$$f(y \mid \beta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ - \frac{(y - x\beta)^2}{2\sigma^2} \right\}$$

$$l(\beta, \sigma^2) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(y - x\beta)^2}{2\sigma^2}$$

$$\frac{\partial l(\beta, \sigma^2)}{\partial \beta} = + \frac{2x(y - x\beta)}{2\sigma^2} = \frac{x(y - x\beta)}{\sigma^2}$$

$$\frac{\partial^2 l(\beta, \sigma^2)}{\partial \beta^2} = -\frac{x^2}{\sigma^2} \quad \Rightarrow \quad \pi(\beta) \propto \sqrt{-E\left(-\frac{x^2}{\sigma^2}\right)} = \underbrace{\sqrt{\frac{x^2}{\sigma^2}}}_{\text{constant}}$$

$$\propto 1$$

$$\frac{\partial l(\beta, \sigma^2)}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{(y - x\beta)^2}{2(\sigma^2)^2}$$

$$\frac{\partial^2 l(\beta, \sigma^2)}{\partial (\sigma^2)^2} = + \frac{1}{2(\sigma^2)^2} - \frac{(y - x\beta)^2}{(\sigma^2)^3}$$

$$\Rightarrow \quad \pi(\sigma^2) \propto \left\{ -E\left(\frac{1}{2(\sigma^2)^2} - \frac{(y - x\beta)^2}{(\sigma^2)^3} \right) \right\}^{1/2}$$

$$= \left\{ -\frac{1}{2(\sigma^2)^2} + \frac{\sigma^2}{(\sigma^2)^3} \right\}^{1/2}$$

$$\propto \frac{1}{\sigma^2}$$

$$\Rightarrow \quad \pi(\theta) = \pi(\beta, \sigma^2) \underset{\substack{\uparrow \\ \text{a priori independence}}}{=} \pi(\beta) \cdot \pi(\sigma^2) \propto 1 \times \frac{1}{\sigma^2} = \frac{1}{\sigma^2}$$

$$f(x_1, \ldots, x_n \mid \theta, \sigma^2) \quad \propto \quad (\sigma^2)^{-n/2} \exp\left(- \frac{S^2}{2\sigma^2} - \frac{n(\bar{x}-\theta)^2}{2\sigma^2} \right)$$

where $\quad S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \quad$ and $\quad \bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$

ⓐ

$$p(\theta \mid \sigma^2, x) \quad \propto \quad p(\theta, \sigma^2 \mid x)$$

$$\propto \quad \pi(\theta, \sigma^2) \; f(x_1, \ldots, x_n \mid \theta, \sigma^2)$$

$$\propto \quad \frac{1}{\sigma^2} \cdot \left(\frac{1}{\sigma^2}\right)^{-n/2} \cdot \exp\left(- \frac{S^2}{2\sigma^2} - \frac{n(\bar{x}-\theta)^2}{2\sigma^2} \right)$$

$$\propto \quad \exp\left(- \frac{n(\bar{x}-\theta)^2}{2\sigma^2} \right)$$

$$\Rightarrow \quad \theta \mid \sigma^2, x \quad \sim \quad N(\bar{x}, \sigma^2/n)$$

ⓑ

$$p(\sigma^2 \mid x) \quad = \quad \int_{\mathbb{R}} p(\theta, \sigma^2 \mid x) \; d\theta$$

$$\propto \int (\sigma^2)^{-n/2 - 1} \exp\left(- \frac{S^2}{2\sigma^2} - \frac{n(\bar{x}-\theta)^2}{2\sigma^2} \right) d\theta$$

$$\propto \quad (\sigma^2)^{-n/2 - 1} \; (\sigma^2)^{\frac{1}{2}} \; \exp\left(- \frac{S^2}{2\sigma^2} \right)$$

$$\Rightarrow \quad \sigma^2 \mid x \quad \sim \quad IG\left(\frac{n-1}{2}, \frac{S^2}{2} \right)$$

(c)

$$P(\theta \mid X) = \int_0^\infty p(\theta, \sigma^2 \mid X)\, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-n/2-1} \exp\left(-\frac{1}{\sigma^2} \frac{S^2 + n(\bar{X}-\theta)^2}{2} \right) d\sigma^2$$

$$\propto \left\{ \frac{S^2 + n(\bar{X}-\theta)^2}{2} \right\}^{-n/2}$$

$$\propto \left\{ 1 + \frac{n(\bar{X}-\theta)^2}{S^2} \right\}^{-n/2} \qquad -\frac{n}{2} + \frac{1}{2} - \frac{1}{2}$$

$\Rightarrow \quad \theta \mid X \sim t_{n-1}$ w/ $\begin{cases} \text{location parameter} & \bar{X} \\ \text{scale parameter} & V = \sqrt{\dfrac{S}{n(n-1)}} \end{cases}$

$$\frac{n}{S^2} = \frac{1}{(n-1)\cdot V^2}$$

$$\Rightarrow \quad V^2 = \frac{S^2}{n(n-1)}$$

(d) We use the Gibbs

① draw σ^2 for $IG\left(\frac{n-1}{2}, \frac{S^2}{2} \right)$

② draw θ from $N(\bar{X}, \sigma^2/n)$

(e)

$$p(y \mid \mathbf{x}) = \int_0^{\infty}\int_{-\infty}^{\infty} p(y \mid \theta, \sigma^2)\; p(\theta, \sigma^2 \mid \mathbf{x})\; d\theta\, d\sigma^2$$

$$\propto \int_0^{\infty}\int_{-\infty}^{\infty} (\sigma^2)^{-1/2} \cdot \exp\left(-\frac{(y-\theta)^2}{2\sigma^2}\right) \cdot$$

$$\times \quad (\sigma^2)^{-n/2-1} \exp\left(-\frac{s^2}{2\sigma^2} - \frac{n(\bar{x}-\theta)^2}{2\sigma^2}\right) d\theta\, d\sigma^2$$

$$\propto \int_0^{\infty} (\sigma^2)^{-1/2-1/2-1} \exp\left(-\frac{y^2}{2\sigma^2} - \frac{s^2}{2\sigma^2} - \frac{n\bar{x}^2}{2\sigma^2}\right)$$

$$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma^2}+\frac{n}{\sigma^2}\right)\left\{\theta - \left(\frac{1}{\sigma^2}+\frac{n}{\sigma^2}\right)\cdot\left(\frac{y}{\sigma^2}+\frac{n\bar{x}}{\sigma^2}\right)\theta\right\}\right] d\theta \; d\sigma^2$$

$$= \int_0^{\infty} (\sigma^2)^{-n/2-1} \exp\left(-\frac{s^2}{2\sigma^2} - \frac{n(y-\bar{x})^2}{2(n+1)\sigma^2}\right) d\sigma^2$$

$$\propto \left[\frac{s^2 + \frac{n(y-\bar{x})^2}{(n+1)}}{2}\right]^{-n/2}$$

$$\propto \left[1 + \frac{(y-\bar{x})^2}{s^2(n+1)/n}\right]^{-n/2}$$

$$y \mid \mathbf{x} \sim t_{n-1}\left(\bar{x}, \sqrt{\frac{s^2(n+1)}{n(n-1)}}\right)$$