AMS 245 Homework #1 **Mary Silva**
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1. **Prove the conditions for the existence of a Gaussian process.**

The likelihood is given by

$$f_{s_1,...,s_k}(x_1,...,x_k|\boldsymbol{\mu},\boldsymbol{\Sigma}) = (2\pi)^{-k/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$
$$\Rightarrow F_{s_1,...,s_k}(x_1,...,x_k) = \int_{-\infty}^{x_1}...\int_{-\infty}^{x_k}(2\pi)^{-k/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}d_{x_k},...,d_{x_1}$$

Thus we need to show first that $F_{s_1,...,s_k}(x_1,...,x_k) = F_{s_{\pi_1},...,s_\pi}(x_{\pi_1},...,x_{\pi_k})$ for any permutation π. To do this, we need to show the following statements are true

$$|\boldsymbol{\Sigma}|^{-1/2} = |\boldsymbol{\Sigma}_\pi|^{-1/2}$$
$$(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) = (\boldsymbol{x_\pi}-\boldsymbol{\mu_\pi})'\boldsymbol{\Sigma}_\pi^{-1}(\boldsymbol{x_\pi}-\boldsymbol{\mu_\pi})$$

Let $\boldsymbol{x_\pi} = (x_{\pi_1},...,x_{\pi_k})$ and $\boldsymbol{\mu} = (\mu_{\pi_1},...,\mu_{\pi_1})$. Let $\boldsymbol{P}$ be a permutation matrix such that $x_\pi = \boldsymbol{Px}$, $\mu_\pi = \boldsymbol{P\mu}$, and $\boldsymbol{\Sigma}_\pi = \boldsymbol{P\Sigma}_\pi\boldsymbol{P}'$. Since $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{P}$ is orthogonal, which means $\boldsymbol{PP}' = \boldsymbol{I}$ and $|\boldsymbol{P}| = 1$. Then

$$|\boldsymbol{\Sigma}_\pi| = |\boldsymbol{P}||\boldsymbol{\Sigma}||\boldsymbol{P}'| = |\boldsymbol{\Sigma}|$$
and
$$\boldsymbol{\Sigma}_\pi = (\boldsymbol{P}')^{-1}\boldsymbol{\Sigma}_\pi^{-1}(\boldsymbol{P})^{-1} = \boldsymbol{P\Sigma}_\pi^{-1}\boldsymbol{P}'$$

The last statement holds since by definition of orthogonality $\boldsymbol{P}^{-1} = \boldsymbol{P}'$ and $(\boldsymbol{P}')^{-1} = \boldsymbol{P}$. Thus,

$$\begin{aligned}(\boldsymbol{x_\pi}-\boldsymbol{\mu_\pi})'\boldsymbol{\Sigma}_\pi^{-1}(\boldsymbol{x_\pi}-\boldsymbol{\mu_\pi}) &= (\boldsymbol{Px}-\boldsymbol{P\mu})'\boldsymbol{P\Sigma}_\pi^{-1}\boldsymbol{P}'(\boldsymbol{Px}_\pi-\boldsymbol{P\mu}_\pi)\\ &= (\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{P}'\boldsymbol{P\Sigma}_\pi^{-1}\boldsymbol{P}'\boldsymbol{P}(\boldsymbol{x}-\boldsymbol{\mu})\\ &= (\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{I\Sigma}_\pi^{-1}\boldsymbol{I}(\boldsymbol{x}-\boldsymbol{\mu})\\ &= (\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}_\pi^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\end{aligned}$$

Thus,

$$\begin{aligned}F_{s_1,...,s_k}(x_1,...,x_k) &= \int_{-\infty}^{x_1}...\int_{-\infty}^{x_k}f_{s_1,...,s_k}(x_1,...,x_k|\boldsymbol{\mu},\boldsymbol{\Sigma})d_{x_k},...,d_{x_1}\\ &= \int_{-\infty}^{x_1}...\int_{-\infty}^{x_k}(2\pi)^{-k/2}|\boldsymbol{\Sigma}_\pi|^{-1/2}\exp\left\{-\frac{1}{2}(\boldsymbol{x}_\pi-\boldsymbol{\mu}_\pi)'\boldsymbol{\Sigma}_\pi^{-1}(\boldsymbol{x}_\pi-\boldsymbol{\mu}_\pi)\right\}\\ &= F_{s_{\pi_1},...,s_\pi}(x_{\pi_1},...,x_{\pi_k})\end{aligned}$$

For the second condition, $F_{s_1,...,s_{k-1}}(x_1,...,x_{k-1}) = F_{s_1,...,s_{k-1}}(x_1,...,x_{k-1},\infty)$, we need to integrate out x_k:

$$\begin{aligned}F_{s_1,...,s_{k-1},s_k}(x_1,...,x_{k-1},\infty) &= \lim_{x_k\to\infty}\int_{-\infty}^{x_1}...\int_{-\infty}^{x_{k-1}}\int_{-\infty}^{x_k}f_{s_1,...,s_k}(x_1,...,x_{k-1},x_k)d_{x_k}d_{x_{k-1}}...d_{x_1}\\ &= \int_{-\infty}^{x_1}...\int_{-\infty}^{x_{k-1}}f_{s_1,...,s_k}(x_1,...,x_{k-1})dx_{k-1}...dx_1\\ &= F_{s_1,...,s_{k-1}}(x_1,...,x_{k-1})\end{aligned}$$

Thus the conditions of Kolmogorov existence theorem are satisfied and the conditions for the existence of a GP holds.

2. **Consider an isotropic correlation function. Consider a transformation that produces geometric anisotropy. Prove that the resulting correlation function is positive definite.**

   We obtain Geometric anisotropy by transforming the covariance function such that

$$C_K(s,s') = C\left(\sqrt{(s-s')'K(s-s')}\right)$$

   We need to show that $C_K(s,s')$ is positive definite. i.e.

$$\forall n, s_j \in S \text{ and } c_j \in \mathcal{R}, j = 1, ..., n, \sum_{i,j} c_i c_j C_k(s,s') \geq 0$$

   We have that

$$\sum_{i,j} c_i c_j C_k(s,s') = \sum_{i,j} c_i c_j C\left(\sqrt{(s-s')'K(s-s')}\right)$$

   which equals zero if $s = s'$ and is greater than 0 otherwise since C_K is a valid covariance matrix and depends only on distance by definition.

3. **Plot all the covariograms and variograms in the tables of the second set of slides. Take the variance to be 1, and take the range parameter to be such that the correlation is .05 at a distance of one unit**

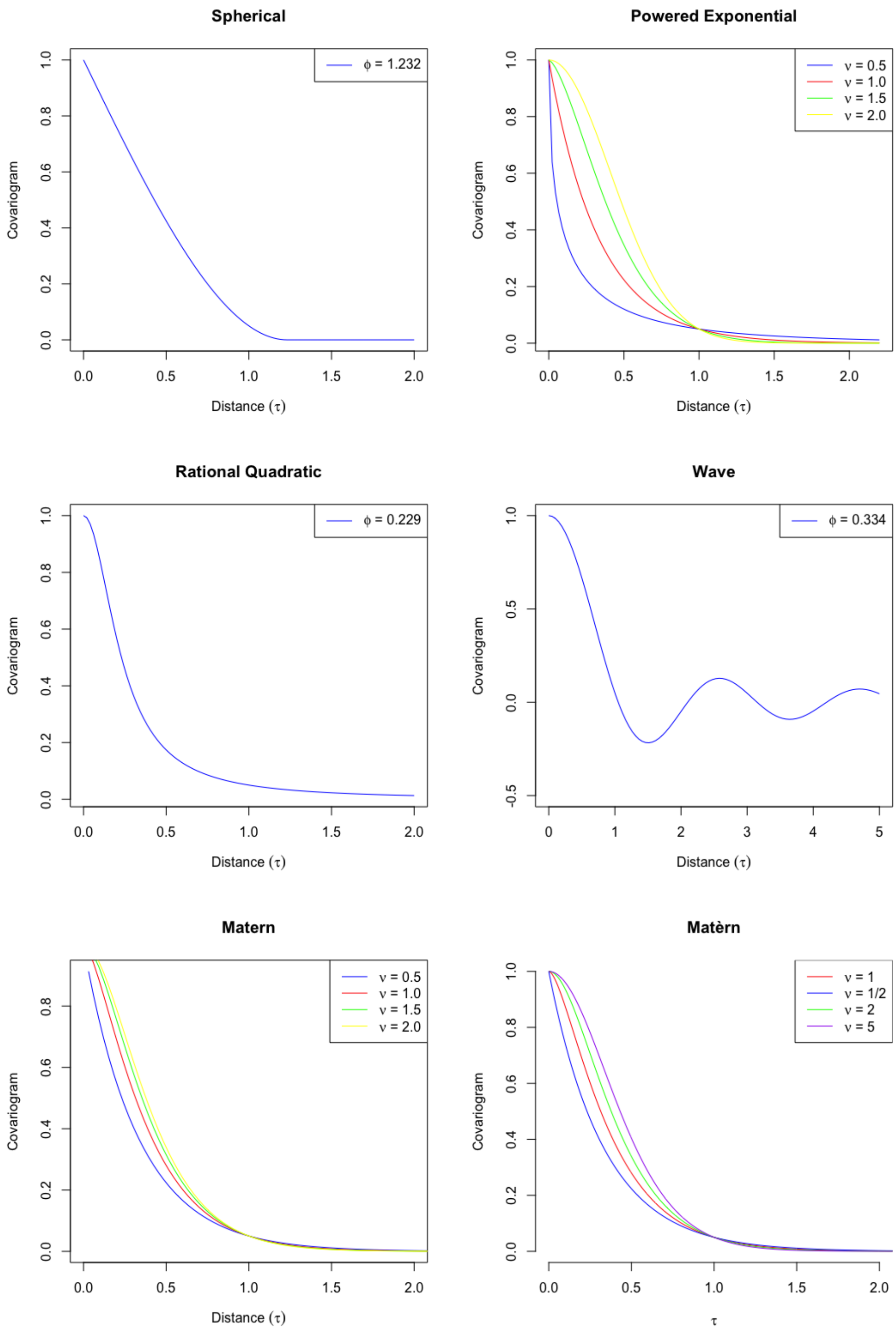

Figure 1: Covariograms with variance set to be 1, and take the range parameter to be such that the correlation is .05 at a distance of one unit. For the last Matèrn plot I used the `geoR` package.

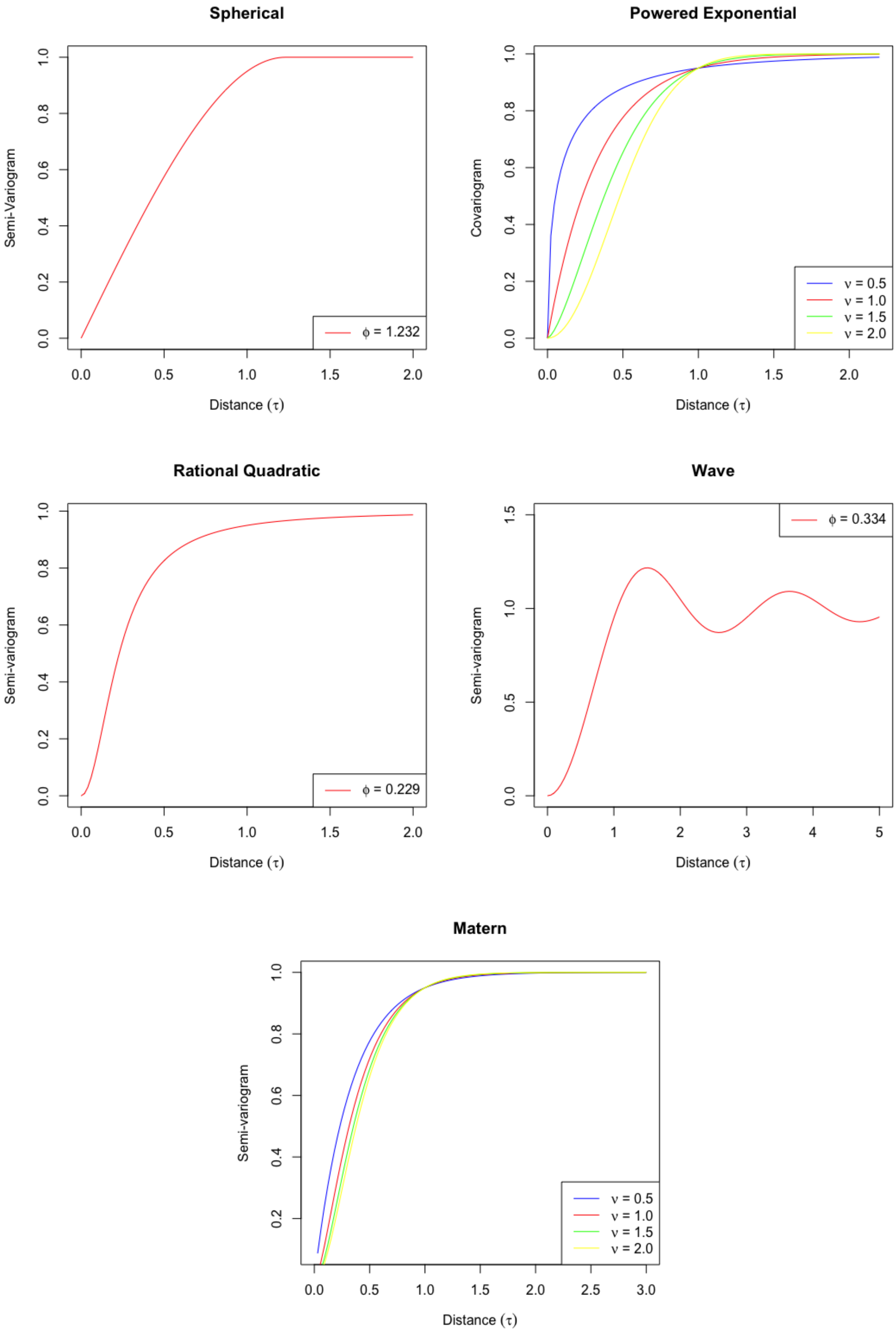

Figure 2: Semivariograms with variance set to be 1, and take the range parameter to be such that the correlation is .05 at a distance of one unit.

4. **Assume that the correlation functions in the previous point correspond to one dimensional Gaussian processes. Simulate one 100-points realization of the process corresponding to each of the plotted functions.**

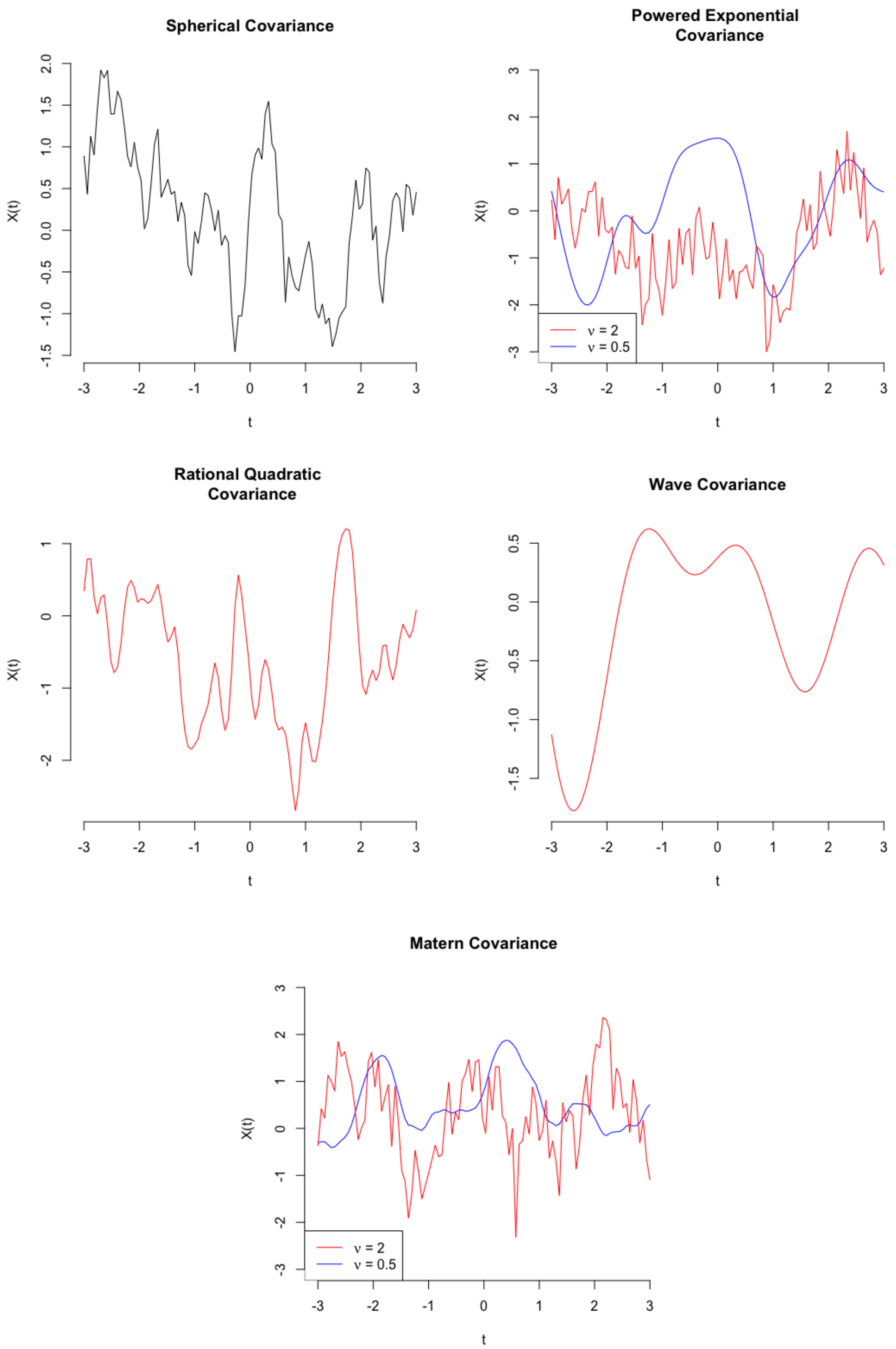

Figure 3: Simulated one 100-points realization of the process corresponding to each of the plotted functions.

Figure 3 shows that the spherical model produces a less smooth plot. The squared exponential with a higher $\nu = 2$ produces a more smooth plot, while a lower $\nu = 0.5$, produces a much more eratic plot. So if the goal is to produce a smoother proces, $\nu > 1.5$ is ideal. Similar, the Matèrn using a larger ν produces a smoother process.

5. **Write explicitly the correlation function of a Matèrn with $\nu = 1/2, 3/2$ and $5/2$.**

$$
\begin{aligned}
C_{1/2}(\tau) &= \sigma^2 \exp\left\{-\frac{\tau}{\phi}\right\} \\
C_{3/2}(\tau) &= \sigma^2 \left(1 + \frac{\sqrt{3}\tau}{\phi}\right) \exp\left\{-\frac{\sqrt{3}\tau}{\phi}\right\} \\
C_{5/2}(\tau) &= \sigma^2 \left(1 + \frac{\sqrt{5}\tau}{\phi} + \frac{5\tau^2}{3\phi^2}\right) \exp\left\{-\frac{\sqrt{5}\tau}{\phi}\right\}
\end{aligned}
$$

Please see attached notes for derivation.