qLearn Week 3: Single Qubit Quantum Gates

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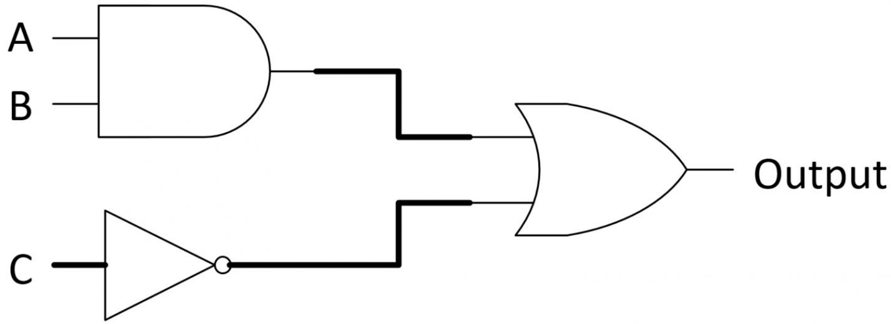
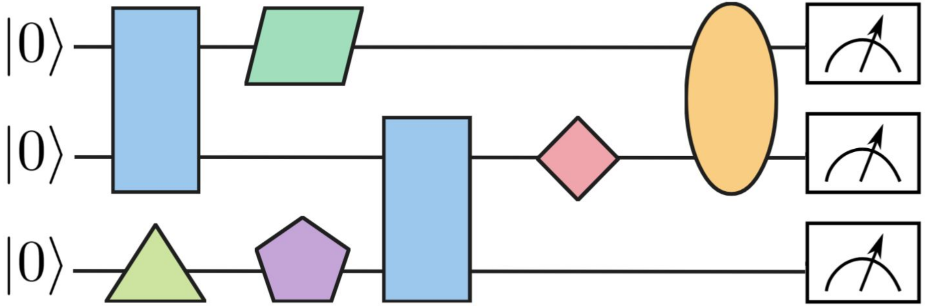
Last Week Recap

Classical Circuits - Boolean Logic

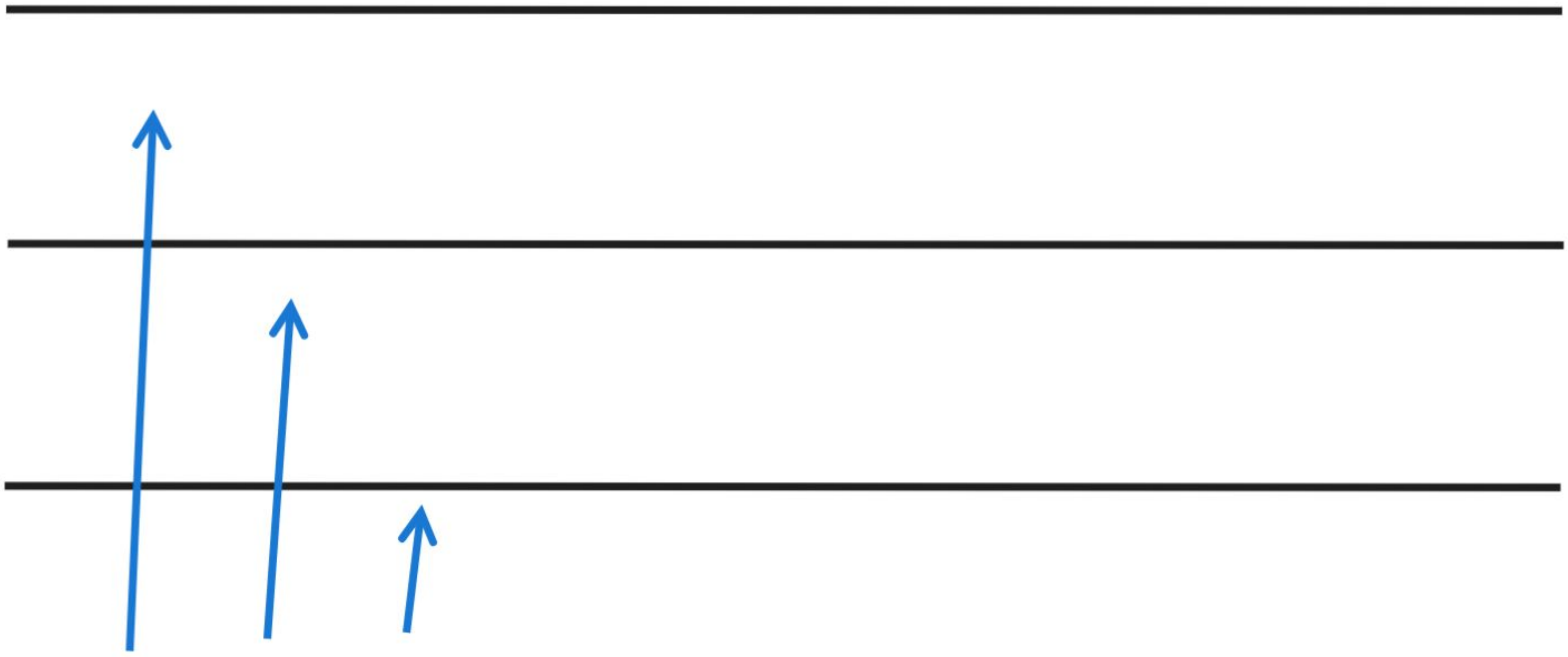


- Binary logic (0 and 1 inputs and outputs)
- <u>**Physical**</u> gates and wiring
- Different types of gates indicate different operations
- Every computer calculation and operation is a result of these circuits

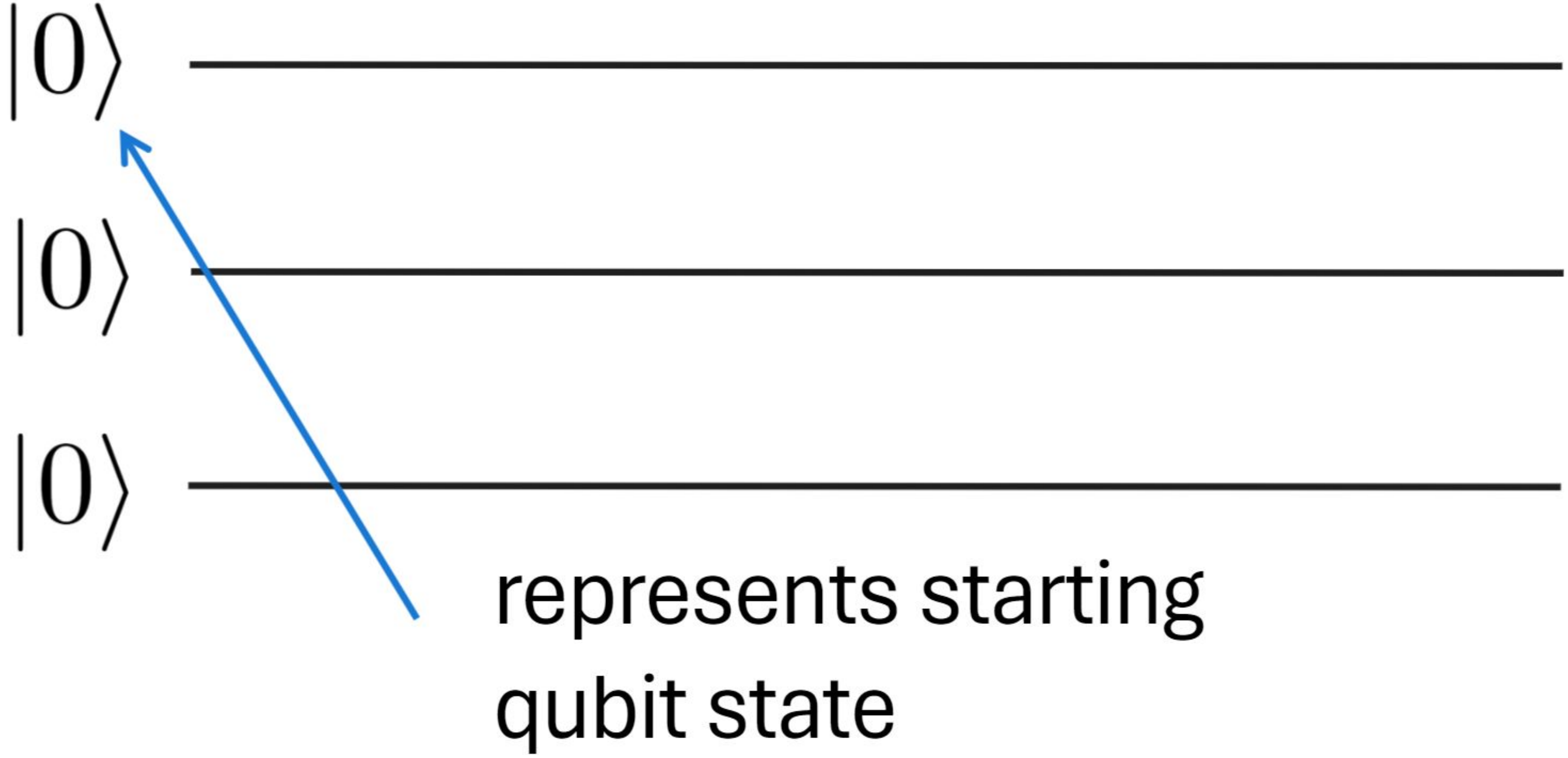
Quantum Circuits

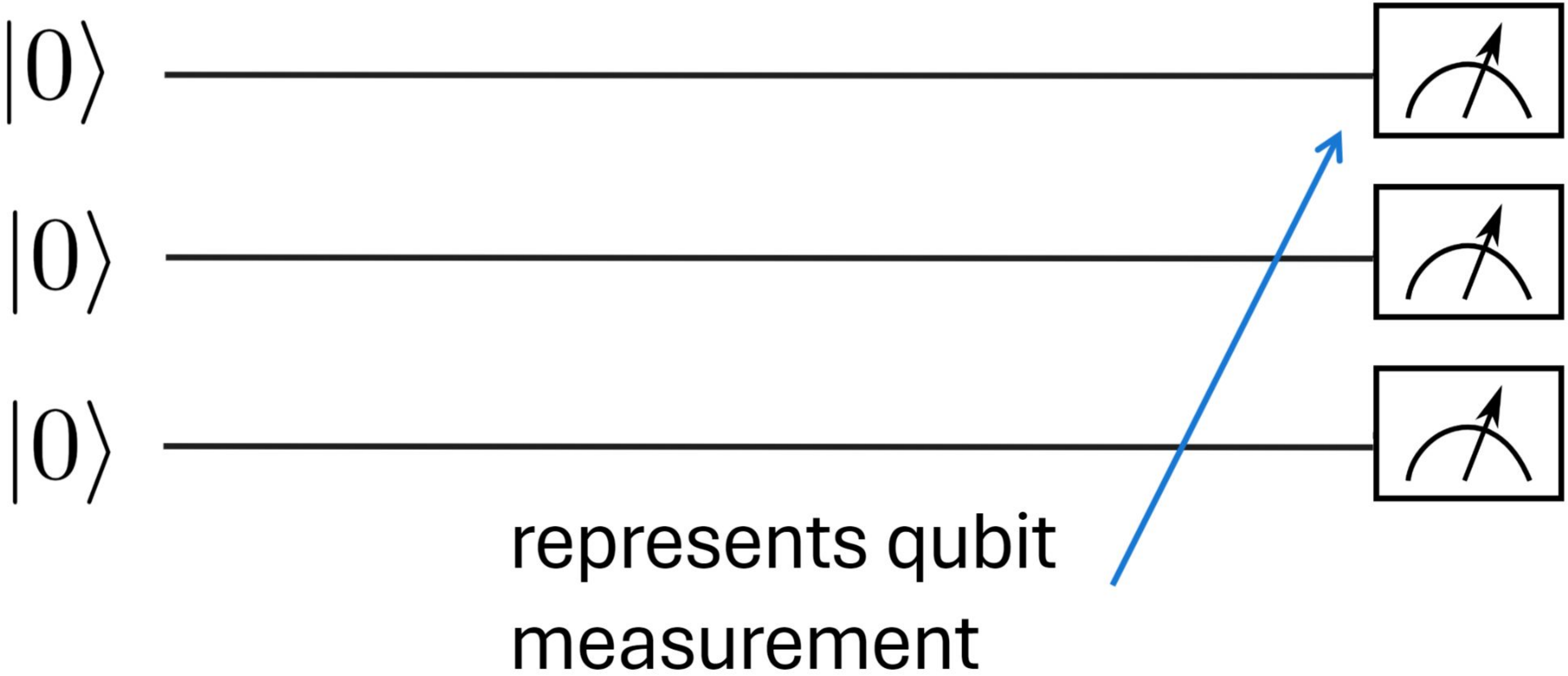


- **<u>Visual</u>** representation of operations being made on a qubit (very complicated physics, learn more in hardware section!!!)
- Represented by linear algebraic transformations mathematically
- Contains information about quantum algorithms, visual representation of them

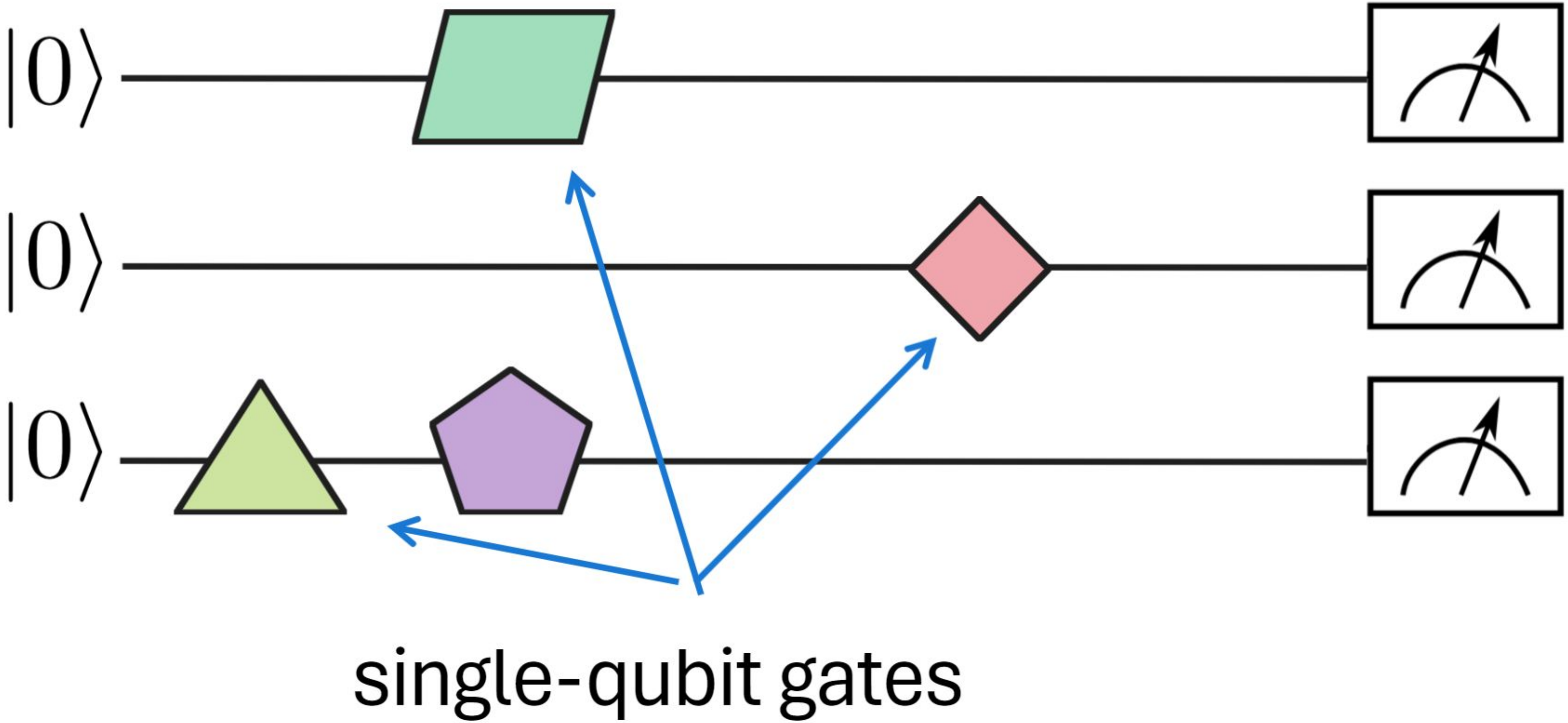


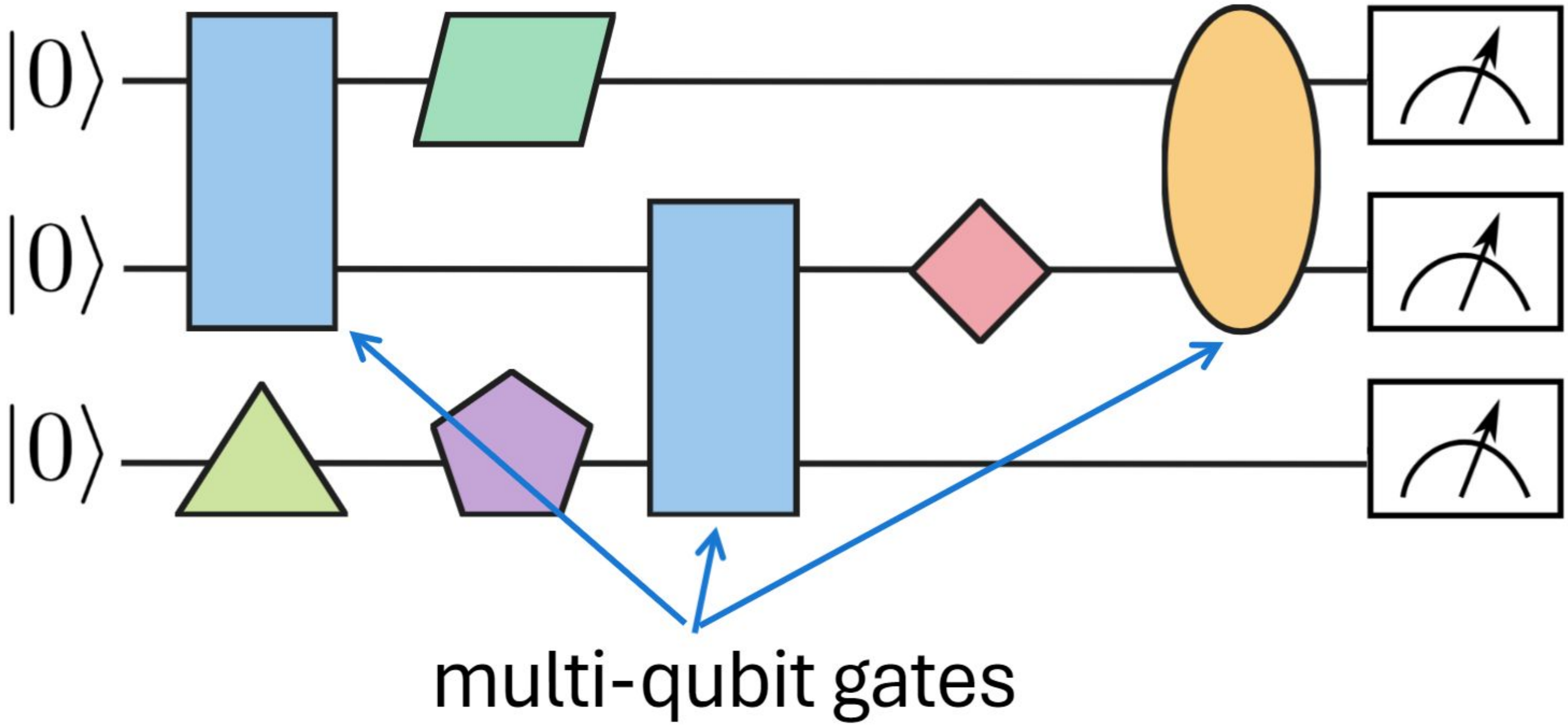
'wires' represent individual qubits


|0⟩
|0⟩
|0⟩
represents starting
qubit state

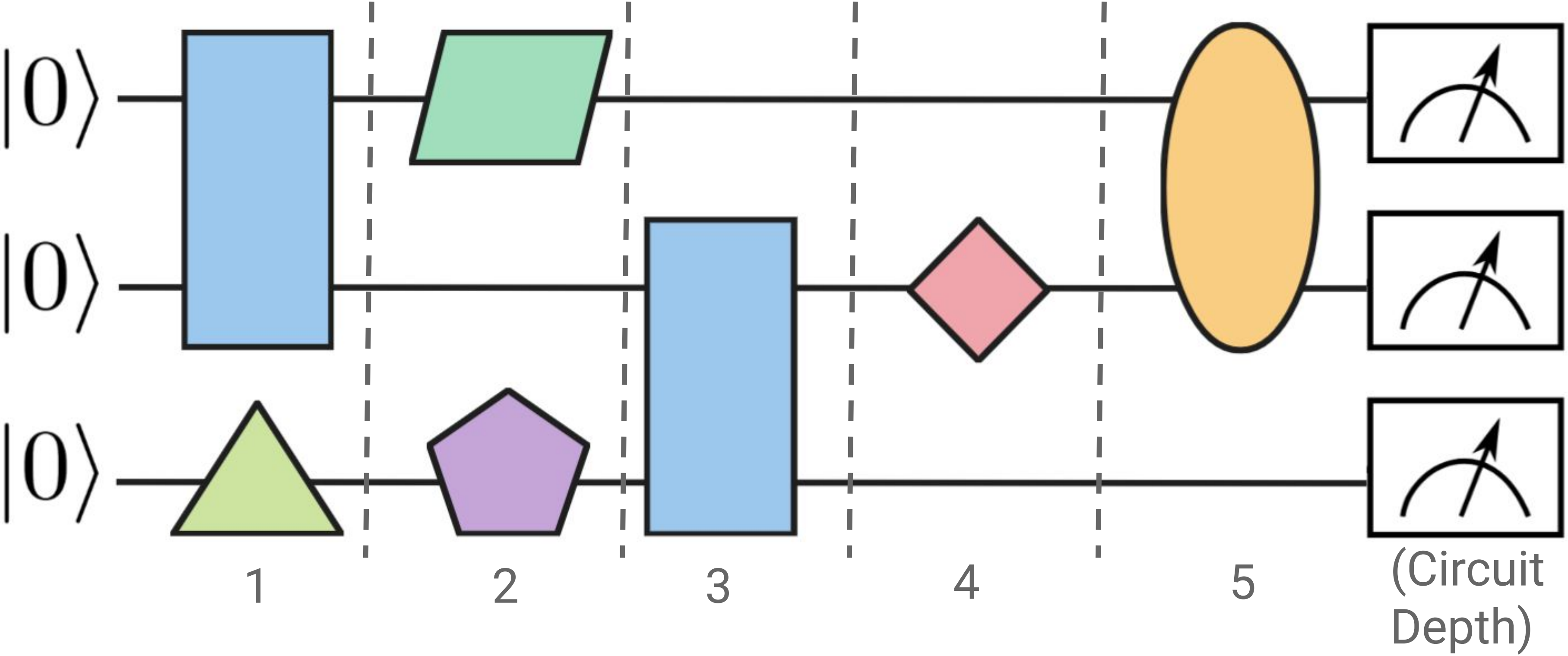


**NOTE: Does not always have to go at end of circuit!!! We will see examples later

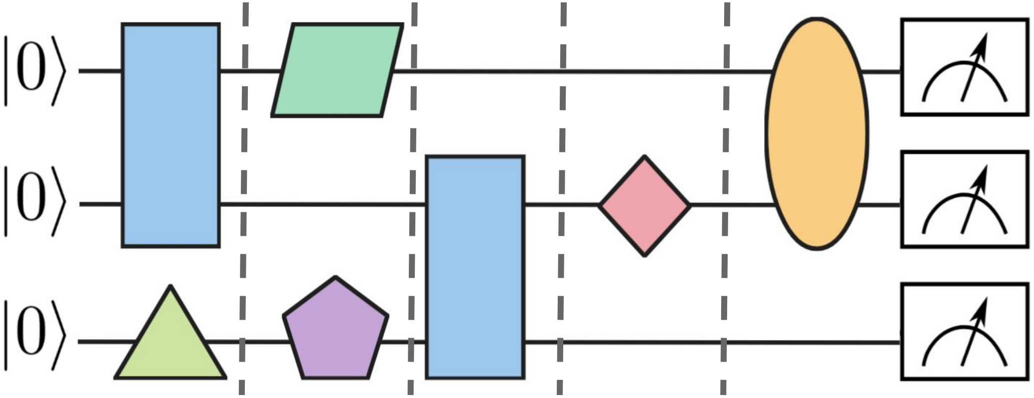

$|0\rangle$
$|0\rangle$
$|0\rangle$
single-qubit gates


$|0\rangle$
$|0\rangle$
$|0\rangle$
multi-qubit gates

Circuits are read **left to right** (just like you're reading this!)

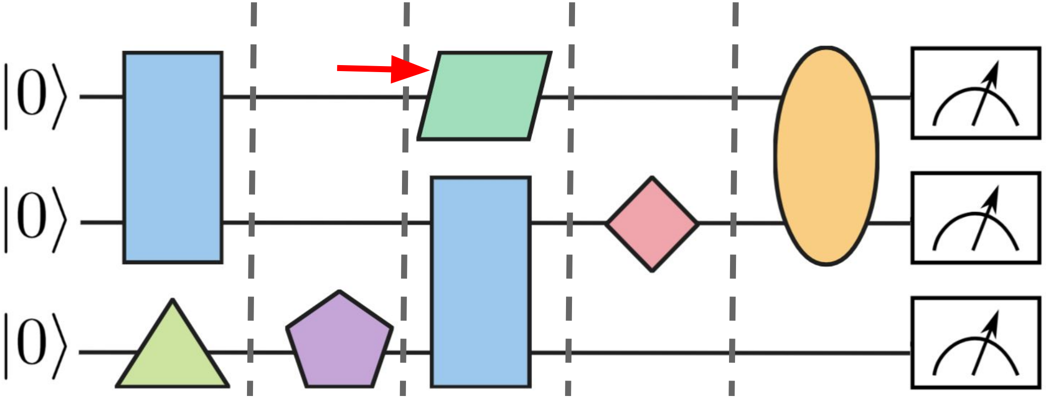


Gates acting on **seperate** qubits act in **parallel**



We do this for simplicity and resource conservation

Is **EQUAL** to



New Topic: Unitary Matrices!!

Key Vocab:

1) Identity: Square matrix with 1s on diagonal and 0s elsewhere
2) Transpose^T: flipped matrix values over its diagonal
3) Conjugate* (for this lecture's sake): sign of imaginary part of a complex number is flipped

As previously mentioned, qubit operations are represented **mathematically** as **unitary matrices**

What is a unitary matrix and why do we use them?

- Qubit state vectors are normalized, as in they **always** have length 1
- Unitary matrices have special properties that, when applied to qubit state vectors, preserve their length

A $n \cdot n$ matrix U is unitary if

$$UU^\dagger = U^\dagger U = I_n$$

Where I_n represents the n-dimensional identity, and $U^\dagger$ is the notation for the conjugate transpose of U

How do these represent qubit transformations???

Parameterization of a unitary matrix!!

Proof: Representing a Unitary in Polar Form

For unitary with complex elements

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Write each element in their polar forms

$$a = e^{i\alpha}\cos(\theta/2), b = -e^{i(\beta-\gamma)}\sin(\theta/2)$$

$$c = e^{i\gamma}\sin(\theta/2), d = e^{i(\beta-\alpha)}\cos(\theta/2)$$

$$U = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & -e^{i(\beta-\gamma)}\sin(\theta/2) \\ e^{i\gamma}\sin(\theta/2) & e^{i(\beta-\alpha)}\cos(\theta/2) \end{pmatrix}$$

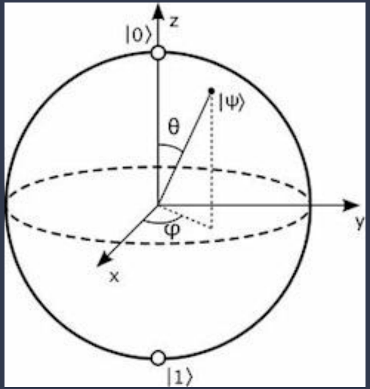
Using substitutions: $\phi = \beta - \alpha - \gamma, \omega = \gamma - \alpha$

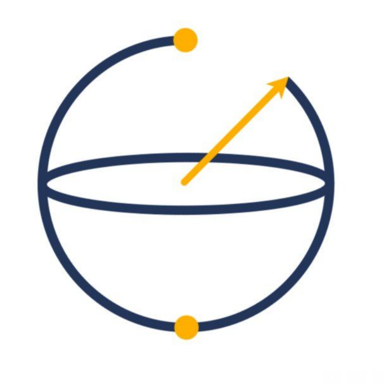
(and phases, will discuss soon!!)

We obtain:

$$U(\phi, \theta, \omega) = \begin{pmatrix} e^{-i(\phi+\omega)/2}\cos(\theta/2) & -e^{i(\phi-\omega)/2}\sin(\theta/2) \\ e^{-i(\phi-\omega)/2}\sin(\theta/2) & e^{i(\phi+\omega)/2}\cos(\theta/2) \end{pmatrix}$$

The Bloch Sphere



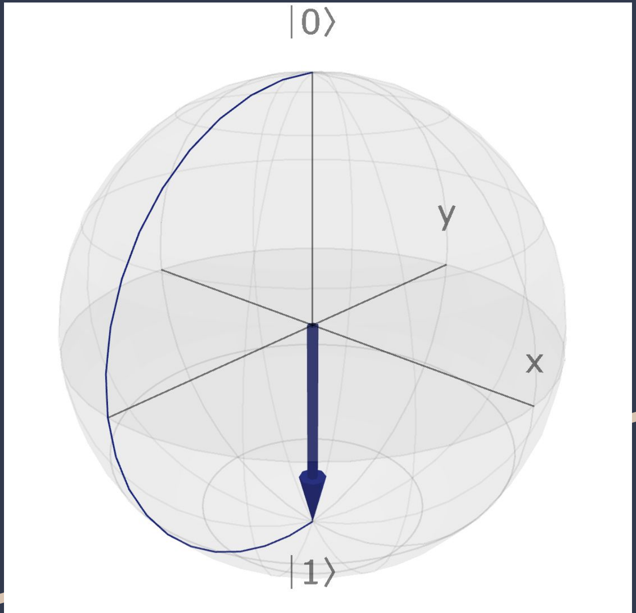


- Who is Bloch?
- **3-D Visualization** of qubit state
- Representation of qubit state and operations

https://bloch.kherb.io/

Single-Qubit Quantum Gates

Pauli X Gate



** Reversible

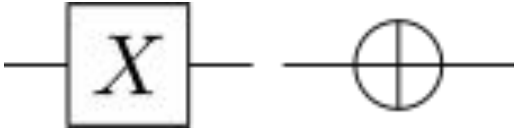
-> Similar to a classical NOT Gate

$$X|0\rangle = |1\rangle$$

$$X|1\rangle = |0\rangle$$

-> Geometrically, 180° rotation around x-axis in Bloch sphere

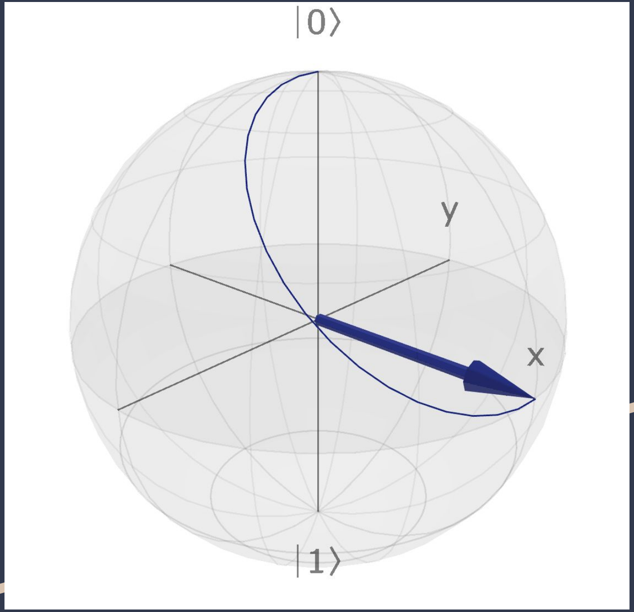
->Gate representation



-> Matrix representation

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

H (Hadamard) Gate



** Reversible

-> Creates a uniform superposition of states

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

-> Geometrically, reflection over the diagonal of the XZ plane in Bloch sphere

->Gate representation

-> Matrix representation

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Code Break!

A Note on Phases

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$
$$\alpha = ae^{i\theta}, \beta = be^{i\varphi}$$
$$|\psi\rangle = ae^{i\theta}|0\rangle + be^{i\varphi}|1\rangle$$
$$= e^{i\theta}(a|0\rangle + be^{i(\varphi-\theta)}|1\rangle)$$
$$= a|0\rangle + be^{i(\varphi-\theta)}|1\rangle$$

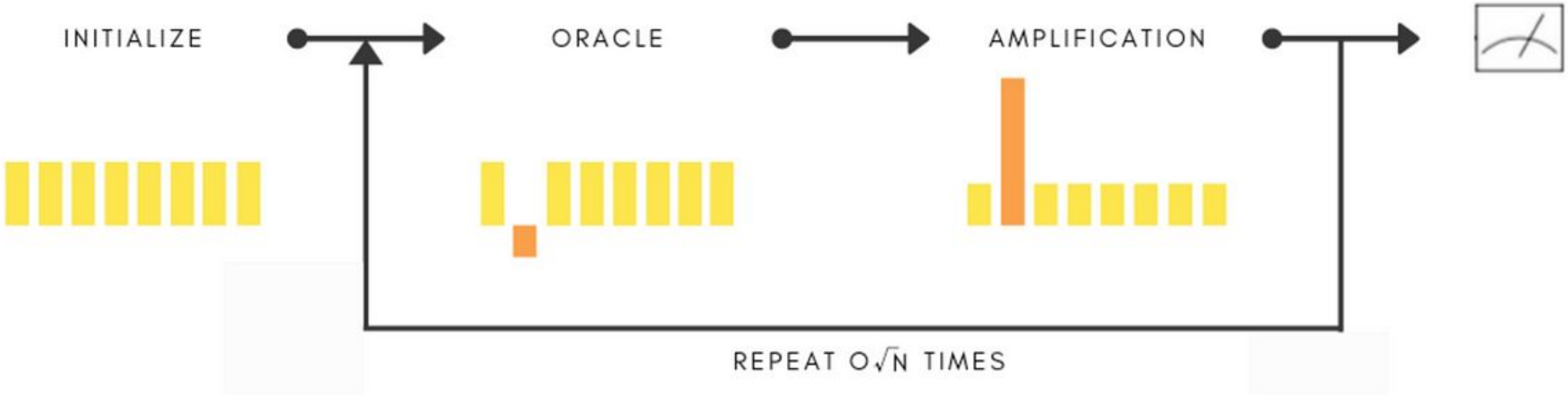
Global Phase:

- Transformation that multiplies the entire state by a complex number of unit magnitude
- Note: No observable impact on measurement probabilities

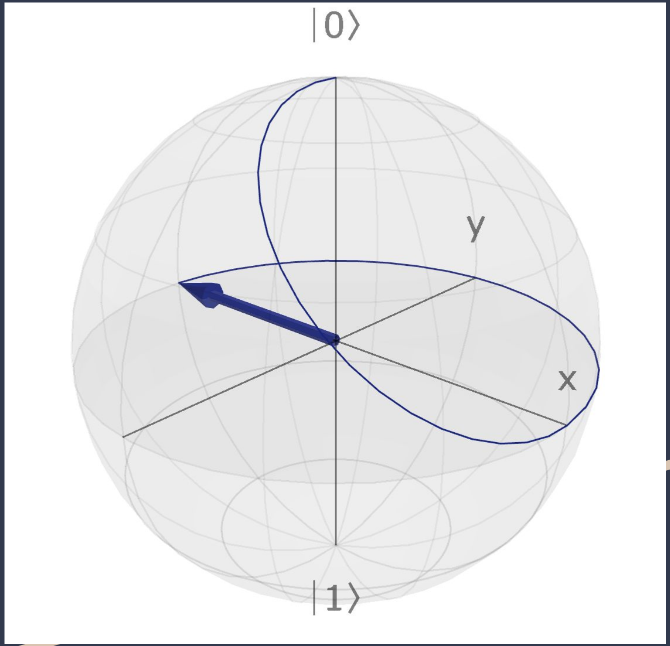
Relative Phase:

- Changes the phase relationship between the components of a superposition state
- Non-uniform impact on probabilities, thus affects measurement outcomes

Example of Relative Phase Use in Algorithm



Pauli Z Gate



** Reversible

-> 'Flips' the phase of the state

$$Z|0\rangle \rightarrow |0\rangle$$

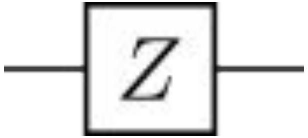
$$Z|1\rangle \rightarrow -|1\rangle$$

$$Z|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

$$Z|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

-> Geometrically, 180° rotation around Z-axis

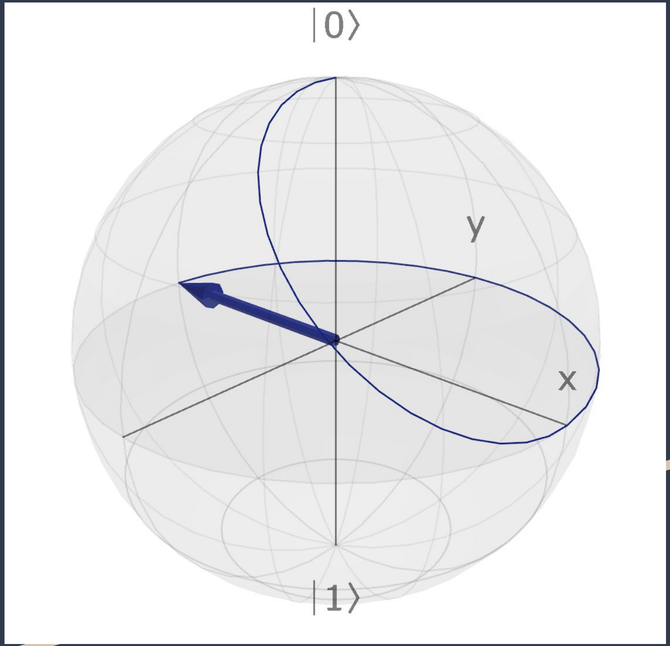
->Gate representation



-> Matrix representation

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

RZ, S, T Gates

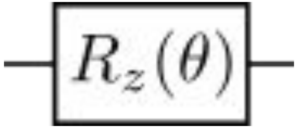


-> 'Rotates' the phase of the state

$$RZ(\omega)|\psi\rangle = \alpha|0\rangle + \beta e^{i\omega}|1\rangle$$

-> Geometrically, ω rotation around Z-axis

->Gate representation



-> Matrix representation

$$RZ(\omega) = \begin{bmatrix} e^{-i\frac{\omega}{2}} & 0 \\ 0 & e^{i\frac{\omega}{2}} \end{bmatrix}$$

-> Pauli Z is just RZ(π)

-> S is RZ(π/2)

-> T is RZ(π/4)

RX, RY,Y Gates

$$|\psi\rangle = a|0\rangle + be^{i\phi}|1\rangle$$

$$a^2 + b^2 = 1, a = cos(\theta/2), b = sin(\theta/2)$$

$$|\psi\rangle = cos(\theta/2)|0\rangle + sin(\theta/2)e^{i\phi}|1\rangle$$

$$RX(\theta)|0\rangle = \cos\frac{\theta}{2}|0\rangle - i\sin\frac{\theta}{2}|1\rangle$$

$$RX(\theta)|1\rangle = -i\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle$$

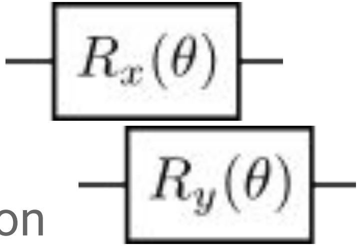
$$RY(\theta)|0\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$$

$$RY(\theta)|1\rangle = -\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle$$

-> Used in precisely altering qubit state

-> Geometrically, ω rotation around respective axes

->Gate representation



-> Matrix representation

$$RX(\theta) = \begin{bmatrix} \cos(\frac{\theta}{2}) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}$$

$$RY(\theta) = \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}$$

Code Break!

$$U(\phi, \theta, \omega) = \begin{pmatrix} e^{-i(\phi+\omega)/2}\cos(\theta/2) & -e^{i(\phi-\omega)/2}\sin(\theta/2) \\ e^{-i(\phi-\omega)/2}\sin(\theta/2) & e^{i(\phi+\omega)/2}\cos(\theta/2) \end{pmatrix}$$

$$RX(\theta) = \begin{bmatrix} \cos(\frac{\theta}{2}) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}$$

$$RY(\theta) = \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}$$

$$RZ(\omega) = \begin{bmatrix} e^{-i\frac{\omega}{2}} & 0 \\ 0 & e^{i\frac{\omega}{2}} \end{bmatrix}$$

Universal Gates

$$U(\phi, \theta, \omega) = \begin{pmatrix} e^{-i(\phi+\omega)/2}\cos(\theta/2) & -e^{i(\phi-\omega)/2}\sin(\theta/2) \\ e^{-i(\phi-\omega)/2}\sin(\theta/2) & e^{i(\phi+\omega)/2}\cos(\theta/2) \end{pmatrix}$$

-> Example: Show that the unitary can be expressed using only 3 gates from the set {RZ, RY}

$$Rz(\phi) = \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}$$

$$Ry(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

$$Rz(\omega) = \begin{bmatrix} e^{-i\omega/2} & 0 \\ 0 & e^{i\omega/2} \end{bmatrix}$$

$$U(\phi, \theta, \omega) = Rz(\omega)Ry(\theta)Rz(\phi)$$

Equivalent Gate Identities

1) $H = Ry(\frac{\pi}{2})Rx(\pi)$

2) $Ry(\theta) = Rz(-\frac{\pi}{2})Rx(\theta)Rz(\frac{\pi}{2})$

3) $Rx(\theta) = Rz(\frac{\pi}{2})Ry(\theta)Rz(-\frac{\pi}{2})$

4) $Rz(\theta) = Ry(\frac{\pi}{2})Rx(\theta)Ry(-\frac{\pi}{2})$

5) $U(\phi, \theta, \omega) = Rz(\omega)Ry(\theta)Rz(\phi)$

Quantum Gates Cheat Sheet

See you next week!