Definitions

1.1 Gaussian random field is a random field where all the finite dimensional distributions, $F(s_1,\dots,s_n)$ are multivariate normal distr's, for any choice of n and $s_1,\dots,s_n$.

→ All we need to specify a Gaussian Process, GP. are the mean functions $m(s)$ and Covariance function $C(s,s')$ at location s and s'.

1.2 $C(s,s')$ is positive definite if for any positive integer n, $s_j \in S$ and $c_j \in \mathbb{R}$ for $j=1,\dots,n$, $\sum_{i,j} c_i c_j C(s_i,s_j) \geq 0$

→ $C(s,s')$ is a valid covariance function if it is positive defini

→ Determining if a f^n is pd. is difficult. Most commonly used tools to obtain classes of valid cov. f^ns is spectral analysis.

→ Very often the correlation f^n is used: $\rho(s,s') = C(s,s')/\sqrt{C(s,s)C(s'}$

→ Notice $C(s,s')$ defines a variance f^n $\sigma^2(s) = C(s,s)$.

1.3 A random field is strictly stationary if for any finite collection of sites $s_1,\dots,s_n$ and any $u \in S$, the joint distr. of $(X(s_1),\dots,X(s_n))$ is the same as that of $(X(s_1+u),\dots,X(s_n+u))$.

→ Weak stationarity: A stochastic process is weakly stationary if its mean function is constant (i.e. $m(s)=m$ $\forall s \in S$) and its covariance f^n is invariant under time shifts (i.e. $\forall s_i, s_j \in S$, $Cov(X(s_i), X(s_j+u)) = C(s_i - s_j)$, a f^n of $s_i - s_j$).

→ For GP, weak stationarity and strong/strict stationarity are the same

1.4 Assume that $E(X(s+u)-X(s))=0$, then the variogram is defined as

$$E(X(s+u)-X(s))^2 = Var(X(s+u)-X(s))$$

→ The process is intrinsically stationary if the variogram depends only on u:

→ We can write $Var(X(s+u)-X(s)) = 2\gamma(u)$ where $\gamma(u)$ is called the semi-variogram.

Notice that the former def'n (variogram) is based on the second moment difference $X(s+u)-X(s)$. If the cov. of the process exists, then $\gamma(u)=C(0)-C(u)$. So, we can recover the semi-variogram from the cov. f^n. Also, if the process is <u>weakly stationary</u> then it is intrinsically stationary. If the semi-variogram is given, we need an additional condition on γ to obtain the cov f^n. In fact we have that

$$C(u)=C(0)-\gamma(u)=\lim_{\|h\|\to\infty}\gamma(h)-\gamma(u)$$

This limit is valid only if the association between two locations vanishes as the locations becomes infinitely distant.

Clearly the limit may not exist, so strict ~~stat~~ stationarity does not imply weak stationarity.

MSE Prediction: when pred'ing T a RV using obs vals y, $\hat{T}$ is pred'or. the $MSE(\hat{T})=E(T-\hat{T})^2$; $MSE(\hat{T})$ minimized at $\hat{T}=E(T|y)$

CONDIT NORMAL: $\begin{pmatrix}\underline{Y}_1\\ \underline{Y}_2\end{pmatrix}\sim N\left(\begin{pmatrix}\underline{\mu}_1\\ \underline{\mu}_2\end{pmatrix},\begin{pmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{pmatrix}\right)$ $E(\underline{Y}_1|\underline{Y}_2)=\underline{\mu}_1+\Omega_{12}\Omega_{22}^{-1}(\underline{Y}_2-\underline{\mu}_2)$

$V(\underline{Y}_1|\underline{Y}_2)=\Omega_{11}-\Omega_{12}\Omega_{22}^{-1}\Omega_{21}$, $Y_1=Y(s_0)$ (unobs val), $\underline{Y}_2=\underline{y}$

then $\Omega_{11}=\sigma^2+\tau^2$, $\Omega_{12}=\gamma^T$ $\Omega_{22}=\Sigma=\sigma^2 H(\phi)+\tau^2 I$

$\gamma^T=(\sigma^2\rho(\phi;d_{01}),\ldots,\sigma^2\rho(\phi;d_{01}))$; $d_{ij}=\|s_i-s_j\|$ dist betw. s_i,s_j

<u>Thus</u> $E(Y(s_0)|\underline{y})=x_0^T\beta+\gamma^T\Sigma^{-1}(\underline{y}-X\beta)$; $V(Y(s_0)|\underline{y})=\sigma^2+\tau^2-\gamma^T\Sigma^{-1}\gamma$

corr f^n for WS stoch proc. w/ non-neg var is a

$C(t)=\int_{-\infty}^{\infty}\exp(iut)f(u)du$ f-spect. dens. f^n.

Definitions

1.5 A stationary random field is <u>isotropic</u> if the covariance f^n depends on distance alone, i.e. $C(s,s') = C(\tau)$ where $\tau = \|s - s'\|$

→ Nugget: $\psi^2 = \lim_{\tau \to 0^+} \gamma(\tau)$

→ sill: $\psi^2 + \sigma^2 = \lim_{\tau \to \infty} \gamma(\tau)$

→ range: when $\gamma(\tau)$ reaches the sill for a finite value of τ, the inverse of such value, which is a f^n of ϕ, is called range

1.5a We can obtain <u>geometric anisotropy</u> by considering the norm $\|s\|_K = \sqrt{s'Ks}$ for a positive definite matrix K. If ρ is a valid correlatio f^n for an isotropic random field, we can define $\rho_K(s,s') = \rho(\|s-s'\|_K)$

1.5b Let $u = (u_1, \dots, u_K)$, $K \le n$, $u_i \in \mathbb{R}^{n_i}$, then

$$\rho(h) = \rho(h_1) \cdots \rho_K(h_K)$$

is a valid correlation f^n if and only if each ρ_i is a valid correlation f^n and $\sum_i n_i = n$. ρ is said to be a <u>separable</u> correlation f^n.

1.6a A random field X has <u>continuous sample paths w/ probability one</u> in B if, for every sequence s_n such that $\|s_n - s\| \to 0$ as $n \to \infty$, then

$$P(\omega : |X(s_n, \omega) - X(s, \omega)| \to 0, \text{ as } n \to \infty, \forall s \in B) = 1$$

1.6b A random field X is <u>almost surely continuous</u> in B if for every sequence s_n such that $\|s_n - s\| \to 0$ as $n \to \infty$, then

$$P(\omega : |X(s_n, \omega) - X(s, \omega)| \to 0 \text{ as } n \to \infty) = 1 \quad \forall s \in B.$$

1.6c A random field X is <u>mean square continuous</u> in B if for every sequence s_n such that $\|s_n - s\| \to 0$ as $n \to \infty$, then

$$E(|X(s_n) - X(s)|^2) \to 0 \quad \text{as } n \to \infty \ \forall s \in B$$

provided the expectation exists.

Definitions

1.6d A random field X is continuous in probability in B if for every sequence s_n s.t. $\|s_n - s\| \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} P(\omega : |X(s_n,\omega) - X(s,\omega)| > \delta) = 0 \quad \forall \delta > 0 \text{ and } \forall s \in B$$

→ For GPs, mean square continuity is a necessary and almost sufficient condition for sample path continuity.

Spectral Density for exponential correlation can be calculated as

$$\begin{aligned} f(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-\phi|\tau| - ik\tau\}\, d\tau \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{0} \exp\{(\phi - ik)\tau\}\, d\tau + \int_{0}^{\infty} \exp\{(\phi - ik)\tau\}\, d\tau\right) \\ &= \frac{\phi}{\pi(\phi + k^2)} \end{aligned}$$

so if ϕ is very large compared to k, the value of $f(k)$ is almost constant. i.e. we can approximate the spectum of X by a constant.

White Noise: We define white noise as a GP with constant spectrum. this corresponds to a corr. whose mass is all concentrated at 0.

Spectral dn for Gaussian corr: $f(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-\phi\tau^2 - i\tau k\}\, d\tau = \frac{1}{\sqrt{\pi}} e^{-k^2/4\phi}$

Theorems

1.1 Assume that $EX(t)$ is continuous. Then, a random field $X(t)$ is mean square continuous at t if and only if its covariance function $C(s,s')$ is continuous at $s=s'=t$. Proof: Abrahamsen (1997).

→ Corrolary A stationary random field $X(s)$ is mean square continuous at $s \in S$ if and only if its correlation function $\rho(h)$ is continuous at 0.

→ Notice that the above result implies that when a nugget is added to an isotropic correlation f^n (like the ones in the table from the slides), the resulting random field is not mean square continuous.

1.2 Consider a random field $X(s)$ with covariance f^n C and expectation f^n sufficiently smooth. If the derivative

$$\frac{\partial^{2\nu} C(s,t)}{\partial s_1^{\nu_1} \cdots \partial s_n^{\nu_n} \partial t_1^{\nu_1} \cdots \partial t_n^{\nu_n}} \qquad \cdots\cdots (1.2.1)$$

where $\nu = \sum_i \nu_i$, exists and is finite $\forall\, i=1,\dots,n$; at (s,s), then $X(s)$ is $|\nu|$ times differentiable at s. More over, if the covariance f^n of

$$\frac{\partial^{2\nu} X(s)}{\partial s_1^{\nu_1} \cdots \partial s_n^{\nu_n}}$$

is given by (1.2.1). Proof: Cramér and Leadbetter (1967)

1.3a Bochner's Thm A real f^n on $\mathbb{R}^n$ is positive definite if and only if is can be represented as the Fourier transform of a non-negative bounded measure.

1.3. A real f^n $\rho(\tau)$ on $\mathbb{R}^n$ is a correlation f^n if and only if it can be represented in the form $\rho(\tau) = \int_{\mathbb{R}^n} e^{i\tau' k}\, dF(k)$

where $F(k)$ on $\mathbb{R}^n$ is an n-dimensional distribution function.

see 1.3a → ρ is a corr f^n ⇔ it can be expressed as the characteristic f^n of some n-dim. R.V. Since ρ is real valued, the Fourier integral simplifies to $\rho(\tau) = \int \cos(\tau' k)\, dF(k)$ →

→ When F is continuous a spectral density exists and

$$\rho(\tau) = \int e^{i\tau' \kappa} f(\kappa)\, d\kappa = \int \cos(\tau' \kappa)\, f(\kappa)\, d\kappa$$

The spectral density can be obtained from the corr. using the inverse Fourier transform, thus

$$f(\kappa) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\tau'\kappa} \rho(\tau)\, d\tau$$

→ A general strategy for determining if a given f^n is a valid correlation is to evaluate its spectral density and check if its non-negative $\forall \kappa \in \mathbb{R}^n$.

→ A strategy for creating valid correlation f^ns is to consider a non-negative f^n as a spectral density and find its Fourier transform.

→ For isotropic corr. f^ns, the Wiener-Khintchine's Th'm takes a simpler form. This is because the n-dim. Fourier integral can be replaced by a one dim. integral.

Param est'ion: for mean, assume linear parametric form $\mu(s) = \beta_0 + \sum_{j=1}^{p} \beta_j d_j(s)$ where β_j, $j = 0, \ldots, p$ are unknown $d_j(s)$ = expl. v'bles. Letting D be covariate matrix, we have LSE soln of Norm. eqns: $D'D\hat{\beta} = D'X$ Where X is observed realizations of random field. Resids are $R = X - D\hat{\beta}$. Then LSE is soln to $D'V^{-1}D\hat{\beta} = D'V^{-1}X$ (X not multiple of V) V is cov. matrix.

$L(\beta, \phi, \sigma^2) \propto (\sigma^2)^{-n/2} |V(\phi)|^{-1/2}$, $L(\phi, \sigma^2) \propto \int_{\mathbb{R}^k} L(\beta, \phi, \sigma^2)\, d\beta = (\sigma^2)^{-n/2} |V|^{-1/2} \times (\sigma^2)^{-k/2} |D^T V^{-1} D|^{-1/2} \times \exp\left\{-\frac{s^2(\phi)}{2\sigma^2}\right\}$

$s^2(\phi) = (X - D\hat{\beta})' V(\phi)^{-1} (X - D\hat{\beta})$, $D'V^{-1}D\hat{\beta} = D'V^{-1}X$.

$L(\phi) \propto |V|^{-1/2} |D'V^{-1}D|^{-1/2} (s(\phi)^2)^{-(m-k)/2}$

$\underline{X} = (X(s_1), \ldots X(s_n))'$ to obtain lin. predictor $\hat{X}(s_0)$ that minimizes $(*)\ \underline{E(\hat{X}(s_0) - X(s_0))^2}$. We have $\hat{X}(s_0) = \lambda_0 + \underline{\lambda}'\underline{X}$. $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)'$. Plugging in $(*) = V(\underline{\lambda}'\underline{X} - X(s_0)) + \underbrace{(\lambda_0 + \underline{\lambda}'\underline{\mu} - \mu(s_0))^2}_{0}$. Let $\sigma^2 = V(X(s))$, $\underline{\sigma} = \mathrm{Cov}(\underline{X}, X(s_0))$ and $\Sigma = \mathrm{cov}(\underline{X})$, then $(*) = \sigma^2 + \underline{\lambda}'\Sigma\underline{\lambda} - 2\underline{\sigma}'\underline{\lambda}$ which has min @ $\lambda'\Sigma = \underline{\sigma}'$

Thus optimal $\lambda_0 = \mu(s_0) - \underline{\lambda}'\underline{\mu}$ and $\underline{\lambda} = \Sigma^{-1}\underline{\sigma}$. Optimal $X(s_0) = \mu(s_0) + \underline{\sigma}'\Sigma^{-1}(\underline{X} - \underline{\mu})$

Simple kriging →

Theorems

1.4 A real function $\rho(\tau), \tau \in \mathbb{R}$ is correlation f^n if and only if

$$\rho(\tau) = 2^{(n-2)/2}\,\Gamma\left(\frac{n}{2}\right)\int_0^{\infty} \frac{J_{(n-2)/2}(k\tau)}{(k\tau)^{(n-2)/2}}\, d\Phi(k)$$

where Φ is a distribution f^n on $\mathbb{R}$ and J are Bessel f^n of the first kind.

→ When a spectral isotropic density exists it is related to Φ by the formula
$$\Phi(k) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^{k} \omega^{n-1} f(\omega)\, d\omega$$

1.3a The corr. f^n of a stationary random field is the characteristic f^n of some n-dimensional random variable X. Conversely, the characteristic f^n of any random variable is a corr. f^n for stationary random field in $\mathbb{R}^n$. i.e. given a corr. f^n ρ, we can write

$$\rho(\tau) = E e^{i\tau' X} \quad \text{for some r.v. } X.$$

1.5 If ρ is a stationary correlation f^n that is continuous everywhere except possibly at zero, then

$$\rho(\tau) = a\rho_w(\tau) + b\rho_c(\tau), \quad a, b \geq 0$$

where $\rho_w(0)=1$ and $\rho(\tau)=0$, if $\tau \neq 0$; ρ_c is a stationary correlation f^n that is continuous everywhere. i.e. the covariance of a stationary correlation f^n is decomposed into a linear combination of white noise corr. and a continuous corr.

Th'm If ρ is a stat. corr f^n that is cont. everywhere except poss at 0, then $\rho(\tau) = a\rho_w(\tau) + b\rho_c(\tau)$, $a,b\geq 0$ where $\rho_w(0)=1$ and $\rho_w(\tau)=0$, if $\tau \neq 0$. ρ_c is contin. corr f^n.

Th'm a real f^n $\rho(\tau)$ is an corr f^n $\Leftrightarrow \rho(\tau) = 2^{(n-2)/2}\int_0^{\infty} \frac{J_{(n-2)/2}(k\tau)}{(k\tau)^{(n-2)/2}}\, d\Phi(k)$ where Φ is a distr. f^n $\in \mathbb{R}$ and J_i is Besel f^n 1st kind.

Ex: • $\rho(\tau) \propto \exp\{-a\tau\}; a, \tau > 0$ (Exp) has spec. dens: $f(k) = \frac{1}{(k^2+a^2)^{(n+1)/2}}$

• $\rho(\tau) \propto \exp\{-a\tau^2\}$ then $f(k) \propto \exp\{-k^2/4a\}$ $\forall k > 0$

• Matérn: $\rho(\tau) \propto (a\tau)^{\nu} K_\nu(a\tau)$ then $f(k) \propto 1/(k^2+a^2)^{(\nu+n)/2} > 0$ $\forall k$

Theorems

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1.5 If ρ is a stationary correlation f^n that is continuous everywhere except possibly at zero, then

$$\rho(t) = a\rho_w(\tau) + b\rho_c(\tau), \quad a, b \geq 0$$

where $\rho_w(0) = 1$ and $\rho(\tau) = 0$, if $\tau \neq 0$; ρ_c is a stationary correlation f^n that is continuous everywhere. i.e. the covariance of a stationary correlation f^n is decomposed into a linear combination of white noise corr. and a continuous corr.