Winter 19 – AMS206B Homework 1 Solution

1. Let $X \sim \text{Exp}(\lambda)$, where $\text{E}(X) = 1/\lambda$. What is the pmf (probability mass function) of $Y = \lfloor X \rfloor$ (the floor of X)? Do you recognize it as a distribution that you have studied in the past?

   We first find $\Pr(Y = y) = \Pr(y \le X < y - 1) = \int_y^{y+1} \lambda e^{-\lambda x} dx = e^{-\lambda y}(1 - e^{-\lambda})$, $y = 0, 1, 2, \ldots$. We observe it is the probability function for the geometric distribution with parameter $e^{-\lambda}$, i.e., $Y \sim \text{Geometric}(e^{-\lambda})$.

2. Let X_1 and X_2 be two independent random variables such that $X_i \sim \text{Ga}(a_i, b)$ for any $a_1, a_2, b > 0$. Define $Y = X_1/(X_1 + X_2)$ and $Z = (X_1 + X_2)$.

   (a) Find the joint pdf for Y and Z and show that these two random variables are independent.

   $Y = X_1/(X_1 + X_2)$ and $Z = (X_1 + X_2)$, $X_1, X_2 \in \mathbb{R}^+$ are differentiable one-to-one transformations. We observe $X_1 = YZ$ and $X_2 = Z(1 - Y)$. We use a change of variables. We have $0 < Y < 1$ and $Z \in \mathbb{R}^+$, and find

$$\begin{aligned} g(y, z) &= f(yz, z(1-y))|J| \\ &= \frac{b^{a_1}}{\Gamma(a_1)}(yz)^{a_1-1}e^{-byz}\frac{b^{a_2}}{\Gamma(a_2)}((1-y)z)^{a_2-1}e^{-b(1-y)z}z \\ &\propto \underbrace{y^{a_1-1}(1-y)^{a_2-1}}_{\text{function of } y \text{ only}}\underbrace{z^{a_1+a_2-1}e^{-bz}}_{\text{function of } z \text{ only}}, y \in (0,1) \text{ and } 0 < z. \end{aligned}$$

   Since the joint pdf of Y and Z can be expressed as a product of a function of Y only and a function of Z only, Y and Z are independent.

   (b) Find the marginal pdf of Z. Do you recognize this pdf as belonging to some family that you know?

   From part (a), we have $f(z) \propto z^{a_1+a_2-1}e^{-bz}$, $z \in \mathbb{R}^+$ and recognize it is proportional to the pdf of $\text{Gamma}(a_1 + a_2, b)$. Thus, $Z \sim \text{Gamma}(a_1 + a_2, b)$.

   (c) Find the marginal pdf of Y. Do you recognize this pdf as belonging to some family that you know?

   From part (a), we have $f(y) \propto y^{a_1-1}(1-y)^{a_2-1}$, $0 < y < 1$ and recognize it is proportional to the pdf of $\text{Be}(a_1, a_2)$. Thus, $Y \sim \text{Be}(a_1, a_2)$.

   (d) Compute $\text{E}(Y^k)$ for any $k > 0$.

   We have

$$\text{E}(Y^k) = \int_0^\infty y^k \frac{1}{B(a_1, a_2)} y^{a_1-1}(1-y)^{a_2-1} dy = \frac{B(a_1 + k, a_2)}{B(a_1, a_2)}.$$

(e) What does this result imply if $a_i = b = 1$?

We observe $Y \sim \text{Be}(1,1) = \text{Unif}(1,1)$. For $Z = X_1 + \ldots, X_n$, $Z \sim \text{Gamma}(n, 1)$, that is, Erlang distribution.

3. Consider three independent random variables X_1, X_2 and X_3 such that $X_i \overset{indep}{\sim} \text{Gamma}(a_i, b)$, $i = 1, 2, 3$. Let

$$\boldsymbol{Y} = (Y_1, Y_2, Y_3) = \left(\frac{X_1}{X_1 + X_2 + X_3}, \frac{X_2}{X_1 + X_2 + X_3}, \frac{X_3}{X_1 + X_2 + X_3} \right).$$

(a) Show that $\boldsymbol{Y} \sim \text{Dirichlet}(a_1, a_2, a_3)$, a Dirichlet distribution.

First, we consider the joint density of the three independent Gamma-distributed RVs:

$$p(x_1, x_2, x_3) = \prod_{i=1}^{3} p(x_i) = \prod_{i=1}^{3} \frac{x_i^{a_i - 1} e^{-x_i/b}}{\Gamma(a_i) b^{a_i}} = \frac{e^{\sum_{i=1}^3 x_i/b} \prod_{i=1}^3 x_i^{a_i-1}}{b^{\sum_{i=1}^3 a_i} \prod_{i=1}^3 \Gamma(a_i)}.$$

Since $Y_i = X_i / \sum_{i=1}^3 X_i$, we find

$$\begin{aligned} X_1 &= Y_1 Z \\ X_2 &= Y_2 Z \\ X_3 &= Y_3 Z = (1 - Y_1 - Y_2) Z, \end{aligned}$$

where $Z = \sum_{i=1}^3 X_i$. To obtain the joint distribution of (Y_1, Y_2, Z), we find the Jacobian for this change of variables. The matrix is

$$J = \begin{bmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} & \frac{dx_1}{d_z} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} & \frac{dx_2}{d_z} \\ \frac{dx_3}{dy_1} & \frac{dx_3}{dy_2} & \frac{dx_3}{d_z} \end{bmatrix} = \begin{pmatrix} Z & 0 & Y_1 \\ 0 & Z & Y_2 \\ -Z & -Z & (1 - Y_1 - Y_2) \end{pmatrix}$$

So, the Jacobian, $|J|$ is Z^2.

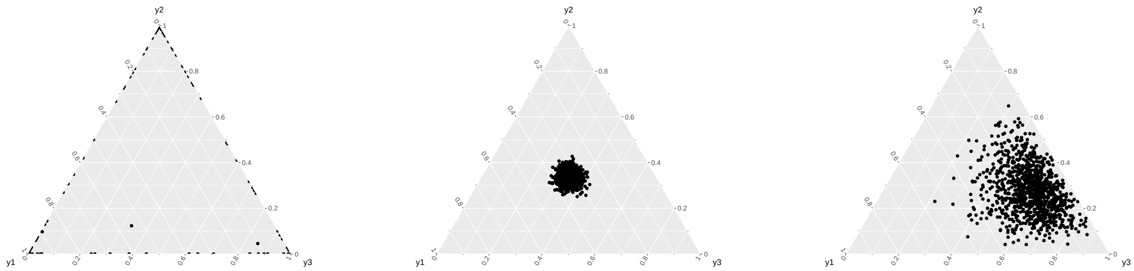
$$p(Y_1, Y_2, Z) = \frac{(y_1 z)^{a_1 - 1} e^{-y_1 z/b} (y_2 z)^{a_2 - 1} e^{-y_2 z/b} \{(1 - y_1 - y_2) z\}^{a_3 - 1} e^{-(1 - y_1 - y_2) z/b}}{b^{\sum_{i=1}^3 a_i} \prod_{i=1}^3 \Gamma(a_i)} z^2,$$

where $0 < y_1, y_2 < 1$, $y_1 + y_2 < 1$ and $0 < z$.
By letting $y_3 = 1 - y_1 - y_2$,

$$p(Y_1, Y_2, Z) = \frac{y_1^{a_1 - 1} y_2^{a_2 - 1} y_3^{a_3 - 1} z^{\sum_{i=1}^3 a_i - 1} e^{z \overbrace{\sum_{i=1}^3 y_i}^{=1} /b}}{b^{\sum_{i=1}^3 a_i} \prod_{i=1}^3 \Gamma(a_i)}.$$

We now integrate out z to obtain $p(y_1, y_2, y_3)$.

$$\begin{aligned} p(y_1, y_2, y_3) &= \int_{\mathbb{R}} \frac{y_1^{a_1 - 1} y_2^{a_2 - 1} y_3^{a_3 - 1} \overbrace{z^{\sum_{i=1}^3 a_i - 1} e^{-z/b}}^{\text{kernel for Gamma}(\sum_{i=1}^3 a_i, b)}}{b^{\sum_{i=1}^3 a_i} \prod_{i=1}^3 \Gamma(a_i)} dz \\ &= \frac{\Gamma(\sum_{i=1}^3 a_i)}{\prod_{i=1}^3 \Gamma(a_i)} y_1^{a_1 - 1} y_2^{a_2 - 1} y_3^{a_3 - 1}. \end{aligned}$$



(a) $\boldsymbol{a} = (0.01, 0.01, 0.01)$ (b) $\boldsymbol{a} = (100, 100, 100)$ (c) $\boldsymbol{a} = (3, 5, 10)$

Figure 1: $\boldsymbol{y} = (y_1, y_2, y_3)$ simulated from Dirichlet distribution $\text{Dir}(\boldsymbol{a})$.

Thus, $\boldsymbol{Y} = (Y_1, Y_2, Y_3) \sim \text{Dir}(a_1, a_2, a_3)$.

(b) How can this result be used to generate random variables according to a Dirichlet distribution? Write a simple function in `R` or `Matlab` (your choice) that takes as inputs n, the number of trivariate vectors to be generated, and $\boldsymbol{a} = (a_1, a_2, a_3)$ and generates a matrix of size $n \times 3$ whose rows correspond to independent samples from a Dirichlet distribution with parameter (a_1, a_2, a_3).

Use each of $\boldsymbol{a} = (0.01, 0.01, 0.01)$, $(100, 100, 100)$, and $(3, 5, 10)$ and comment how the density of $\boldsymbol{Y}$ changes over $\boldsymbol{a}$.

```
#a is a vector of length p; a=(a_1, a_2, ..., a_p)
#n is the sample size

> dirichlet <- function(a, n){
        p <- length(a)
        y <- array(NA, dim=c(n, p)) #Each row of y is iid sample from Dir(a)

        for (i in 1:n) {
              tmp <- rgamma(p, a, 1)
              y[i, ] <- tmp / sum(tmp)
        }

        return(y)
  }
```

Fig 1 illustrates trinary plots of 1000 simulated $\boldsymbol{y}$ with different $\boldsymbol{a}$. For $a < 1$, $\boldsymbol{y}$ are at the vertices and edges, meaning either one or two of y_1, y_2 or y_3 are close to zero. For $a = 100$, all y_i are close to 1/3 (i.e., small variance) as shown in Fig 1(b). For $\boldsymbol{a} = (3, 5, 10)$, $\boldsymbol{y}$ are around the mean $(3/18, 5/18, 10/18)$ and its variance is larger than that with $a = 100$.

4. Y follows an inverse Gamma distribution with shape parameter a and scale parameter b ($Y \sim \mathrm{IG}(a,b)$) if $Y = 1/X$ with $X \sim \mathrm{Gamma}(a,b)$ (assume the Gamma distribution is parameterized so that $\mathrm{E}(X) = ab$.

   (a) Find the density of Y.

   Since $X \sim \mathrm{Gamma}(a,b)$,

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx), \text{for } x > 0.$$

   Let $y = 1/x$. Then $x = 1/y$ and $\frac{dx}{dy} = -1/y^2$. Therefore,

$$p(y) = \frac{b^a}{\Gamma(a)} \left(\frac{1}{y}\right)^{a-1} \exp(-b/y) \left| -\frac{1}{y^2} \right| = \frac{b^a}{\Gamma(a)} y^{-(a+1)} \exp(-b/y), \ y > 0,$$

   which is an inverse Gamma with shape a and rate b, $y \sim \mathrm{IG}(a,b)$.

   (b) Compute $\mathrm{E}(Y^k)$. Do you need to impose any constrain on the problem for this expectation to exists?

   We have

$$\mathrm{E}(Y^k) = \int_{\mathbb{R}^+} y^k \frac{b^a}{\Gamma(a)} y^{-(a+1)} \exp(-b/y) dy = \frac{b^a}{\Gamma(a)} \frac{\Gamma(a-k)}{b^{a-k}} = \frac{b^k \Gamma(a-k)}{\Gamma(a)}.$$

   That is, $\mathrm{E}(Y^k) = b^k \Gamma(a-k)/\Gamma(a)$ for $a - k > 0$.

   (c) Compare $\mathrm{E}(Y^k)$ to $1/\mathrm{E}(X^k)$ (hint: look at the ratio of the two quantities).

   We first find

$$\mathrm{E}(X^k) = \frac{b^a \Gamma(a+k)}{\Gamma(a) b^{a+k}} = \frac{\Gamma(a+k)}{\Gamma(a) b^k}.$$

   Then,

$$\zeta \equiv \frac{1/\mathrm{E}(X^k)}{\mathrm{E}(Y^k)} = \frac{\Gamma(a)\Gamma(a)}{\Gamma(a+k)\Gamma(a-k)}$$

   which implies that $\zeta = 1 \Leftrightarrow k = 0$. That is, none of the moments of X is invariant to the reciprocal transformation.

5. Y follows a log normal distribution with parameters μ and σ^2 (denotes as $Y \sim \text{Log-N}(\mu, \sigma^2)$ if $Y = \exp(X)$ where $X \sim \mathrm{N}(\mu, \sigma^2)$).

   (a) Find the density of Y.

   We have $Y = \exp(X) \in \mathbb{R}^+$. We use a change of a variable and find

$$g(y) = f(\log(y)) |\frac{1}{y}| = \frac{1}{\sqrt{2\pi\sigma^2} y} \exp\left\{ -\frac{(\log(y) - \mu)^2}{2\sigma^2} \right\}, \quad y \in \mathbb{R}^+.$$

(b) Compute the mean and the variance of Y.

Observe the $k-$th moment of Y, $\mathrm{E}^Y(Y^k) = \mathrm{E}^X(e^{kX})$ and find

$$\begin{aligned}
\mathrm{E}^X(e^{kX}) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} + kx\right\} dx \\
&= \exp\left\{-\frac{\mu^2}{2\sigma^2} + \frac{(\mu+k\sigma^2)^2}{2\sigma^2}\right\} \\
&= \exp\left(k\mu + k^2\sigma^2/2\right).
\end{aligned}$$

Thus, $\mathrm{E}(Y) = \exp\left(\mu + \sigma^2/2\right)$ and $\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \mathrm{E}(Y)^2 = \exp(2\mu+\sigma^2)\{\exp(\sigma^2)-1\}$.

6. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)$ with $X \sim \mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ and set $\boldsymbol{Z}_1 = (X_1, \ldots, X_q)$ and $\boldsymbol{Z}_2 = (X_{q+1}, \ldots, , X_p)$ with $1 < q < p$. Show that

$$\boldsymbol{Z}_1 \mid \boldsymbol{Z}_2 \sim \mathrm{N}_q\left(\boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\boldsymbol{Z}_2 - \boldsymbol{\mu}_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right),$$

where $\boldsymbol{\mu}_k$ and $\Sigma_{k\ell}$ denote the blocks of $\boldsymbol{\mu}$ and Σ where the rows correspond to the variables in $\boldsymbol{Z}_k$ and the columns to the variables in $\boldsymbol{Z}_\ell$.

Using some results of the inverse of a partitioned matrix, we find for symmetric and $\Sigma > 0$,

$$\Sigma^{-1} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \Sigma_{11.2}^{-1} & -\Sigma_{11.2}^{-1}\Sigma_{12}\Sigma_{22}^{-1} \\ -\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1} & \Sigma_{22.1}^{-1} \end{bmatrix} = \begin{bmatrix} \Sigma_{11.2}^{-1} & -\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22.1}^{-1} \\ -\Sigma_{22.1}^{-1}\Sigma_{21}\Sigma_{11}^{-1} & \Sigma_{22.1}^{-1} \end{bmatrix},$$

where $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ and $\Sigma_{22.2} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$. We have

$$\begin{aligned}
g(\boldsymbol{z}_1 \mid \boldsymbol{z}_2) &\propto f(\boldsymbol{z}_1, \boldsymbol{z}_2) \\
&\propto \exp\left\{-\frac{1}{2}\begin{bmatrix} \boldsymbol{z}_1 - \boldsymbol{\mu}_1 \\ \boldsymbol{z}_2 - \boldsymbol{\mu}_2 \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \boldsymbol{z}_1 - \boldsymbol{\mu}_1 \\ \boldsymbol{z}_2 - \boldsymbol{\mu}_2 \end{bmatrix}\right\} \\
&\propto \exp\Big[-\frac{1}{2}\Big\{(\boldsymbol{z}_1 - \boldsymbol{\mu}_1)'\Sigma_{11.2}^{-1}(\boldsymbol{z}_1 - \boldsymbol{\mu}_1) - (\boldsymbol{z}_1 - \boldsymbol{\mu}_1)'\Sigma_{11.2}^{-1}\Sigma_{12}\Sigma_{22}^{-1}(\boldsymbol{z}_2 - \boldsymbol{\mu}_2) \\
&\qquad -(\boldsymbol{z}_2 - \boldsymbol{\mu}_2)'\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1}(\boldsymbol{z}_1 - \boldsymbol{\mu}_1)\Big\}\Big] \\
&\propto \exp\Big[-\frac{1}{2}\Big\{\boldsymbol{z}_1'\Sigma_{11.2}^{-1}\boldsymbol{z}_1 - \boldsymbol{z}_1'\left(\Sigma_{11.2}^{-1}\boldsymbol{\mu}_1 + \Sigma_{11.2}^{-1}\Sigma_{12}\Sigma_{22}^{-1}(\boldsymbol{z}_2 - \boldsymbol{\mu}_2)\right) \\
&\qquad -(\boldsymbol{\mu}_1'\Sigma_{11.2}^{-1} + (\boldsymbol{z}_2 - \boldsymbol{\mu}_2)'\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1})\boldsymbol{z}_1\Big\}\Big].
\end{aligned}$$

We recognize the kernel for $\mathrm{N}_q(\Sigma_{11.2}\left(\Sigma_{11.2}^{-1}\boldsymbol{\mu}_1 + \Sigma_{11.2}^{-1}\Sigma_{12}\Sigma_{22}^{-1}(\boldsymbol{z}_2 - \boldsymbol{\mu}_2)\right), \Sigma_{11.2})$.
That is, $\boldsymbol{Z}_1 \mid \boldsymbol{Z}_2 \sim \mathrm{N}_q\left(\boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\boldsymbol{Z}_2 - \boldsymbol{\mu}_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$.

7. Show that if $X \sim \mathrm{Exp}(\beta)$, then

(a) $Y = X^{1/\gamma}$ has a Weibull distribution with parameters γ and β with $\gamma > 0$ a constant.

We have $Y = X^{1/\gamma} \in \mathbb{R}^+$. We use a change of a variable and find

$$g(y) = f(y^\gamma)|\gamma y^{\gamma-1}| = \beta\gamma y^{\gamma-1} \exp\left(-\beta y^\gamma\right), \quad y \in \mathbb{R}^+.$$

We also observe $\mathrm{E}^Y(Y^k) = \mathrm{E}^X(X^{k/\gamma}) = \frac{\Gamma(1+k/\gamma)}{\beta^{k/\gamma}}$ (use the gamma kernel). We find $\mathrm{E}^Y(Y) = \frac{\Gamma(1+1/\gamma)}{\beta^{1/\gamma}}$ and $\mathrm{Var}(Y) = \mathrm{E}^X(X^{2/\gamma}) - \mathrm{E}^X(X^{1/\gamma})^2 = \left\{\Gamma(1+2/\gamma) - (\Gamma(1+1/\gamma))^2\right\}/\beta^{2/\gamma}$.

(b) $Y = (2X/\beta)^{1/2}$ has the Rayleigh distribution.

We have $Y = (2X/\beta)^{1/2} \in \mathbb{R}^+$. We use a change of a variable and find

$$g(y) = f(\beta y^2/2)|\beta y| = \beta^2 y \exp\left(-\beta^2 y^2/2\right), \quad y \in \mathbb{R}^+.$$

We also observe $\mathrm{E}^Y(Y^k) = \mathrm{E}^X((2X/\beta)^{k/2}) = \frac{2^{k/2}\Gamma(1+k/2)}{\beta^k}$ (use the gamma kernel). We find $\mathrm{E}^Y(Y) = \sqrt{2}\Gamma(1.5)/\beta$ and $\mathrm{Var}(Y) = \mathrm{E}^X(2X/\beta) - \mathrm{E}^X((2X/\beta)^{1/2})^2 = 2(1 - \Gamma(1.5)^2)/\beta^2$.

For both parts, derive the form of the pdf, verify that is a pdf, and calculate the mean and the variance.

8. Let $Y \mid X \sim \mathrm{Poisson}(X)$ and let $X \sim \mathrm{Exp}(\lambda)$. What is the marginal distribution of Y?

We find

$$m(y) = \int_{\mathbb{R}^+} \frac{e^{-x}x^y}{y!}\lambda e^{-\lambda x}dx = \frac{\Gamma(y+1)\lambda}{y!(1+\lambda)^{y+1}} = \frac{\lambda}{(1+\lambda)^{y+1}}, y = 0, 1, 2, \ldots.$$

9. (Robert 1.37) Let $x \sim \mathrm{N}(\theta, \sigma^2)$, $y \sim \mathrm{N}(\rho x, \sigma^2)$ with ρ known. Assume a prior of the form $\pi(\theta, \sigma^2) = 1/\sigma^2$. Find the predictive density of y given x.

We check the marginal distribution $\int_{\mathbb{R}} m(x)dx < 0$.

$$\begin{aligned}\int_{\mathbb{R}} m(x)dx &= \int_{\mathbb{R}^+}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)(\sigma^2)^{-1/2-1}d\theta d\sigma^2 \\ &= \int_{\mathbb{R}^+} (\sigma^2)^{-1}d\sigma^2 = \infty.\end{aligned}$$

Thus, $\theta, \sigma^2 \mid x$ cannot be defined and $y \mid x$ cannot.

10. (Robert) If $y \sim \text{Binomial}(n, \theta)$ and $x \sim \text{Binomial}(m, \theta)$, and $\theta \sim \text{Beta}(\alpha, \beta)$. Find the predictive distribution of y given x.

   Assume conditional independence of X and Y given θ. We know that $\theta \mid x \sim \text{Be}(\alpha + x, \beta + m - x)$ and then have for $y = 0, \ldots, n$,

$$
\begin{aligned}
f(y \mid x) &= \int_0^1 f(y \mid \theta)\pi(\theta \mid x) d\theta \\
&= \int_0^1 \binom{n}{y} \theta^y (1-\theta)^{n-y} \frac{\theta^{\alpha+x-1}(1-\theta)^{\beta+m-x-1}}{B(\alpha+x, \beta+m-x)} d\theta \\
&= \frac{\binom{n}{y} B(\alpha+x+y, \beta+m-x+n-y)}{B(\alpha+x, \beta+m-x)}, \quad y = 0, 1, 2, \ldots
\end{aligned}
$$

   that is, $y \mid x \sim \text{Beta-Binomial}(n, \alpha + x, \beta + m - x)$.

11. (Robert) Give the posterior and the marginal distributions in the following cases:

   (a) $x \mid \sigma^2 \sim \text{N}(0, \sigma^2)$ and $1/\sigma^2 \sim \text{Gamma}(1, 2)$.

   We know $1/\sigma^2 \sim \text{Gamma}(1, 2) \;\Rightarrow \sigma^2 \sim \text{IG}(1, 2)$.

$$
\begin{aligned}
m(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi}} (\sigma^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \frac{2}{\Gamma(1)} (\sigma^2)^{-2} \exp\left(-\frac{2}{\sigma^2}\right) d\sigma^2 \\
&= \frac{2\Gamma(3/2)}{\sqrt{2\pi}(2 + x^2/2)^{3/2}}, \quad x \in \mathbb{R}.
\end{aligned}
$$

   We have

$$
\pi(\sigma^2 \mid x) \propto (\sigma^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma^2}\right) (\sigma^2)^{-2} \exp\left(-\frac{2}{\sigma^2}\right), 0 < \sigma^2
$$

   that is, $\sigma^2 \mid x \sim \text{IG}(3/2, 2 + x^2/2)$ ($\Rightarrow \quad 1/\sigma^2 \mid x \sim \text{Gamma}(3/2, 2 + x^2/2)$).

   (b) $x \mid p \sim \text{Negative-Binomial}(10, p)$ and $p \sim \text{Beta}(1/2, 1/2)$.

   Following the parameterization of CR p522, we have

$$
\begin{aligned}
m(x) &= \int_0^\infty \binom{n+x+1}{x} p^n (1-p)^x \frac{1}{B(1/2, 1/2)} p^{-1/2}(1-p)^{-1/2} dp \\
&= \frac{\binom{n+x+1}{x} B(n+1/2, x+1/2)}{B(1/2, 1/2)}, x = 0, 1, \ldots.
\end{aligned}
$$

   We have

$$
\pi(p \mid x) \propto p^n (1-p)^x p^{-1/2} (1-p)^{-1/2}, 0 < p < 1,
$$

   that is, $p \mid x \sim \text{Be}(n + 1/2, x + 1/2)$.