln(n_1) - v_{e2}\ln(n_2) = 0$$

Using Lagrange multipliers and solving, we get:

$$n_1 = \frac{\eta v_{e1} - 1}{\eta v_{e1}\varepsilon_1}$$

$$n_2 = \frac{\eta v_{e2} - 1}{\eta v_{e2}\varepsilon_2}$$

Using the iterative method, we solve:

$$v_{e1}\ln\left(\frac{\eta v_{e1} - 1}{\eta v_{e1}\varepsilon_1}\right) + v_{e2}n_2 = \frac{\eta v_{e2} - 1}{\eta v_{e2}\varepsilon_2} = v_{\text{total}}$$

**Steps:**

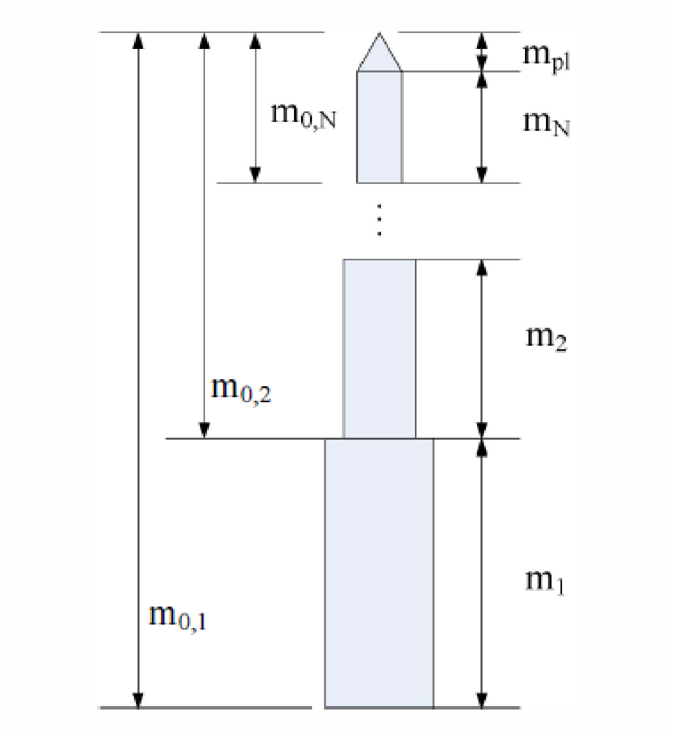


Figure 4: Mass definitions for serial staging

1. First, find $\eta$:
   - Solve for $\eta$ using the Lagrange multiplier method (iterative solution), based on the mission $\Delta V$ and the given propellant-specific constants for each stage.
2. Find $n_1$ and $n_2$:
   - Compute the mass ratios for each stage:

$$n_1 = \frac{1 + \eta C_1}{\eta C_1 \varepsilon_1}$$

$$n_2 = \frac{1 + \eta C_2}{\eta C_2 \varepsilon_2}$$

3. Find the payload mass $m_{\text{pl}}$:

    - Calculate the optimal payload mass using the following equation:

    $$m_{\text{pl}} = \frac{M_0}{(1 - \varepsilon_1 n_1)(1 - \varepsilon_2 n_2)} \cdot \frac{(1 - \varepsilon_1)(1 - \varepsilon_2) n_1 n_2}{}$$

    where $M_0$ is the total initial mass (liftoff mass).

4. Find the mass of the second stage $m_2$:

    - The second stage mass can be calculated as:

    $$m_2 = \frac{m_{\text{pl}}}{n_2 - 1} \cdot \frac{1}{1 - n_2 \varepsilon_2}$$

5. Find the structural and propellant masses for stage 2:

    - The structural mass of stage 2 is:

    $$m_{s2} = \varepsilon_2 m_2$$

    - The propellant mass of stage 2 is:

    $$m_{p2} = m_2 - m_{s2}$$

6. Find the mass of the first stage $m_1$:

    - The first stage mass is given by:

    $$m_1 = \frac{m_{\text{pl}} + m_2}{n_1 - 1} \cdot \frac{1}{1 - n_1 \varepsilon_1}$$

7. Finally, find the structural and propellant masses for stage 1:

    - The structural mass of stage 1 is:

    $$m_{s1} = \varepsilon_1 m_1$$

    - The propellant mass of stage 1 is:

    $$m_{p1} = m_1 - m_{s1}$$

| Parameter | Value | Unit | Description |
|---|---|---|---|
| Total $\Delta v_{\text{required}}$ | 9.754562677 | km/s | Total Delta-V required (from 1a) |
| Liftoff mass | 672470 | kg | Given liftoff mass |
| $g_0$ | 9.80665 | m/s$^2$ | Standard gravitational acceleration |
| $I_{sp1}$ | 278 | s | Specific impulse of Stage 1 |
| $I_{sp2}$ | 302 | s | Specific impulse of Stage 2 |
| $\epsilon_1$ | 0.0513 | - | Stage 1 structural fraction |
| $\epsilon_2$ | 0.0404 | - | Stage 2 structural fraction |
| $\eta$ | 0.492948557 | - | Staging parameter (solved using Excel Solver) |
| $V_{e1}$ | 2.7262487 | km/s | Exhaust velocity of Stage 1 (calculated) |
| $V_{e2}$ | 2.9616083 | km/s | Exhaust velocity of Stage 2 (calculated) |
| $\Delta v_1$ | 3.986553136 | km/s | Stage 1 Delta-V |
| $\Delta v_2$ | 5.768009598 | km/s | Stage 2 Delta-V |
| Total $\Delta v_{\text{calc}}$ | 9.754562734 | km/s | Calculated total Delta-V |
| $n_1$ | 4.315810776 | - | Stage 1 scaling factor |
| $n_2$ | 7.01179462 | - | Stage 2 scaling factor |

Table 2: Delta-V and Vehicle Specifications

| Parameter | Value | Unit | Description |
|---|---|---|---|
| Stage 1 Mass (W/O Mpl) | 544592.1256 | kg | Stage 1 mass excluding payload |
| Stage 1 Structural Mass | 27937.57604 | kg | Stage 1 structural mass |
| Stage 1 Propellant Mass | 516654.5496 | kg | Stage 1 propellant mass |
| Stage 2 Mass (W/O Mpl) | 114256.29 | kg | Stage 2 mass excluding payload |
| Stage 2 Structural Mass | 4615.954116 | kg | Stage 2 structural mass |
| Stage 2 Propellant Mass | 109640.3359 | kg | Stage 2 propellant mass |
| Payload Mass | 13621.58439 | kg | Payload mass |
| Mass Check (Should Equal Liftoff Mass) | 672470 | kg | Total mass check |

Table 3: Stage and Payload Mass Breakdown

## 1.9 Part (d): Payload Mass for Hohmann Transfer

A Hohmann transfer is performed using a parking orbit at 250 km altitude. The $\Delta V$ required for the transfer and the final payload mass can be calculated using the Hohmann transfer equation.

## Given Data

- Earth's radius, $R_E = 6378.137\,\text{km}$
- Gravitational parameter, $\mu = 398600.4418\,\text{km}^3/\text{s}^2$
- Initial altitude: $h_1 = 250\,\text{km}$
- Final altitude: $h_2 = 400\,\text{km}$
- Direct Ascent $\Delta V$: $9.7546\,\text{km/s}$

### Orbit Radii

$$r_1 = R_E + h_1 = 6378.137 + 250 = 6628.137\,\text{km}$$

$$r_2 = R_E + h_2 = 6378.137 + 400 = 6778.137\,\text{km}$$

**Semi-Major Axis of Transfer Orbit**

$$a = \frac{r_1 + r_2}{2} = \frac{6628.137 + 6778.137}{2} = 6703.137\,\text{km}$$

**Velocity in Initial Circular Orbit (at 250 km)**

Using the vis-viva equation:

$$v_{c1} = \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{398600.4418}{6628.137}} = 7.7548\,\text{km/s}$$

**Velocity at Perigee of Transfer Orbit (starting point)**

$$v_{t1} = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a}\right)} = \sqrt{398600.4418\left(\frac{2}{6628.137} - \frac{1}{6703.137}\right)} = 7.7981\,\text{km/s}$$

**First Impulse (Delta-V$_1$)**

$$\Delta v_1 = v_{t1} - v_{c1} = 7.7981 - 7.7548 = 0.0433\,\text{km/s}$$

**Velocity at Apogee of Transfer Orbit (at 400 km)**

$$v_{t2} = \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a}\right)} = \sqrt{398600.4418\left(\frac{2}{6778.137} - \frac{1}{6703.137}\right)} = 7.6255\,\text{km/s}$$

**Velocity in Final Circular Orbit (at 400 km)**

$$v_{c2} = \sqrt{\frac{\mu}{r_2}} = \sqrt{\frac{398600.4418}{6778.137}} = 7.6686\,\text{km/s}$$

**Second Impulse (Delta-V$_2$)**

$$\Delta v_2 = v_{c2} - v_{t2} = 7.6686 - 7.6255 = 0.0431\,\text{km/s}$$

**Total Delta-V for Hohmann Transfer**

$$\Delta V_{\text{Hohmann}} = \Delta v_1 + \Delta v_2 = 0.0433 + 0.0431 = 0.0864\,\text{km/s}$$

**New Total V required at 250 km**

The total $\Delta V$ required for the Hohmann transfer orbit, taking into account various losses and the velocity gain, is given by:

$$\Delta V_{\text{new}} = v_{c1} + \text{losses} + \Delta V_{\text{HT}} - \Delta V_{\text{gain}}$$

Where:

$$v_{c1} = 7.759\,\text{km/s} \quad \text{(velocity in initial circular orbit)}$$

$$\text{losses} = \text{gravity losses} + \text{drag losses} + \text{thrust losses} + \text{steering losses}$$

$$\Delta V_{\text{gain}} = 0.431\,\text{km/s}$$

The total losses at 250 km are calculated as:

$$\text{Total losses at 250 km} = 1.737746852 + 0.0277 + 0.0728472 + 0.243 = 2.081294052\,\text{km/s}$$

Thus, the total $\Delta V$ required for the Hohmann transfer orbit at 250 km is:

$$\Delta V_{\text{new}} = 7.759 + 2.081294052 + \text{V for HT} - 0.43123421 = 9.491189961\,\text{km/s}$$

### Delta-V Savings

Compared to direct ascent:

$$\Delta V_{\text{saved}} = 9.7546 - 9.4912 = 0.2634\,\text{km/s}$$

### Conclusion

By performing a Hohmann transfer with a parking orbit at 250 km, a Delta-V saving of approximately $0.2634\,\text{km/s}$ is achieved, enabling a significant increase in payload mass compared to direct ascent.

$$\text{New Payload Mass} = 13621.584\,\text{kg} + 1785.085\,\text{kg} = 15406.669\,\text{kg}$$

The percentage increase in payload is:

$$\text{Percentage Increase} = \frac{1785.085}{13621.584} \times 100 = 13.10\%$$

## 1.10 Part (e): Burnout Parameters for Trajectory

The trajectory parameters including time, altitude, velocity, and downrange at burnout can be calculated using the equations of motion for both stages, taking into account the changing mass and thrust over time.

Using the Tsiolkovsky rocket equation:

$$\Delta V = V_e \ln\left(\frac{m_0}{m_1}\right) \quad \Rightarrow \quad \frac{m_1}{m_0} = e^{-\Delta V/V_e}$$

Where:

- $m_0 = 672,470\,\text{kg}$ is the initial mass (liftoff mass),
- $\Delta V_1 = 4,199\,\text{m/s}$ is the change in velocity for stage 1,
- $V_{e1} = 2,726\,\text{m/s}$ is the exhaust velocity for stage 1.

$$\frac{m_1}{m_0} = e^{-4199/2726} \approx 0.215 \quad \Rightarrow \quad m_1 = 0.215 \times 672,470 \approx 144,594\,\text{kg}$$

Thus, the mass at burnout for Stage 1 is approximately $144,594\,\text{kg}$.

## Time to Stage 1 Burnout

The thrust ($T$) for Stage 1 is $8.2\,\text{MN} = 8,200,000\,\text{N}$, and the specific impulse ($I_{sp1}$) is 278 s.

$$\dot{m}_1 = \frac{T}{I_{sp1} \cdot g_0} = \frac{8,200,000}{278 \cdot 9.81} \approx 3,018\,\text{kg/s}$$

The time to burnout is given by:

$$\Delta t_1 = \frac{672,470 - 144,594}{3018} \approx 175\,\text{seconds}$$

## Altitude, Velocity, and Downrange at Stage 1 Burnout

The altitude at Stage 1 burnout is approximately $75\,\text{km}$.

The velocity at Stage 1 burnout is approximately $4.2\,\text{km/s}$ (from $\Delta V_1$).

The downrange at Stage 1 burnout is estimated as:

$$\text{Downrange} = \frac{\text{avg speed}}{2} \times \Delta t_1 = \frac{4.2\,\text{km/s}}{2} \times 175\,\text{seconds} = 367\,\text{km}$$

### Stage 2 Burnout Calculations

Using the Tsiolkovsky rocket equation for Stage 2:

$$\frac{m_2}{m_1} = e^{-\Delta V_2/V_{e2}} \quad \Rightarrow \quad \frac{m_2}{144,594} = e^{-5829/2963} \approx 0.14$$

Thus, the mass at burnout for Stage 2 is:

$$m_2 = 0.14 \times 144,594 \approx 20,243\,\text{kg}$$

## Time to Stage 2 Burnout

The thrust for Stage 2 is $1.2\,\text{MN} = 1,200,000\,\text{N}$, and the specific impulse ($I_{sp2}$) is 302 s.

$$\dot{m}_2 = \frac{1,200,000}{302 \cdot 9.81} \approx 405.4\,\text{kg/s}$$

The time to burnout for Stage 2 is:

$$\Delta t_2 = \frac{144,594 - 20,243}{405.4} \approx 306\,\text{seconds}$$

## Altitude, Velocity, and Downrange at Stage 2 Burnout

The altitude at Stage 2 burnout is approximately 250 km.

The velocity at Stage 2 burnout is approximately 7.8 km/s (typical LEO orbital velocity).

The downrange at Stage 2 burnout is estimated as:

$$\text{Downrange} = \frac{\text{avg speed}}{2} \times \Delta t_2 = \frac{(4.2 + 7.8)\,\text{km/s}}{2} \times 306\,\text{seconds} = 1,836\,\text{km}$$

Thus, the total downrange is:

$$\text{Total downrange} = 367\,\text{km} + 1,836\,\text{km} = 2,203\,\text{km}$$

## Final Results

| Parameter | Stage 1 Burnout | Stage 2 Burnout |
|---|---|---|
| Time (s) | 175 | 306 |
| Altitude (km) | 75 | 250 |
| Velocity (km/s) | 4.2 | 7.8 |
| Downrange (km) | 367 | 2,203 |

## 1.11 Part (f): Propellant and Liftoff Mass for Increased Payload

For carrying 1.4 times the payload calculated in (1c), the propellant and liftoff mass are recalculated using the same approach, with adjustments for the increased payload.

Given:

- $M_{\text{payload, old}} = 13621.58439\,\text{kg}$ (from part (1c))
- New payload mass: $M_{\text{payload, new}} = 1.4 \times M_{\text{payload, old}}$
- Stage 1 mass: $M_{\text{stage 1}} = 762428.9759\,\text{kg}$
- Stage 2 mass: $M_{\text{stage 2}} = 159958.806\,\text{kg}$
- Old liftoff mass: $M_{\text{liftoff, old}} = 672470\,\text{kg}$

**New payload mass**

$$M_{\text{payload, new}} = 1.4 \times 13621.58439\,\text{kg} = 19070.21815\,\text{kg}$$

**New Stage Masses with 1.4 times Payload Increase**

| **Stage** | **Structural Mass (kg)** | **Propellant Mass (kg)** | **Total Mass (kg)** |
|---|---|---|---|
| Stage 1 | 39112.60646 | 723316.3694 | 762428.9759 |
| Stage 2 | 6462.335762 | 153496.4702 | 159958.806 |
| Payload (new) | 19070.21815 | | |
| **New Liftoff Mass** | 936009.3663 | | |

Table 4: New stage masses with the increase in payload mass

**New total liftoff mass**

$$M_{\text{liftoff, new}} = 19070.21815 + 762428.9759 + 159958.806 = 936009.3663\,\text{kg}$$

**Difference in liftoff mass**

$$\Delta M_{\text{liftoff}} = 936009.3663 - 672470 = 263539.3663\,\text{kg}$$

**Percentage increase in liftoff mass**

$$\text{Percentage increase} = \frac{\Delta M_{\text{liftoff}}}{M_{\text{liftoff, old}}} \times 100 = \frac{263539.3663}{672470} \times 100 = 39.19\%$$

Thus, the new liftoff mass is 936009.3663 kg, with a 39.19% increase in liftoff mass compared to the original liftoff mass.

# 2 QUESTION 2

Describe the equations of motion for the launch vehicle described in Table 1 under the influence of gravitational, aerodynamic and thrust forces in planar ascent. Using calculus of variations and Pontryagin's principle to devise an optimal control problem to minimize the fuel consumption while achieving the target orbit by assuming thrust-to-weight ratio as control variable in the vector form. Derive the Hamiltonian function along with the costates and its equations and explain the condition of optimality. Introduce variations in atmospheric density, thrust magnitude, wind disturbances and initial launch conditions using a Monte Carlo simulation approach. Discuss the sensitivity of initial launch parameters and optimal control law on the maximum Q and acceleration loads, rate of climb, and degree of climb to these uncertainties.

# PART 1: Derivation of Launch Vehicle Equations of Motion

During the atmospheric ascent of a launch vehicle (LV), the vehicle is subjected to several forces, namely thrust, aerodynamic forces (lift and drag), gravitational force, and buoyancy. The resulting motion is governed by Newton's second law, where the sum of forces equals mass times acceleration. This document derives the planar equations of motion for the launch vehicle.

We model the ascent using a normal-tangential (local) coordinate frame attached to the vehicle's center of mass. The necessary equations of motion for numerical simulation are derived starting from inertial frame principles.

## 2.1 Coordinate Systems

### 2.1.1 Inertial Frame

The inertial frame is centered at the planet's center (point C), with:

- Horizontal (downrange) direction: $\mathbf{u_x}$
- Vertical (altitude) direction: $\mathbf{u_h}$

The position vector is:

$$\mathbf{r} = (R_p + h)\,\mathbf{u_h}$$

### 2.1.2 Local (Moving) Frame

A moving frame is attached at point G (center of mass of the vehicle):

- Tangential unit vector: $\mathbf{u_t}$ (aligned with the velocity vector)
- Normal unit vector: $\mathbf{u_n}$ (perpendicular to velocity vector)
- Out-of-plane unit vector: $\mathbf{u}_\perp$ (defined by $\mathbf{u_t} \times \mathbf{u_n}$)

## 2.2 Kinematics

### 2.2.1 Angular Velocity of the Local Frame

The angular velocity $\boldsymbol{\omega_l}$ of the local frame is:

$$\boldsymbol{\omega_l} = -\frac{v\cos\gamma}{R_p + h}\,\mathbf{u}_\perp$$

where $\gamma$ is the flight path angle.

### 2.2.2 Absolute Acceleration

From dynamics:

$$\mathbf{a} = \left(\frac{d\mathbf{v}}{dt}\right)_l + \boldsymbol{\omega_l} \times \mathbf{v}$$

Expanding:

$$\left(\frac{d\mathbf{v}}{dt}\right)_l = \dot{v}\mathbf{u_t} + v\dot{\gamma}\mathbf{u_n}$$

$$\boldsymbol{\omega_l} \times \mathbf{v} = -\frac{v^2\cos\gamma}{R_p + h}\mathbf{u_n}$$

Thus:

$$\mathbf{a} = \dot{v}\mathbf{u_t} + \left(v\dot{\gamma} - \frac{v^2\cos\gamma}{R_p + h}\right)\mathbf{u_n}$$

### 2.2.3 Components of Acceleration

- Tangential acceleration:

$$\mathbf{a_t} = \dot{v}\mathbf{u_t}$$

- Normal acceleration:

$$\mathbf{a_n} = \left(v\dot{\gamma} - \frac{v^2\cos\gamma}{R_p + h}\right)\mathbf{u_n}$$

## 2.3 Forces Acting on the Launch Vehicle

The forces acting are:

- Thrust force $T$
- Gravitational force $W = mg$
- Aerodynamic forces: drag $D$ and lift $L$
- Buoyancy force $B$

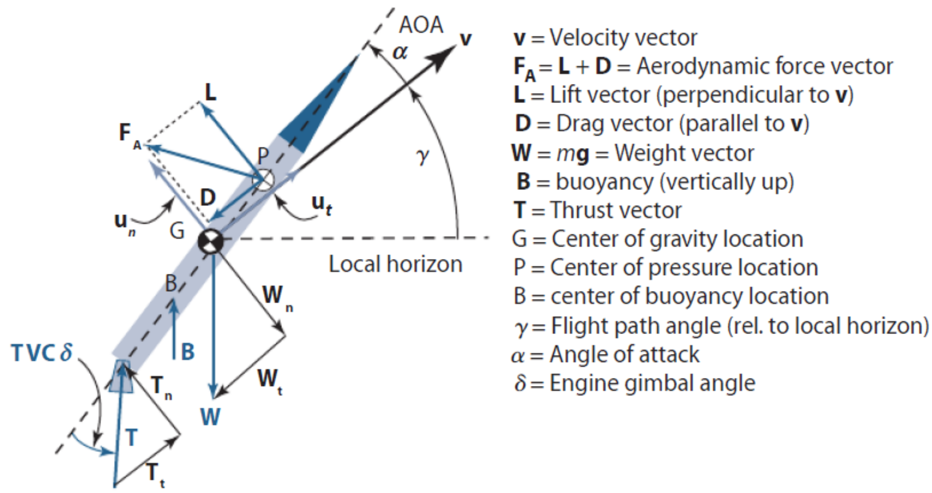


Figure 5: Launch vehicle forces and local frame orientation.

### 2.3.1 Aerodynamic Forces

$$D = \frac{1}{2} C_D \rho S v^2 \tag{10}$$

$$L = \frac{1}{2} C_L \rho S v^2 \tag{11}$$

where air density varies with altitude:

$$\rho(h) = \rho_0 \exp\left(-\frac{h}{H_s}\right) \tag{12}$$

### 2.3.2 Weight Force

- Gravity variation with altitude:

$$g(h) = g_0 \left(\frac{R_p}{R_p + h}\right)^2$$

- Components:

$$W_t = -mg \sin\gamma \quad , \quad W_n = -mg \cos\gamma$$

### 2.3.3 Thrust Force

- Thrust generation:

$$T = \dot{m} v_e + A_e (P_e - P_1)$$

- Components:

$$T_t = T \cos(\alpha + \delta) \quad , \quad T_n = T \sin(\alpha + \delta)$$

### 2.3.4 Buoyancy Force

- Components:

$$B_t = B \sin\gamma \quad , \quad B_n = B \cos\gamma$$

#### 2.3.5 Position Kinematics

$$\dot{x} = v \cos \gamma \tag{13}$$

$$\dot{h} = v \sin \gamma \tag{14}$$

## 2.4 Equations of Motion

### 2.4.1 Tangential and Normal Force Balance

- Tangential direction:

$$m\dot{v} = T_t - D + B_t + W_t$$

- Normal direction:

$$m\left(v\dot{\gamma} - \frac{v^2 \cos \gamma}{R_p + h}\right) = L + T_n + B_n + W_n$$

### 2.4.2 Simplified Equations under Assumptions

Under the following assumptions:

- Small angle of attack ($\alpha$) and thrust vector angle ($\delta$)
- Neglect Lift ($L \approx 0$) and Buoyancy ($B \approx 0$)

The final simplified equations become:

$$\dot{v} = \frac{T - D}{m} - g \sin \gamma$$

$$\dot{\gamma} = -\frac{g}{v} \cos \gamma + \frac{v}{R_p + h} \cos \gamma$$

$$\dot{m} = -\frac{T}{I_{sp} g_0}$$

where:

- $I_{sp}$ = Specific impulse (s)
- $g_0$ = Standard gravity ($9.80665\,\mathrm{m/s^2}$)

## 2.5 Planet Rotation Effects

- Launch site's initial inertial velocity due to planet rotation:

$$v_{\text{launch site}} = v_{\text{equator}} \cos \lambda$$

- For air-launched vehicles:

$$v_{\text{inertial}} = v_{\text{launch site}} + v_{\text{drop}}$$

## 2.6 Final Set of Differential Equations

The final system of equations governing the planar ascent are:

$$\boxed{\begin{aligned} \dot{v} &= \frac{T-D}{m} - g\sin\gamma \\ \dot{\gamma} &= -\frac{g}{v}\cos\gamma + \frac{v}{R_p+h}\cos\gamma \\ \dot{m} &= -\frac{T}{I_{sp}g_0} \\ \dot{x} &= v\cos\gamma \\ \dot{h} &= v\sin\gamma \end{aligned}}$$

# PART 2: Optimal Control Problem Formulation

In order to minimize the fuel consumption while achieving a specified target orbit, we formulate an optimal control problem. The thrust-to-weight ratio is considered as the control variable. The system dynamics are governed by the following equations:

## 2.7 State Variables

The state vector is defined as:

$$\mathbf{x} = \begin{bmatrix} h \\ x \\ v \\ \gamma \\ m \end{bmatrix}$$

where:

- $h$ = altitude,
- $x$ = downrange distance,
- $v$ = velocity,
- $\gamma$ = flight path angle,
- $m$ = mass of the vehicle.

## 2.8 Control Variable

The control input is the thrust-to-weight ratio:

$$u(t) = \frac{T(t)}{m(t)g_0}$$

## 2.9 Equations of Motion

The equations of motion are rewritten incorporating the control $u(t)$:

$$\dot{h} = v \sin\gamma \tag{15}$$
$$\dot{x} = \frac{v\cos\gamma}{1+\frac{h}{R_p}} \tag{16}$$
$$\dot{v} = g_0(u - \frac{D}{mg_0}) - g(h)\sin\gamma \tag{17}$$
$$\dot{\gamma} = \frac{L}{mv} + \left(\frac{v}{R_p+h} - \frac{g(h)}{v}\right)\cos\gamma \tag{18}$$
$$\dot{m} = -\frac{umg_0}{I_{sp}g_0} = -\frac{um}{I_{sp}} \tag{19}$$

where:

$$g(h) = g_0\left(\frac{R_p}{R_p+h}\right)^2$$

and $D$ and $L$ are the aerodynamic drag and lift, respectively.

## 2.10 Cost Functional

The performance index to minimize is:

$$J = -m(t_f)$$

where $t_f$ is the final time at orbit insertion.

## 2.11 Hamiltonian Formulation

Define the costate vector:

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_h \\ \lambda_x \\ \lambda_v \\ \lambda_\gamma \\ \lambda_m \end{bmatrix}$$

The Hamiltonian is:

$$\begin{aligned} \mathcal{H} = \ & \lambda_h v\sin\gamma + \lambda_x \frac{v\cos\gamma}{1+\frac{h}{R_p}} \\ & + \lambda_v\left(g_0\left(u - \frac{D}{mg_0}\right) - g(h)\sin\gamma\right) \\ & + \lambda_\gamma\left(\frac{L}{mv} + \left(\frac{v}{R_p+h} - \frac{g(h)}{v}\right)\cos\gamma\right) \\ & - \lambda_m\frac{um}{I_{sp}} \end{aligned} \tag{20}$$

### 2.12 Costate Equations

The costates evolve according to:

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}$$

Explicitly:

$$\dot{\lambda}_h = -\frac{\partial \mathcal{H}}{\partial h} \tag{21}$$

$$\dot{\lambda}_x = -\frac{\partial \mathcal{H}}{\partial x} = 0 \quad \text{(no explicit dependence)} \tag{22}$$

$$\dot{\lambda}_v = -\frac{\partial \mathcal{H}}{\partial v} \tag{23}$$

$$\dot{\lambda}_\gamma = -\frac{\partial \mathcal{H}}{\partial \gamma} \tag{24}$$

$$\dot{\lambda}_m = -\frac{\partial \mathcal{H}}{\partial m} \tag{25}$$

### 2.13 Optimality Condition

The optimality condition for the control $u(t)$ is given by the equation:

$$\frac{\partial \mathcal{H}}{\partial u} = 0$$

From the Hamiltonian, we obtain the following relation:

$$\lambda_v g_0 - \lambda_m \frac{m}{I_{sp}} = 0$$

This leads to:

$$\lambda_v g_0 = \lambda_m \frac{m}{I_{sp}}$$

Solving for the thrust-to-weight ratio $u^*(t)$, we get:

$$u^*(t) = \frac{\lambda_v I_{sp}}{m\lambda_m} g_0$$

This represents the optimal control law as a function of the costates $\lambda_v$, $\lambda_m$, the vehicle mass $m$, the specific impulse $I_{sp}$, and the gravitational acceleration $g_0$.

## 3 Uncertainty Modeling and Monte Carlo Simulation

In practical launch scenarios, there are uncertainties in atmospheric conditions, thrust generation, wind disturbances, and initial conditions. These uncertainties are modeled using a Monte Carlo simulation.

## 3.1 Sources of Uncertainty

- **Atmospheric density variation:**
$$\rho(h) \to \rho(h)(1 + \epsilon_\rho)$$
- **Thrust magnitude perturbation:**
$$T(t) \to T(t)(1 + \epsilon_T)$$
- **Wind disturbances:** Random lateral and longitudinal wind speeds are superimposed.
- **Initial launch conditions:** Deviations in initial mass, velocity, and flight path angle:
$$\mathbf{x}_0 \to \mathbf{x}_0 + \Delta\mathbf{x}_0$$

where $\epsilon_\rho$, $\epsilon_T$, and $\Delta\mathbf{x}_0$ are random variables sampled from normal distributions.

## 3.2 Monte Carlo Simulation Methodology

In order to analyze the sensitivity of the launch vehicle's trajectory to uncertainties, a Monte Carlo simulation with 5000 samples was performed. The following uncertainties were introduced:

- Atmospheric density perturbed by a Gaussian noise with 5% standard deviation.
- Thrust magnitude perturbed by a Gaussian noise with 3% standard deviation.
- Wind disturbance modeled as random lateral and longitudinal velocities with 20 m/s standard deviation.
- Initial launch conditions perturbed within realistic ranges:
  - Altitude perturbation: $\pm 100$ m,
  - Velocity perturbation: $\pm 10$ m/s,
  - Flight-path angle perturbation: $\pm 1$ degree.

The simulation methodology was as follows:

1. Generate $N = 1000$ random samples for the uncertainties.
2. For each sample, integrate the perturbed equations of motion forward for 300 seconds using a fixed optimal control law based on thrust-to-weight ratio.
3. For each trajectory, record key quantities:
   - Maximum dynamic pressure ($Q_{max}$),
   - Maximum acceleration ($a_{max}$),
   - Maximum rate of climb ($\dot{h}_{max}$),
   - Maximum degree of climb ($\gamma_{max}$).
4. Perform statistical analysis on the recorded results:
   - Calculate the mean and variance,
   - Estimate 95% confidence intervals for each metric.

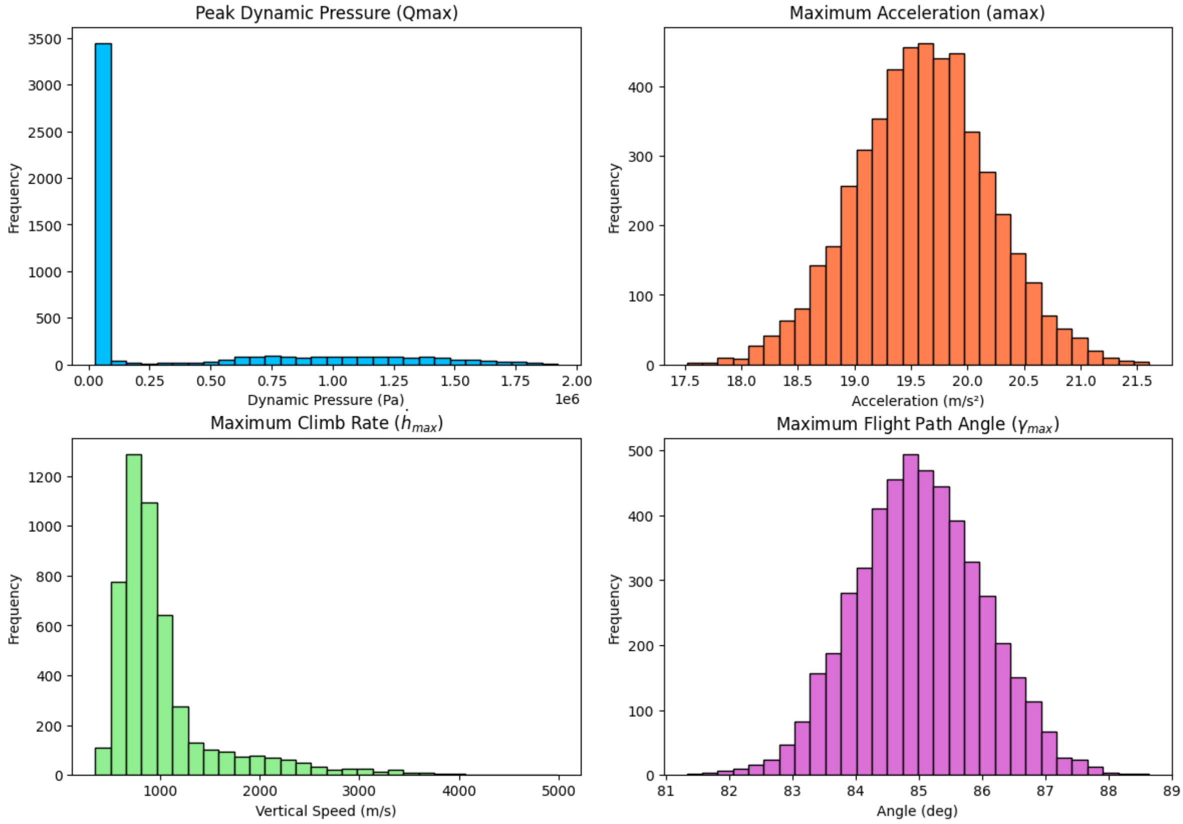


Figure 6: Histograms of key trajectory parameters.

## 3.3 Sensitivity Analysis and Discussion

### 3.3.1 Maximum Dynamic Pressure ($Q_{max}$)

The results indicate that $Q_{max}$ is highly sensitive to atmospheric density perturbations and initial velocity deviations. An increase in atmospheric density or headwind disturbances can significantly raise the peak dynamic pressure, leading to higher aerodynamic loading. This emphasizes the need for structural margins near the Max Q region.

### 3.3.2 Maximum Acceleration ($a_{max}$)

The maximum acceleration is mainly affected by uncertainties in thrust generation and vehicle mass. Higher-than-expected thrust outputs or lower vehicle mass lead to increased accelerations, potentially exceeding design G-load limits. Thrust modeling errors must be tightly controlled.

### 3.3.3 Rate of Climb and Degree of Climb

The maximum rate of climb ($\dot{h}_{max}$) and maximum degree of climb ($\gamma_{max}$) are particularly sensitive to:

- Wind disturbances,
- Initial launch angle errors.

Wind-induced lateral velocity deviations alter the vertical ascent trajectory, while launch angle perturbations directly influence both vertical and horizontal motion, affecting orbit insertion accuracy.

### 3.3.4 Robustness of Optimal Control Law

The optimal control law $u^*(t)$, derived for the nominal trajectory, demonstrates varying robustness across different flight phases:

- Greater sensitivity to thrust perturbations at low altitudes (where atmospheric drag is significant).
- Greater sensitivity to atmospheric density variations near the Max Q phase.
- Lateral wind disturbances causing off-nominal flight path angles, potentially requiring mid-course trajectory corrections.

## 3.4 Summary

Despite the presence of significant uncertainties, the simulation results suggest that the optimal control law derived from calculus of variations and Pontryagin's Minimum Principle provides a strong baseline. However, for operational robustness, it is recommended to include:

- Margin policies on critical parameters,
- Adaptive or feedback control strategies to mitigate deviations during ascent.

Overall, the Monte Carlo analysis confirms the importance of rigorous uncertainty modeling and robust control design in achieving reliable launch vehicle performance.

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