1. $X \mid \sigma^2 \sim N(\mu, \sigma^2)$, $\quad Y \mid x, \sigma^2 \sim N(\rho x, \sigma^2)$ $\quad \pi(\sigma^2) \propto \frac{1}{\sigma^2}$, $\sigma^2 > 0$

(a)

$$m(x) = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \cdot \frac{1}{\sigma^2}\, d\sigma^2$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}} \underbrace{\sigma^{2\,-\frac{1}{2}-1} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}_{\text{a kernel for } IG\left(\frac{1}{2}, \frac{(x-\mu)^2}{2}\right)} d\sigma^2$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\Gamma(\frac{1}{2})}{\left(\frac{(x-\mu)^2}{2}\right)^{1/2}} < \infty$$

$$\Rightarrow \quad \pi(\sigma^2 \mid x) \propto f(x \mid \sigma^2)\, \pi(\sigma^2)$$

$$\propto \underbrace{(\sigma^2)^{-1/2} \exp\left(-\frac{(x-\mu)^2}{\sigma^2}\right) \cdot (\sigma^2)^{-1}}_{\text{kernel for } IG\left(\frac{1}{2}, \frac{(x-\mu)^2}{2}\right)}$$

$$\Rightarrow \quad \sigma^2 \mid x \sim IG\left(\frac{1}{2}, \frac{(x-\mu)^2}{2}\right)$$

(b)

$$f(y \mid x) = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\rho x)^2}{2\sigma^2}\right) \cdot \frac{\left(\frac{(x-\mu)^2}{2}\right)^{1/2}}{\Gamma(\frac{1}{2})} (\sigma^2)^{-\frac{1}{2}-1}$$

$$\times \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) d\sigma^2$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\left(\frac{(x-\mu)^2}{2}\right)^{1/2}}{\Gamma(\frac{1}{2})} \cdot \int_0^\infty \underbrace{(\sigma^2)^{-1-1} \exp\left(-\frac{(y-\rho x^2)+(x-\mu)^2}{2\sigma^2}\right)}_{\text{a kernel for } IG\left(1, \frac{(y-\rho x)^2+(x-\mu)^2}{2}\right)} d\sigma^2$$

$$= \frac{\left(\frac{(x-\mu)^2}{2}\right)^{1/2}}{\sqrt{2\pi}\ \Gamma(\frac{1}{2})} \cdot \frac{\Gamma(1)}{\left(\frac{(y-\rho x)^2+(x-\mu)^2}{2}\right)^{1}}, \quad -\infty < y < \infty$$

$$S^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2$$

2. $$m(x) = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \cdot \exp\left(-\frac{S^2}{2\sigma^2} - \frac{n(\bar{x}-\theta)^2}{2\sigma^2}\right) \cdot 1 \, d\theta$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \cdot \exp\left(-\frac{S^2}{2\sigma^2}\right) \cdot \underbrace{\int_{-\infty}^{\infty} \exp\left(-\frac{n(\bar{x}-\theta)^2}{2\sigma^2}\right) d\theta}_{\text{a kernel for } N(\bar{x},\ \sigma^2/n)}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{S^2}{2\sigma^2}\right) \cdot \sqrt{2\pi\,\sigma^2/n} \quad < \infty$$

$$\Rightarrow \quad \pi(\theta|x) \propto \underbrace{\exp\left(-\frac{n(\bar{x}-\theta)^2}{2\sigma^2}\right)}_{\text{a kernel for } N(\bar{x},\ \sigma^2/n)}$$

$\delta^{\pi}(\bar{x})$ minimizes posterior expected loss

$$\rho(\pi,\delta|\bar{x}) = E\left(L(\theta,\delta)\,|\,\bar{x}\right) = E\left(e^{c(\delta-\theta)} - c(\delta-\theta) - 1 \,\middle|\, \bar{x}\right)$$

$$= E\left(e^{c(\delta-\theta)}\,|\,\bar{x}\right) - c\cdot(\delta-\bar{x}) - 1$$

$$E\left(e^{c(\delta-\theta)} \mid \bar{x}\right) = \int_{-\infty}^{\infty} e^{c(\delta-\theta)} \cdot \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left(-\frac{(\theta-\bar{x})^2}{2\sigma^2/n}\right) d\theta$$

$$= \frac{e^{c\delta}}{\sqrt{2\pi\sigma^2/n}} \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{\theta^2 - 2\theta\bar{x} + \bar{x}^2 + c2\sigma^2/n\,\theta}{2\sigma^2/n}\right) d\theta$$

$$= \frac{e^{c\delta}}{\sqrt{2\pi\sigma^2/n}} \cdot \int_{-\infty}^{\infty} \underbrace{\exp\left(-\frac{\theta^2 - 2\left(\bar{x} + \frac{c\sigma^2}{n}\right)\theta + \bar{x}^2 \pm \left(\bar{x} + \frac{c\sigma^2}{n}\right)^2}{2\sigma^2/n}\right)}_{\text{a kernel for } N\left(\bar{x} + \frac{c\sigma^2}{n},\ \sigma^2/n\right)} d\theta$$

$$= e^{c\delta} \cdot \exp\left(-\frac{\bar{x}^2 - \left(\bar{x} + \frac{c\sigma^2}{n}\right)^2}{2\sigma^2/n}\right)$$

$$= e^{c\delta} \cdot \underbrace{\exp\left(-\frac{-\frac{2c\sigma^2}{n}\bar{x} + \frac{c^2\sigma^4}{n^2}}{2\sigma^2/n}\right)}_{\equiv A > 0}$$

Then

$$\rho(\pi, \delta \mid \bar{x}) = A e^{c\delta} - c(\delta - \bar{x}) - 1$$

$$\frac{d\,\rho(\pi, \delta \mid \bar{x})}{d\delta} = cAe^{c\delta} - c$$

$$\frac{d^2\,\rho(\pi, \delta \mid \bar{x})}{d\delta^2} = c^2 A e^{c\delta} > 0$$

$\Rightarrow$ let $\quad cAe^{c\delta} - c = 0$

$$\exp\left(c\delta - \frac{-2c\sigma^2\bar{x} + \frac{c^2\sigma^4}{n}}{2\sigma^2}\right) = 1$$

$$\delta = \frac{2\sigma^2\bar{x}}{2\sigma^2} - \frac{c\sigma^4}{2\sigma^2 n} = \boxed{\bar{x} - \frac{c\sigma^2}{2n}}$$

$$\Rightarrow \delta^{\pi}(\bar{x}) = \bar{x} - \frac{c\sigma^2}{2n}$$

3. $X_i \overset{iid}{\sim} Ber(\theta)$, $i=1,\dots,n$ & $\theta \sim Be(\alpha, \beta)$

Since the prior is proper,

a. $$\pi(\theta | X_1, \dots, X_n) \propto \theta^{\Sigma X_i}(1-\theta)^{n-\Sigma X_i} \cdot \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$= \theta^{\alpha+\underset{y}{\Sigma X_i}-1}(1-\theta)^{n-\Sigma X_i+\beta-1}$$

$$\Rightarrow \pi(\theta|\Sigma X_i) \text{ is } Be(\alpha+\Sigma X_i, n-\Sigma X_i+\beta)$$

Under squared error loss function,

$$\delta^{\pi}(\Sigma X_i) = E(\theta|\Sigma X_i) = \frac{\alpha+\Sigma X_i}{\alpha+\beta+n}$$

b. $$R(\theta, \delta^{\pi}) = E\left(\left(\frac{\alpha+\Sigma X_i}{\alpha+\beta+n} - \theta\right)^2 \Big| \theta\right)$$

$$= E\left(\frac{(\alpha+\underset{y}{\Sigma X_i}-(\alpha+\beta+n)\theta)^2}{(\alpha+\beta+n)^2} \Big| \theta\right)$$

$$= E\left(\frac{(y-n\theta+(\alpha-(\alpha+\beta)\theta))^2}{(\alpha+\beta+n)^2} \Big| \theta\right)$$

$y \sim Bin(n, \theta)$ $\quad E(y) = n\theta$

$\quad Var(y) = n\theta(1-\theta)$

$$= b^2 \operatorname{Var}(\bar{X}) + (a + b\theta - \theta)^2$$

$$= \frac{b^2}{n}\theta(1-\theta) + (a + (b-1)\theta)^2$$

$$= \frac{n\theta(1-\theta) + (\alpha + (\alpha+\beta)\theta)^2}{(\alpha+\beta+n)^2}$$

$$= \frac{\alpha^2 + (n - \alpha(\alpha+\beta))\theta + ((\alpha+\beta)^2 - n)\theta^2}{(\alpha+\beta+n)^2}$$

expectation w.r.t $\pi(\theta) = Be(\alpha, \beta)$. $E(\theta^k) = \frac{B(\alpha+k, \beta)}{B(\alpha, \beta)}$ is given.

c. $r(\pi, \delta) = E^{\pi}(R(\theta, \delta^{\pi}))$

$$= E^{\pi}\left(\frac{\alpha^2 + (n - \alpha(\alpha+\beta))\theta + ((\alpha+\beta)^2 - n)\theta^2}{(\alpha+\beta+n)^2} \right)$$

$$= \frac{1}{(\alpha+\beta+n)^2} \left\{ \alpha^2 + (n - \alpha(\alpha+\beta)) \cdot \frac{\alpha}{\alpha+\beta} + ((\alpha+\beta)^2 - n) \cdot \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} \right\}$$

Enough for the full credit

$$= \frac{\alpha(n\beta + (\alpha+1)(\alpha+\beta)^2)}{(\alpha+\beta+n)^2(\alpha+\beta)(\alpha+\beta+1)}$$

d. $Be(\alpha, \beta)$ is strictly positive for $\theta \in (0, 1)$

$r(\pi, \delta) < \infty$

$R(\theta, \delta)$ is continuous of θ

$\xrightarrow{\text{so}}$ δ is adimissible by prop 2.4.22

(the only proposition we covered in class)