qLearn Week 4: Multi-Qubit Quantum Gates & Entanglement

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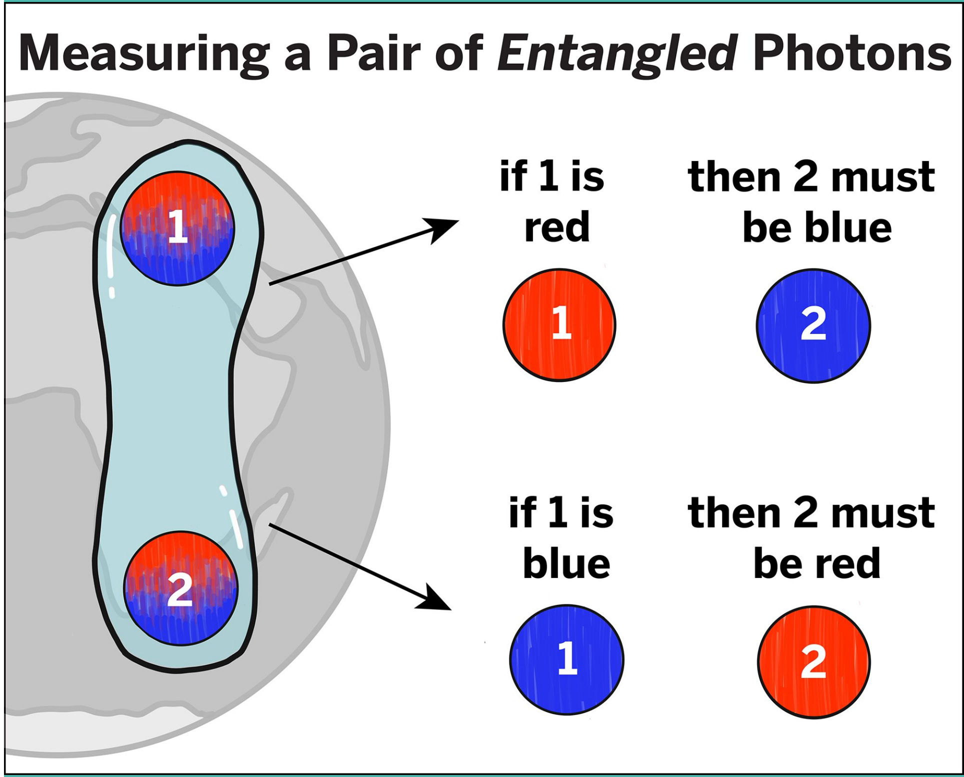
Last Week Recap

The Concept of Quantum Entanglement



(Short) Math Recap

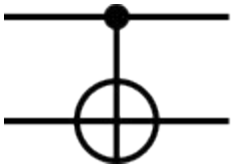
For quantum states $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$

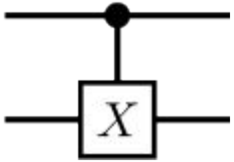
$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix}$$

$$= \begin{pmatrix} a\binom{c}{d} \\ b\binom{c}{d} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

- Tensor Product: operation that combines vector spaces into a new, larger vector space
- We can use it to describe the combinations of multiple quantum systems
- What does this all mean???

CNOT (Controlled Not) Gate



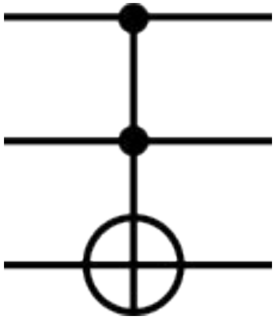


$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- Simplest multi-qubit Gate
- Conditional operation: conditionally flips the target qubit based on state of the control qubit
- CNOT Truth Table:

$$|00\rangle \longrightarrow |00\rangle$$
$$|01\rangle \longrightarrow |00\rangle$$
$$|10\rangle \longrightarrow |11\rangle$$
$$|11\rangle \longrightarrow |10\rangle$$

Toffoli (CCNOT) Gate

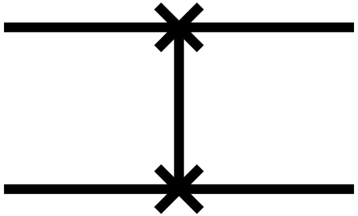


$$\mathrm{CCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- Controlled-Controlled-NOT
- Conditional operation: conditionally flips the target qubit based on state of the control qubits
- Toffoli Truth Table:

$$|00K\rangle \rightarrow |00K\rangle$$
$$|01K\rangle \rightarrow |01K\rangle$$
$$|10K\rangle \rightarrow |10K\rangle$$
$$\cdots$$
$$|11K\rangle \rightarrow |11\overline{K}\rangle$$

SWAP Gate



- ‘Swaps’ the state of two qubits

$$SWAP(|00\rangle) = |00\rangle$$
$$SWAP(|01\rangle) = |10\rangle$$
$$SWAP(|10\rangle) = |01\rangle$$
$$SWAP(|11\rangle) = |11\rangle$$

Challenge: Find the matrix of the SWAP gate

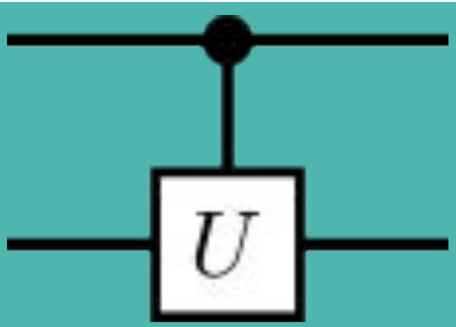
Hint: what basis states are affected?

$$|01\rangle \text{ and } |10\rangle \text{ are the affected states}$$

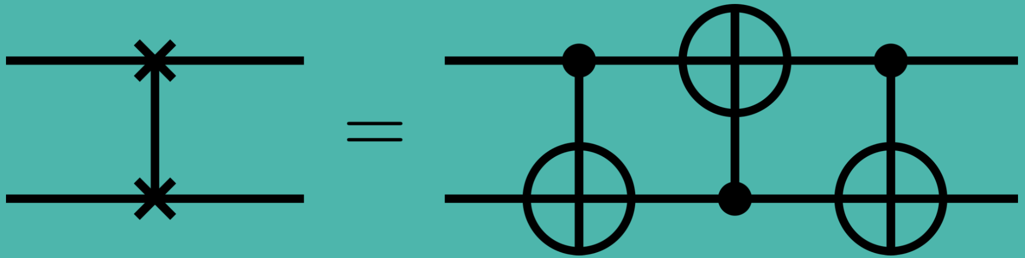
$$\text{SWAP swaps said states}$$

$$\therefore SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Equivalent Gates in Multi-Qubit Systems



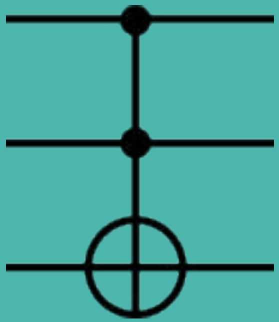
$$CU = \begin{pmatrix} I_2 & 0 \\ 0 & U \end{pmatrix}$$



Universal Gates in Multi-Qubit Systems

$$\{CNOT, RY, RZ\}$$

$$\{CNOT, H, T\}$$



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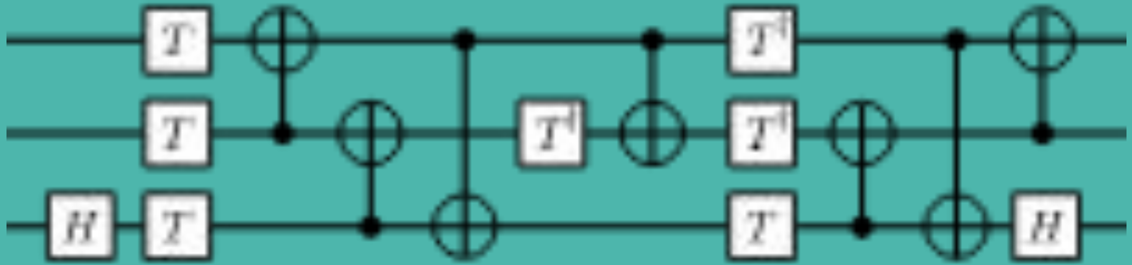

The Magic of Quantum Algorithms

A Multi-Qubit Uniform Superposition

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|\psi\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle$$

$$= \frac{1}{\sqrt{2^n}}(||000...0\rangle + |000...1\rangle ... + |111...1\rangle)$$

The Magicians Tool... the Oracle!

$$f(x) = \begin{cases} 1 \text{ if } \mathrm{x} = \mathrm{s} \\ 0 \text{ otherwise} \end{cases}$$

$$|x\rangle \longrightarrow (-1)^{f(x)}|x\rangle =$$

$$U_f(x) = \begin{cases} -|x\rangle \text{ if } \mathrm{x} = \mathrm{s} \\ |x\rangle \text{ otherwise} \end{cases}$$

Our First ‘real’ Quantum Algorithm, Grover’s Algorithm!

Grover's Algorithm

1) Start with an equal superposition of all possible states (how do we do this?)

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

2) Apply the oracle to flip the phase of the amplitude of the correct state

$$U_f|\psi\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$$

3) Diffusion Operator: inverts the amplitudes about the average of all amplitudes, increase amplitude of the correct state and decreasing the rest

Diffusion Operator in Grover's Algorithm

$$D = 2|\psi\rangle\langle\psi| - I$$

Mean (average) of amplitudes

$$\text{Mean} = \frac{1}{4}(\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) = \frac{1}{4}$$

$$\text{For } |00\rangle : 2\frac{1}{4} - \frac{1}{2} = 0$$

$$\text{For } |01\rangle : 2\frac{1}{4} - (-\frac{1}{2}) = 1$$

$$\text{For } |10\rangle : 2\frac{1}{4} - \frac{1}{2} = 0$$

$$\text{For } |11\rangle : 2\frac{1}{4} - \frac{1}{2} = 0$$