On the Stability of Nonlinear Receding Horizon Control: A Geometric Perspective

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Abstract

The widespread adoption of nonlinear Receding Horizon Control (RHC) strategies by industry has led to more than 30 years of intense research efforts to provide stability guarantees for these methods. However, current theoretical guarantees require that each (generally nonconvex) planning problem can be solved to (approximate) global optimality, which is an unrealistic requirement for the derivative-based local optimization methods generally used in practical implementations of RHC. This paper takes the first step towards understanding stability guarantees for nonlinear RHC when the inner planning problem is solved to first-order stationary points, but not necessarily global optima. Special attention is given to feedback linearizable systems, and a mixture of positive and negative results are provided. We establish that, under certain strong conditions, first-order solutions to RHC exponentially stabilize linearizable systems. Crucially, this guarantee requires that state costs applied to the planning problems are in a certain sense 'compatible' with the global geometry of the system, and a simple counter-example demonstrates the necessity of this condition. These results highlight the need to rethink the role of global geometry in the context of optimization-based control.

1 Introduction

The global stabilization of nonlinear systems is one of the most fundamental and challenging problems in control theory. In principle, the search for a stabilizing controller can be reduced to finding a control Lyapunov function [1], yet the search for such a function may be just as challenging as the search for a controller. Historically, the framing of these equivalent synthesis problems has been guided by two distinct perspectives: geometric control and optimal control.

Geometric control is a broad term that loosely refers to design methodologies which systematically exploit global system structure to achieve a control objective for a specific class of systems [2, 3, 4, 5, 6]. In the context of stabilization, this has led to constructive procedures for synthesizing stabilizing controllers or control Lyapunov functions for many important classes of systems (feedback linearizable, strict feedback, etc). Despite the broad impact of this collection of techniques, the traditional criticism of these approaches is that they often require significant human ingenuity and system-specific analysis to implement.

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Conversely, optimal control offers a way to sidestep these challenges by formulating certain infinite-horizon problems that synthesize optimal stabilizing controllers automatically [7]. Unfortunately, exact solutions to the infinite-horizon problem require solving the Hamilton-Jacobi-Bellman partial differential equation, whose exact solution is generally impractical in state spaces of large or even modest dimensionality [8]. The response to these limitations has been the rise of *receding horizon* or *model-predictive* control strategies [9, 10, 11], which attempt to stabilize the system by solving a sequence of more tractable open-loop optimal control problems along the system trajectory. Providing stability guarantees for receding horizon methods has, however, been an active area of research for theoreticians for more than 30 years, and it remains largely open.

In this paper we aim to link the geometric and optimal control perspectives, showing how such a link provides new insights into RHC stability. Specifically, we show how the running costs used in RHC formulations interact with both the local and global geometric structure of the control system to influence the convergence of derivative-based search algorithms to stabilizing solutions. Our main result (Theorem 1) provides sufficient conditions that ensure that all (approximate) stationary points of the planning problems correspond to exponentially decaying state trajectories. We use this result to provide stability guarantees for nonlinear RHC when the implementation relies on derivative-based descent methods (Theorem 2). This goes beyond prior results which generally require (approximate) global optimality during each planning phase. Our result requires that the planning horizons have sufficient (though modest) length, but is largely independent of the particular terminal cost.

Our results are based on a novel analysis technique in which we approximate each (generally nonconvex) RHC planning problem along a candidate solution via a convex problem obtained from first-order expansion of the system dynamics. Unlike Pontryagin's necessary conditions [7], the traditional way to characterize stationary points in optimal control, this approach does not use a Taylor expansion of the cost function along the nominal trajectory. We accordingly obtain a richer understanding of how the local (first-order) structure of the system dynamics interacts with the global structure that is induced by the RHC cost function.

We focus on feedback linearizable systems [6], the most widely studied class of systems in the geometric control literature. Our main result (Theorem 1) shows that solutions to the RHC planning problems are exponentially decaying in this setting if (1) the running state and input costs are *convex in the linearizing coordinates*, (2) the control cost is sufficiently small relative to the state cost, (3) the Jacobian linearizations of the dynamics along the planned trajectory are uniformly stabilizable, and (4) the nonlinearity in the input is sufficiently small.

These four conditions are quite stringent and one would like to know if they can be weakened. We present counter-examples that show that RHC can fail to globally stabilize the system if either condition (1)—convexity in linearizing coordinates—or condition (2)—sufficiently small control costs—fail. Our counter-examples are surprisingly simple, suggesting that it will be difficult to weaken these conditions. Condition (3) is clearly necessary even for linear systems. Whether condition (4) is necessary remains open. In addition, we establish that there can be stabilizing trajectories which satisfy our conditions, but are not global optima; therefore, the results we establish hold for cost functionals which need not exhibit quasi-convexity or any other obvious form of "hidden" convexity.

More broadly, we highlight the need for a new theory that connects the local geometry of the stationary conditions from optimal control to the global structures identified by the geometric control literature. Beyond further applications to stability theory, such a framework holds the promise of illuminating precisely when provably convergent local optimization methods can reliably produce other desirable global behaviors for controlled systems, such as set-invariance or asymptotic tracking.

Further Background on RHC For basic background on RHC, we refer the reader to any of a number of comprehensive reviews on RHC (see, e.g., [9, 12]). For our purposes, it is sufficient to characterize previous theoretical nonlinear RHC formulations into *constrained* approaches and *unconstrained* approaches. Constrained RHC formulations directly enforce stability by either constraining the terminal predictive state to lie at the origin [13], or using inequality constraints to force the system into a neighborhood containing the origin, and then stabilizing the system using a local controller [14]. The usual critique of these methods [15] is that the satisfaction of the relevant constraints may be overly demanding computationally in an online implementation. An alternative approach is to enforce stability using a global control Lyapunov function (CLF) for the system [11] to constrain the evolution of the predictive state. In contrast, unconstrained approaches implicitly enforce stability by either using an appropriate CLF as the terminal cost [15] or a sufficiently long prediction horizon [10].

As alluded to above, most of these stability guarantees require that a globally optimal solution can be found for each prediction problem. Several approaches provide stability guarantees using sub-optimal solutions, but generally require that an initial feasible solution is available [16], which may be restrictive in high-performance real-time scenarios, or require the availability of a CLF [11, 15], which implies that the stabilization problem has already been solved. Thus, in this paper we study unconstrained RHC formulations which use general terminal costs, and aim to provide stability guarantees which only require that a stationary point of each optimization problem can be found. We feel that this accurately reflects the spirit of optimization-based control—to stabilize the system using a minimal amount of system-specific knowledge—as well as the practical computational constraints facing practitioners.

Global Guarantees for Nonconvex Optimization In recent years, the machine learning and optimization communities have made considerable progress on global minimization of certain favorable nonconvex objective functions. It has long been known that gradient-based optimization methods provably converge to approximate stationary points of smooth possibly nonconvex objective functions [17], a fact that has been applied to iterative control [18]. Recent work has elucidated many scenarios where gradient descent provably converges to global optima. For example, certain formulations of system identification [19], sparse coding [20] and LQR policy search [21] exhibit *quasi-convexity*, where gradients "point" in the direction of global optimizers. Similar guarantees have been obtained under the *strict saddle point condition*, where Hessians at critical points have at least one strictly negative eigenvalue [22, 23, 24]. This property holds for a range of core machine learning problems [25, 26, 27]. Unlike prior work, this paper does not seek to establish convergence to global minimizers, which may not be possible. Instead, we demonstrate that all first-order stationary points (under appropriate conditions) exhibit favorable stability properties, regardless of their global optimality.

2 Preliminaries

This paper studies control systems of the form

$$\dot{x}(t) = F(x(t), u(t)), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ the input, and $\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$ denotes time derivatives. We make the following assumptions about the vector field $F\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$:

Assumption 1. *The origin is an equilibrium point of* (1)*, namely,* $F(0,0) = 0$.

Assumption 2. *The vector field F is continuously differentiable. Furthermore, there exist constants $L_F > 0$ such that for each $x_1, x_2 \in \mathbb{R}^n$ and $u_1, u_2 \in \mathbb{R}^m$ we have:*

$$\|F(x_1,u_1) - F(x_2,u_2)\|_2 \leq L_F \left(\|x_1 - x_2\|_2 + \|u_1 - u_2\|_2\right).$$

Taken together, these standard assumptions support the global existence and uniqueness of solutions to (1) on compact intervals of time [28, Proposition 5.6.5].

The primary object of study in is paper will be finite horizon cost functionals $J_T(\cdot; x_0)\colon \mathcal{U}_T \to \mathbb{R}$ of the form

$$\begin{aligned} J_T(\tilde{u}; x_0) = \int_0^T \left(Q(\tilde{x}(\tau)) + R(\tilde{u}(\tau))\right) \mathrm{d}t + V(\tilde{x}(T)) \\ \text{s.t. } \dot{\tilde{x}}(t) = F(\tilde{x}(t), \tilde{u}(t)), \quad \tilde{x}(0) = x_0, \end{aligned} \tag{2}$$

where $T > 0$ is a finite prediction horizon, $x_0 \in \mathbb{R}^n$ is the initial condition for (1), and the space of admissible inputs is given by $\mathcal{U}_T\colon = \mathcal{L}^2\left([0,T], \mathbb{R}^m\right) \cap \mathcal{L}^\infty\left([0,T], \mathbb{R}^m\right)$. We make the following assumptions about the cost functions:

Assumption 3. *We assume that $Q(\cdot), R(\cdot), V(\cdot)$ are twice-continuously differentiable positive definite convex functions, whose Hessians satisfy the pointwise bounds $\alpha_Q I \preceq \nabla^2 Q \preceq \beta_Q I$, $\alpha_R I \preceq \nabla^2 R \preceq \beta_R I$, and $\alpha_V I \preceq \nabla^2 V \preceq \beta_V I$ for constants $0 < \alpha_Q \leq \beta_Q$, $0 < \alpha_R \leq \beta_R$, and $0 \leq \alpha_V \leq \beta_V$.*

We emphasize that this assumption may not hold under different parameterizations of the control system. The application of our sufficient conditions relies on finding a coordinate system in which this assumption holds.

2.1 (Approximate) Stationary Points

Next, we briefly review a few basic facts from the calculus of variations which are essential for understanding out results. We endow $\mathcal{L}^2\left([0,T], \mathbb{R}^m\right)$ with the usual inner product and norm, denoted $\langle \cdot, \cdot \rangle\colon \mathcal{L}^2([0,T], \mathbb{R}^m) \times \mathcal{L}^2([0,T], \mathbb{R}^m) \to \mathbb{R}$ and $\|\cdot\|_2 : \mathcal{L}^2([0,T], \mathbb{R}^m) \to \mathbb{R}$. Under Assumptions 2 and 3, directional (Fréchet) derivatives of $J_T(\cdot, x_0)$ are guaranteed to exist [28, Theorem 5.6.8] as there is a well-defined gradient at each point in the optimization space. We denote the directional Fréchet derivative of $J_T(\cdot, x_0)$ at the point $\tilde{u} \in \mathcal{U}_T$ in the direction $\delta u \in \mathcal{U}_T$ by $\mathrm{D}J_T(\tilde{u}; x_0; \delta u)$. The gradient $\nabla J_T(\tilde{u}; x_0) \in \mathcal{L}^2([0,T], \mathbb{R}^m)$ is the unique object satisfying, for each $\delta u \in \mathcal{U}_T$,

$$\mathrm{D}J_T(\tilde{u}; x_0; \delta u) = \int_0^T \left\langle \nabla J_T\left(\tilde{u}; x_0)(t)\right), \delta u(t) \right\rangle \mathrm{d}t, \tag{3}$$

or more compactly, $\mathrm{D}J_T(\tilde{u};x_0;\delta u) = \langle \nabla J_T(\tilde{u};x_0), \delta u\rangle$. We refer the reader to [28, Theorem 5.6.8] for computations of these derivatives. Only the subsequent abstract definitions are required to understand our main results:

Definition 1. *We say that an input $\tilde{u}$ is a* **first-order stationary point** *(*`FOS`*) if $\nabla J_T(\tilde{u};x_0) = 0$. We say that $\tilde{u}$ is an ϵ-*`FOS` *if $\|\nabla J_T(\tilde{u};x_0)\|_2 \leq \epsilon$.*

In practice, derivative-based descent algorithms take an infinite number of iterations to converge to exact stationary points, thus our analysis will primarily focus on the approximate stationary points of $J_T(\cdot, x_0)$.

2.2 Jacobian Linearizations and Convex Time-Varying Approximations to $J_T(\cdot, x_0)$

Rather than directly studying the properties of (approximate) stationary points of $J_T(\cdot, x_0)$, we instead analyze convex approximations to $J_T(\cdot, x_0)$ which are constructed using a first-order expansion of the system dynamics along the desired solution. We recall that a first-order expansion of the system dynamics along a nominal state trajectory and input pair $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ starting from initial condition $\tilde{x}(0) = x_0$ can be constructed as follows. Let $\delta u \in \mathcal{U}_T$ be an admissible perturbation to the input and let $\hat{x}(\cdot)$ be the system trajectory corresponding to the input $\hat{u}(\cdot) = \tilde{u}(\cdot) + \delta u(\cdot)$ with the same initial condition. Then by [28, Theorem 5.6.8] a first-order approximation to $\hat{x}(\cdot)$ is given by

$$\hat{x}(\cdot) \approx \bar{x}(\cdot) = \tilde{x}(\cdot) + \delta x(\cdot) \tag{4}$$

where $\delta x\colon [0,T] \to \mathbb{R}^n$ and $\delta\dot{x}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\delta x(t)$ satisfy

$$\delta\dot{x}(t) = \tilde{A}(t)\delta x(t) + \tilde{B}(t)\delta u(t), \quad \delta x(0) = 0, \tag{5}$$

for $\tilde{A}(t) = \frac{\partial}{\partial x}F(\tilde{x}(t), \tilde{u}(t))$ and $\tilde{B}(t) = \frac{\partial}{\partial u}F(\tilde{x}(t), \tilde{u}(t))$.

Definition 2. *Let $(\tilde{x}(\cdot)$,$\tilde{u}(\cdot))$, $\tilde{A}(\cdot)$ and $\tilde{B}(\cdot)$ be defined as above. We refer to the time-varying linear system $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ as the* **Jacobian linearization** *of the vector field F along the trajectory $(\tilde{x}(\cdot), \tilde{u}(\cdot))$.*

The evolution of the estimate $\bar{x}(\cdot)$ is given by

$$\begin{aligned}\dot{\bar{x}}(t) &= \dot{\tilde{x}}(t) + \delta\dot{x}(t)\\ &= F(\tilde{x}(t), \tilde{u}(t)) + \tilde{A}(t)\delta x(t) + \tilde{B}(t)\delta u(t)\\ &= \tilde{A}(t)\bar{x}(t) + \tilde{B}(t)\bar{u}(t) + \tilde{d}(t),\end{aligned}$$

where we call $\tilde{d}(t) = F(\tilde{x}(t), \tilde{u}(t)) - \tilde{A}(t)\tilde{x}(t) - \tilde{B}(t)\tilde{u}(t)$ the *drift* term. We use these dynamics to construct an approximation of $J_T(\cdot;x_0)$, denoted $J_T^{\texttt{jac}}(\cdot;x_0,\tilde{u})\colon \mathcal{U}_T \to \mathbb{R}$, around the nominal solution $(\tilde{x}(\cdot), \tilde{u}(\cdot))$:

$$\begin{aligned}J_T^{\texttt{jac}}(\bar{u};x_0,\tilde{u}) &= \int_0^T \left(Q(\bar{x}(\tau)) + R(\bar{u}(\tau))\right)\mathrm{d}t + V(\bar{x}(T))\\ \text{s.t.}\quad \dot{\bar{x}}(\tau) &= \tilde{A}(t)\bar{x}(\tau) + \tilde{B}(t)\bar{u}(\tau) + \tilde{d}(t), \quad \bar{x}(0) = x_0,\end{aligned} \tag{6}$$

Due to Assumption 3 and the fact that the time-varying dynamics in (6) are affine we observe that $J_T(\cdot;x_0,\tilde{u})$ is strongly convex, and thus has a unique stationary point. The following result motivates this approximation:

Lemma 1. *For any input $\tilde{u}(\cdot) \in \mathcal{U}_T$ we have*

$$J_T(\tilde{u}; x_0) = J_T^{\mathtt{jac}}(\tilde{u}; x_0, \tilde{u}), \quad \textit{and} \quad \nabla J_T(\tilde{u}; x_0) = \nabla J_T^{\mathtt{jac}}(\tilde{u}; x_0, \tilde{u}).$$

Thus, $\tilde{u}$ is a stationary point of $J_T(\cdot, x_0)$ if and only if it is the global minimizer of $J_T^{\mathtt{jac}}(\cdot; x_0, \tilde{u})$. This observation enables us to bound the cost at each stationary point $\tilde{u}$ of $J_T(\cdot; x_0)$ by bounding the cost incurred by the global optimizer of $J_T^{\mathtt{jac}}(\cdot; x_0, \tilde{u})$, a crucial step in our analysis. This analysis technique requires the convexity of Q, R and V stipulated by Assumption 3, meaning our approach requires finding a coordinate system in which this assumption holds.

The following result extends the above discussion to approximate stationary points of $J_T(\cdot, x_0)$:

Lemma 2. *(Approximate* FOS*) Suppose Assumption 3 holds, and that $\tilde{u}$ is an ϵ-*FOS *of $J_T(\cdot; x_0)$. Then,*

$$J_T(\tilde{u}; x_0) \leq \min_u J_T^{\mathtt{jac}}(u; x_0, \tilde{u}) + \tfrac{\epsilon^2}{2\alpha_R}.$$

Proof. Assumption 3 implies that $J_T^{\mathtt{jac}}(\bar{u}; x_0, \tilde{u}) - \alpha_R \|\bar{u}\|^2$ is convex, and thus the Polyak-Łojasiewicz inequality holds: $J_T^{\mathtt{jac}}(\bar{u}; x_0, \tilde{u}) \leq \min_u J_T^{\mathtt{jac}}(u; x_0, \tilde{u}) + \|\nabla J_T^{\mathtt{jac}}(\bar{u}; x_0, \tilde{u})\|_2^2 / 2\alpha_R$. Since $\tilde{u}$ is an ϵ-FOS of $J_T(\cdot; x_0)$ and $\nabla J_T(\tilde{u}; x_0) = \nabla J_T^{\mathtt{jac}}(\tilde{u}; x_0, \tilde{u})$, it follows that $J_T^{\mathtt{jac}}(\tilde{u}; x_0, \tilde{u}) \leq \min_u J_T^{\mathtt{jac}}(u; x_0, \tilde{u}) + \epsilon^2 / 2\alpha_R$. Using $J_T(\tilde{u}; x_0) = J_T^{\mathtt{jac}}(\tilde{u}; x_0, \tilde{u})$ concludes. □

3 Sufficient Conditions for Exponentially Decaying First-Order Stationary Points

We begin our analysis by providing sufficient conditions which ensure that *all* (approximate) stationary points of $J_T(\cdot; x_0)$ decay exponentially to the origin at a rate that is independent of $T \geq 0$ and $x_0 \in \mathbb{R}^n$. These sufficient conditions are stated entirely in terms of properties of the local first-order expansions of the system dynamics introduced above in Section 2.2. The result is developed in Section 3.1, with its proof given in Section 3.2. Section 3.3 applies the result to several systems and investigates two simple counterexamples which highlight the necessity of some of our strong assumptions. In addition, we provide a further counterexample demonstrating that our assumptions permit stationary points which are not globally optimal. Finally, Section 3.4 draws the connection to feedback linearization, connecting our first-order conditions to the global geometry of the problem.

3.1 Sufficient Conditions for Exponentially Decaying First-Order Stationary Points

Informally, our sufficient conditions require that 1) the Jacobian linearization along each nominal system trajectory $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is uniformly stabilizable in an appropriate sense 2) the drift term $\tilde{d}(t)$ in the expansion of the system dynamics along the trajectory can be 'canceled' out by an appropriate choice of input 3) this cancellation does not require too much input and 4) state running cost is sufficiently small when compared to the input running cost.

When these conditions are met, we use $\tilde{\mathcal{V}}(s) := J_{T-s}(\tilde{u}_{[s,T]}, \tilde{x}(s))$ along an approximate stationary pair $(\tilde{u}(\cdot), \tilde{x}(\cdot))$ as a Lyapunov function to verify that the selected trajectory decays to the origin using a standard argument [6, Theorem 5.17]. We apply a (generally suboptimal) control of the form $\bar{u}(\cdot) = \bar{u}_1(\cdot) + \bar{u}_2(\cdot)$ to the time varying dynamics in (6) and bound the resulting cost. Here, the control $\bar{u}_1(\cdot)$ will be used to 'reject' the drift term $\tilde{d}(\cdot)$, while $\bar{u}_2(\cdot)$ is designed to stabilize the linear, drift-free system $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ towards the origin. The existence of these control is guaranteed by conditions 1) and 2) above, which we now formalize with the following assumptions:

Assumption 4. *Along each system trajectory $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ the drift term in (6) satisfies $\tilde{d}(t) \in \mathrm{range}(\tilde{B}(t))$.*

In the language of nonlinear control theory, this Assumption is known as a *matching condition* [6, Chapter 9.4]. Specifically, this assumption ensures that $\tilde{d}(t)$ in (6) can be 'canceled out' by the control $\bar{u}_1(t) = B^{\dagger}\tilde{d}(t)$ so that the control $\bar{u}(\cdot) = \bar{u}_1(\cdot) + \bar{u}_2(\cdot)$ yields the linear dynamics

$$\dot{\bar{x}}(t) = \tilde{A}(t)\bar{x}(t) + \tilde{B}(t)\bar{u}_2(t) \tag{7}$$

when applied to (6). We note the conceptual similarity between this approach and the global feedback linearizing controllers introduced in Section 3.4 below.

The next assumption then implies that the Jacobian linearizations along the trajectories of the system are uniformly stabilizable in the following sense:

Definition 3. (Stabilizability) *We say that the linear system $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ defined on the time interval $[0, T]$ is γ-**stabilizable** if for each $s \in [0, T]$ and $x_0 \in \mathbb{R}^n$*

$$\inf_{\hat{u}(\cdot)} \int_{t=s}^{T} \|\hat{x}(t)\|_2^2 + \|\hat{u}(t)\|_2^2 \mathrm{d}t + \|\hat{x}(T)\|_2^2 \leq \gamma \|x_0\|_2^2, \tag{8}$$

where $\dot{\hat{x}}(t) = \tilde{A}(t)\hat{x}(t) + \tilde{B}(t)\hat{u}(t)$ and $\hat{x}(s) = x_0$.

Thus the parameter $\gamma > 0$ upper bounds the cost of solving an LQR problem (with simple norm costs) along a given Jacobian linearization $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ from a particular initial condition, providing a measure of how difficult it is so stabilize the time varying dynamics. We require that one such parameter holds along all trajectories of the system:

Assumption 5. *There exists $\gamma > 0$ such that for each time horizon $T \geq 0$, $x_0 \in \mathbb{R}^n$ and $\tilde{u} \in \mathcal{U}_T$ the Jacobian linearization $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ of F along the corresponding system trajectory $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is γ-stabilizable.*

The uniformity demanded by Assumption 5 is crucial for finding a rate of decay in Theorem 1 which holds for all $T \geq 0$, $x_0 \in \mathbb{R}^m$ and all stationary points of $J_T(\cdot, x_0)$.

The next assumption ensures that the amount of energy needed to cancel out $\tilde{d}(\cdot)$ does not grow too quickly:

Assumption 6. *There exists $L_x, L_u > 0$ such that for each system trajectory $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ defined on $[0, T]$ we have $\|\tilde{B}^{\dagger}(t)\tilde{d}(t)\| \leq L_x\|\tilde{x}(t)\| + L_u\|\tilde{u}(t)\|$ for each $t \in [0, T]$.*

In particular, if the magnitude of $\tilde{d}(\cdot)$ grows too quickly then it may be too costly to simply 'cancel out' the disturbance term when constructing a suboptimal stabilizing control, and a more system-specific analysis technique may be required to upper bound the cost associated to particular stationary points. The following assumption reveals the extent to which the running costs Q and R can be rescaled to allow for larger L_x and L_u is our analysis:

Assumption 7. *The constants $L_x, L_u > 0$ in are small enough to satisfy $L_u^2 \leq \frac{\alpha_R}{8\beta_R}$ and $L_x^2 \leq \frac{\alpha_Q}{8\beta_R}$, where $\alpha_R, \beta_R, \alpha_Q$ and β_Q are as in Assumption 3.*

In particular, our analysis demonstrates that arbitrarily large values of L_x can be accommodated by increasing the relative weighting of the state running cost compared to the input running cost (so that $\alpha_R \gg \beta_R$). The restriction on L_u is more significant, since the ratio $\frac{\alpha_R}{\beta_R}$ (which bounds the inverse of the condition number of $\nabla^2 R$), can be at most equal to 1. Thus, the assumption always implies that implies that $L_u < \frac{1}{8}$,[1] which effectively places a limit on how nonlinear F can be in the input. We discus specific examples of systems which do and do not satisfy these assumptions in 3.3. Under these assumptions we obtain our main result:

Theorem 1. *Suppose Assumptions 1 to 7 hold. Then for each $T \geq 0$, $x_0 \in \mathbb{R}^n$ and every $\tilde{u} \in \mathcal{U}_T$ which is an ϵ-`FOS` of $J_T(\cdot; x_0)$ the following hold. If $\alpha_V > 0$, then $\forall s \in [0, T]$,*

$$\|\tilde{x}(s)\|^2 \leq \mathcal{C}_0 \cdot (\mathcal{C}_1 e^{-\frac{s}{\mathcal{C}_1}} \cdot \|x_0\|^2 + \mathcal{C}_2 \epsilon^2),$$

where $\mathcal{C}_0 = 6L_F + \frac{2L_F\alpha_Q}{\alpha_R} + \frac{2\alpha_Q}{\alpha_V}$, $\mathcal{C}_1 = 4\gamma \max\{\beta_V, \beta_R, \beta_Q\}$ and $\mathcal{C}_2 = \frac{1}{2\alpha_R}(1 + 8\beta_R \max\{\frac{L_x^2}{\alpha_Q}, \frac{L_u^2}{\alpha_R}\})$. More generally, for any $\alpha_V \geq 0$, it holds that for all $s \in [0, T]$ and $\delta > 0$,

$$\|\tilde{x}(s)\|^2 \leq \mathcal{C}_0^\delta \cdot (\mathcal{C}_1^\delta e^{-\frac{s}{\mathcal{C}_1}} \cdot \|x_0\|^2 + \mathcal{C}_2 \epsilon^2)$$

where $\mathcal{C}_0^\delta := 6L_F + \frac{2L_F\alpha_Q}{\alpha_R} + \min\{\frac{2}{\delta}, \frac{2\alpha_Q}{\alpha_R}\}$, $\mathcal{C}_1^\delta := e^{\frac{\delta}{\mathcal{C}_1}}\mathcal{C}_1$.

Note that when $\epsilon = 0$ taking the square-root of both sides of either bound in the statement of the theorem demonstrates that the stationary point is exponentially decaying. In the context of RHC we provide a rule which ensures that ϵ is small enough each time a planning problem is solved to ensure exponential stability. The two different bounds in the statement of the theorem arise from the fact that we obtain different under bounds for $\tilde{V}(s)$ when there is a terminal cost and when there is not a terminal cost, which affects our estimate for the rate of decay. In particular, when there is no terminal cost it is impossible to uniformly under bound $\tilde{V}(s)$ near the end of the trajectory, and we must account for the fact that the trajectory may actually be increasing at this point (hence the exponential term in $\mathcal{C}_1^\delta$).

3.2 Proof of Theorem 1

Let $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ be as in the statement of the theorem, and let $\tilde{\mathcal{V}}(s) := J_{T-s}(\tilde{u}_{[s,T]}, \tilde{x}(s))$ for $s \in [0, T]$. Then the following bounds hold (under the assumptions of Theorem 1):

Lemma 3. *If $\alpha_V > 0$ the for each $0 \leq s' \leq s \leq T$ we have $\|\tilde{x}(s)\|^2 \leq \frac{1}{\alpha_Q}\mathcal{C}_0 \cdot \tilde{\mathcal{V}}(s')$. Alternatively, if $\alpha_V \geq 0$ then for each $0 \leq \delta \leq T$, $s' \leq T - \delta$ and $s' \leq s \leq T$ we have $\|\tilde{x}(s)\|^2 \leq \frac{1}{\alpha_Q}\mathcal{C}_0^\delta \cdot \tilde{\mathcal{V}}(s')$.*

Lemma 4. *If $\tilde{u}(t)$ is an ϵ-`FOS` of $J_T(\cdot, x_0)$, then for each $s \in [0, T]$ we have $\tilde{\mathcal{V}}(s) \leq \alpha_Q(\mathcal{C}_1\|\tilde{x}(s)\|^2 + \frac{\mathcal{C}_2}{2}\epsilon^2)$.*

[1] We remark that the constant of 1/8 can be made arbitrary close to 1, at the expense of degrading other constants in the proof of Theorem 1.

Proofs of both lemmas above are given in the appendix.

By the Fundamental Theorem of Calculus,

$$\begin{aligned}-\tfrac{\mathrm{d}}{\mathrm{d}s}\tilde{\mathcal{V}}(s) &= Q(\tilde{x}(s)) + R(\tilde{u}(s)) \geq \alpha_Q \|\tilde{x}(s)\|^2 \\ &\geq \tfrac{1}{\mathcal{C}_1}\tilde{\mathcal{V}}(s) - \tfrac{\alpha_Q \mathcal{C}_2 \epsilon^2}{2\mathcal{C}_1},\end{aligned}$$

where the last line uses Lemma 4. Integrating the bound and again invoking Lemma 4,

$$\begin{aligned}\tilde{\mathcal{V}}(s) &\leq \exp(-\tfrac{s}{\mathcal{C}_1})\tilde{\mathcal{V}}(0) + \tfrac{\alpha_Q \mathcal{C}_2 \epsilon^2}{2\mathcal{C}_1}\int_{t=0}^{s} \exp(-\tfrac{t}{\mathcal{C}_1})\mathrm{d}t \\ &\leq \exp(-\tfrac{s}{\mathcal{C}_1})\tilde{\mathcal{V}}(0) + \tfrac{\alpha_Q}{2}\mathcal{C}_2\epsilon^2 \\ &\leq \alpha_Q \cdot (\mathcal{C}_1 e^{-\frac{s}{\mathcal{C}_1}} \cdot \|\tilde{x}(0)\|^2 + \mathcal{C}_2\epsilon^2).\end{aligned} \tag{9}$$

Finally, in the case where $\alpha_V > 0$ Lemma 3 lets us convert the above bound to one on $\|\tilde{x}(s)\|^2$, replacing α_Q with $\mathcal{C}_0$, as desired. In the case where $\alpha_V = 0$, for each $\delta \in [0, T]$ application of Lemma 3 yields the desired result for each $s \in [0, T-\delta]$. For each $s \in [T-\delta, T]$ Lemma 3 yields

$$\begin{aligned}\|\tilde{x}(s)\|_2^2 &\leq \mathcal{C}_0^\delta \cdot (\mathcal{C}_1 e^{-\frac{T-\delta}{\mathcal{C}_1}} \cdot \|\tilde{x}(0)\|^2 + \mathcal{C}_2\epsilon^2) \\ &\leq \mathcal{C}_0^\delta \cdot (e^{\frac{\delta}{\mathcal{C}_1}}\mathcal{C}_1 e^{-\frac{s}{\mathcal{C}_1}} \cdot \|\tilde{x}(0)\|^2 + \mathcal{C}_2\epsilon^2).\end{aligned}$$

3.3 Examples and Counterexamples

We present three counterexamples and one positive example to illustrate the necessity of our conditions and generality of our results. The first counterexample demonstrates the necessity of small control costs. The second demonstrates the necessity of the matching condition between the drift $\tilde{d}(t)$ and range of the linearized B-matrix $\tilde{B}(t)$. The third counterexample constructs an example where our conditions hold, and thus first-order stationary points are stabilizing, but where there exists a `FOS` which is *not a global optimum*. This reveals that our conditions are more generaly than popular forms of "hidden convexity" such as quasi-convexity. Finally, our fourth example describes a system which is not input affine, but for which the regularity conditions outlined above still hold.

3.3.1 Relative Weighting of State and Input Costs

Consider the scalar system $\dot{x} = \sin(x) + u$ and consider the cost:

$$J_T(\cdot, x_0) = \int_0^T q(\|\tilde{x}(t)\|_2^2 + r\|\tilde{u}(t)\|_2^2)\mathrm{d}t + q_f\|\tilde{x}(t)\|_2^2,$$

where $q, r, q_f > 0$ Suppose we select $x_0 = \frac{3\pi}{4}$ and select the input $\tilde{u} \in \mathcal{U}_T$ by setting $\tilde{u}(\cdot) = -\sin(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$. The trajectory $\tilde{x}(\cdot)$ resulting from this data is given by $\tilde{x}(\cdot) \equiv \frac{3\pi}{4}$. Note that the Jacobian linearization along this trajectory is given by

$$\tilde{A}(\cdot) \equiv \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} \quad \text{and} \quad \tilde{B}(\cdot) \equiv 1.$$

By [28, Theorem 5.6.8] we have $\nabla J_T(\tilde{u}, x_0)(t) = \tilde{B}(t)p(t) + 2r\tilde{u}(t) = p(t) + 2r\tilde{u}(t)$ where the costate $p\colon [0,T] \to \mathbb{R}^n$ satisfies $-\dot{p}(t) = \tilde{A}(t)p(t) + 2q\tilde{x}(t) = -\frac{1}{\sqrt{2}}p(t) + \frac{3\pi q}{2}$ with initial condition $p(T) = 2q_f\tilde{x}(T) = \frac{3\pi q_f}{2}$. Thus, if we select q and q_f so that $q = \frac{1}{\sqrt{2}}q_f$ we will have $p(\cdot) \equiv \frac{3\pi q_f}{2}$. Moreover, if we then pick $r = \frac{3\pi q_f}{2\sqrt{2}}$ we see that $\nabla J_T(\tilde{u}; x_0)(t) \equiv 0$, and thus $\tilde{u(\cdot)}$ is a stationary point of $J_T(\cdot, x_0)$. Note that this analysis holds for all time horizons $T > 0$. However, it is straightforward to verify that this system satisfies the conditions of Theorem 1 if we pick $q > 8\alpha_R$. However, stability guarantees for RHC which rely on global optimality would predict that RHC will stabilize this system regardless of the relative scaling of the input and state costs when $T \geq 0$ is sufficiently large [10].

It is interesting to note that it is impossible to find a stationary pair $(\hat{x}(\cdot), \hat{u}(\cdot))$ of $J_T(\cdot, x_0)$ with the property that $\hat{x}(\cdot) \equiv x_0$ if we instead have $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Indeed, if we pick $\tilde{u}(\cdot) = -\sin(x_0)$ so that $\hat{x}(\cdot) \equiv x_0$, then in this case the costate will satisfy the differential equation $-\dot{p}(t) = \hat{A}(t)p(t) + 2q\hat{x}(t)$ where $\hat{A}(\cdot) \equiv \cos(x_0) > 0$. Thus, in this case the costate cannot be a constant function of time, which means that $\hat{u}(\cdot)$ cannot be a stationary point of $J_T(\cdot, x_0)$.

More broadly, consider a general 1-dimensional system $\dot{x} = F(x, u)$ which satisfies Assumptions 4 and 5. One can verify that and trajectory $(\check{x}(\cdot), \check{u}(\cdot))$ of F such that $\check{x}(\cdot) \equiv x_0$ and $\check{u}(\cdot) \equiv \check{u}_0$ (for some $x_0, \check{u} \in \mathbb{R}$) can only be a stationary pair of $J_T(\cdot, x_0)$ if $\frac{\partial}{\partial x}F(x_0, \bar{u}) < 0$ and $\check{d}(\cdot) \equiv \check{d}_0 = F(x_0, \check{u}_0) - \check{A}x_0 - \check{B}u_0$ is such that $sign(d_0) = sign(x_0)$, where $(\check{A}_0, \check{B}_0)$ are the Jacobian linearization of F at $(x_0, \check{u}_0)$. This highlights the need to further study how the geometry of the Jacobian linearizations and drift terms promote or inhibit stabilizing behavior.

3.3.2 Structure of Local Disturbance Term

Let $\eta\colon \mathbb{R} \to \mathbb{R}_+$ be any C^∞ "bump" function, satisfying 1) $\eta(z) = 0$ if $u \leq 1$, 2) $\eta(z) = 1$ if $z \geq 2$, 3) η' is nonnegative and bounded on $[1, 2]$. Consider the system with state $x = (x_1, x_2) \in \mathbb{R}^2$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \eta(x_1) \underbrace{\begin{bmatrix} 10 \\ 0 \end{bmatrix}}_{d}. \tag{10}$$

On $\mathcal{D}_1\colon = \{(x_1, x_2)\colon x_1 \leq 1\}$ the dynamics of (10) are identical to the controllable linear system $\dot{x} = Ax + Bu$, while on $\mathcal{D}_2 = \{(x_1, x_2)\colon x_1 \geq 2\}$ the dynamics are affine and governed by $\dot{x} = Ax + Bu + d$. The reader may verify that these dynamics satisfy Assumption 5 for a suitable $\gamma > 0$, as the Jacobian linearizations along every system trajectory are uniformly controllable, however the system clearly does not satisfy Assumption 4 on $\mathcal{D}_2$ since $d \notin \text{range}(B)$.

Because the dynamics are locally affine in $\mathcal{D}_2$, first order stationary points of natural planning objectives will settle at optimal equilibria for the system with drift, $\dot{x} = Ax + Bu + d$, and fail to converge to the origin.

Formally, consider the cost functional and initial point

$$\begin{aligned} J_T(\cdot; x_0) &= \int_0^T q\|\tilde{x}(t)\|_2^2 + r\|\tilde{u}(t)\|_2^2 dt + q_f\|\tilde{x}_1(T)\|_2^2 \\ x_0 &= (5, -5) \in \mathcal{D}_2 \end{aligned}$$

If we choose $\tilde{u}(\cdot) \equiv 0$ then the corresponding trajectory will be $\tilde{x}(\cdot) \equiv x_0$. Moreover, the costate equation associated to this trajectory is given by $-\dot{p}(t) = A^T p(t) + 2qx_0$ with $p(T) =$

$(2q_f x_1(T), 0)^T = (10q_f, 0)$. The reader may verify that if we choose $q = q_f$ then we will have $p(\cdot) \equiv (10q_f, 0)$, which then implies that $\nabla J_T(\cdot, x_0)(t) = B^T p(t) + 2r\tilde{u}(t) = 0$ for each $t \in [0, T]$, thus $\tilde{u}(\cdot)$ is a stationary point of $J_T(\cdot, x_0)$. This analysis holds *for every* $T > 0$ *and does not depend on the value of* $r > 0$, which can be used to control the relative scaling of the input and state running cost. As shown in Section 3.4, this issue can be overcome by designing the state running costs in a coordinate system which better captures the underlying global geometry of this system.

It is instructive to illustrate the heuristic we used to find this equilibrium. Consider the average cost problem

$$\begin{aligned} \bar{J}(\tilde{u}; x_0) &:= \lim_{T\to\infty} \int_0^T (q\|\tilde{x}(t)\|_2^2 + r\|\tilde{u}(t)\|_2^2)\mathrm{d}t \\ &\quad \text{s.t. } \dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t) + d \end{aligned} \tag{11}$$

which penalizes the long-term average running cost subject to the dynamics the system witnesses on $\mathcal{D}_2$. Note that since $d \notin \mathrm{range}(B)$ we will have $\inf_{\tilde{u}(\cdot)} \bar{J}(\cdot, x_0) > 0$, since the affine dynamics cannot be asymptotically stabilized to the origin. Instead, the average long-term cost is minimized by stabilizing the system to equilibria of the affine dynamics which minimize the pointwise running cost:

$$\min_{x\in\mathbb{R}^2, u\in\mathbb{R}} \left\{ q\|x\|_2^2 + r\|u\|_2^2 \colon 0 = Ax + Bu + d \right\}.$$

The reader may verify that $x = (5, -5)$ and $u = 0$ is the globally optimal solution to the this problem for all values of $q, r > 0$. Thus, intuitively, this point is attractive (for just the running costs) when the original system is in D_2. We add the terminal costs to J_T to ensure that this point is a stationary point for the finite horizon problem.

3.3.3 Stabilizing stationary points need not be global optima.

In this example, we demonstrate that exponentially stabilizing staionary points need not be global optima. Thus, the problems do not exhibit "hidden convexity" or quasi-convexity.

To construct the counterexample, we construct a system which has linear dynamics $\dot{x} = u$ within a region $\mathcal{D}_1$, but which enjoys a beneficial drift term in a region $\mathcal{D}_2$, with dynamics $\dot{x} = u + d$. The two regions are interpolated via a bump function. We construct a cost J_T so there is a `FOS` which remains in $\mathcal{D}_1$, executing the optimal control law for the driftless dynamics. We then show that the cost can be strictly improved with a control law which enters $\mathcal{D}_2$ to take advantage of the drift.

Formally, let $\eta : \mathbb{R} \to \mathbb{R}$ be any bump function with η' bounded nonnegative, with $\eta(z) = 0$ for $z \leq 1$, and $\eta(z) = 1$ for $z \geq 2$. For convenience, assume the normalization $\int_1^2 \eta(z)\mathrm{d}z = 1/2$. Consider the system in $\mathbb{R}^2$ with state $x = (x_1, x_2)$, input $u = (u_1, u_2)$, and, for a parameter $\alpha > 0$ to be manipulated, the dynamics

$$\dot{x} = u + \eta(x_1) \underbrace{\begin{bmatrix} 0 \\ -\alpha \end{bmatrix}}_{d} \tag{12}$$

For time horizon T and parameters $q, r, B > 0$, consider an initial point $x_0 = (0, B)$

$$\begin{aligned} J_T(\tilde{u}; x_0) &= \int_0^T (q\|\tilde{x}(t)\|_2^2 + r\|\tilde{u}(t)\|_2^2)\mathrm{d}t + \sqrt{qr}\|\tilde{x}(T)\|_2^2 \\ &\text{s.t.} \quad \dot{x} = u + \eta(x_1) \cdot d. \end{aligned} \tag{13}$$

Much like the previous example, one can verify that the above system satisfies the requisite stability conditions to ensure that all first-order stationary points are exponentially stabilizing. We now show that if B satisfies

$$\frac{B}{18} \geq \max\left\{1 + \frac{1}{\alpha\sqrt{rq}}, \sqrt{\frac{r}{q}}\right\}, \tag{14}$$

and if T is sufficiently large, then there exists a local optimum of J_T which is not global.

To show this, consider the region $\mathcal{D}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$. On $\mathcal{D}_1$, the dynamics are simply $\dot{x} = u$, and thus a first order stationary point which remains in $\mathcal{D}_1$ is given by the optimal control law from that system, i.e. the optimal law of

$$\bar{J}_T(\tilde{u}; x_0) = \int_0^T (q\|\tilde{x}(t)\|_2^2 + r\|\tilde{u}(t)\|_2^2)\mathrm{d}t + \sqrt{qr}\|\tilde{x}(T)\|_2^2 \quad \text{s.t.} \quad \dot{x} = u. \tag{15}$$

To classify the optimal control law, observe that the optimal law for $\lim_{T\to\infty} J_T(\tilde{u}; x_0)$ can be obtained by solving the Continuous Algebraic Riccati Equation with matrices $A = 0$, $B = I$, yielding a cost-to-go matrix $P = \sqrt{qr}I$, and optimal law $\tilde{u}^\star = -\sqrt{q/r} \cdot \tilde{x}^\star$. Thus, the terminal cost in J_T is equal to the infinite-horizon cost-to-go function $\tilde{x}(T)^\top P \tilde{x}(T)$, $\tilde{u}^\star(t) = -\sqrt{q/r} \cdot \tilde{x}^\star(t)$ is an optimal law for $\bar{J}_T$ as in (15) with value $\bar{J}_T^\star(x_0) = \inf_{\tilde{u}} \bar{J}_T(\tilde{u}; x_0) = \sqrt{rq}B^2$. Note that the law $\tilde{u}^\star(t) = -\sqrt{q/r} \cdot \tilde{x}^\star(t)$, when executed starting at $x_0 = (0, B)$, remains on the Y-axis in $\mathbb{R}^2$, and hence remains in $\mathcal{D}_2$. Thus, this law is also a local optimal of $J_T(\cdot; x_0)$ with cost $\sqrt{rq}B^2$.

To show that $\tilde{u}^\star(t) = -\sqrt{q/r} \cdot \tilde{x}^\star(t)$ is not a global minimizer of $J_T(\cdot; x_0)$, let us construct a control $\bar{u}(t)$ with $J_T(\bar{u}; x_0) < J_T(\tilde{u}^\star; x_0)$. Let $t_0 = 1/(B\sqrt{q/r})$, $t_1 = \alpha(1 - t_0)$ to be selected, consider the control $\bar{u}(t)$

$$\bar{u}u(t) = \begin{cases} (\frac{2}{t_0}, 0) & t \in [0, t_0] \\ (0, 0)) & t \in [t_0, t_0 + t_1] \\ (-\frac{2}{t_0}, 0), 0) & t \in [t_0 + t_1, 2t_0 + t_1] \\ (0, 0) & t \geq t_1 + 2t_0 \end{cases} \tag{16}$$

Let us characterize the trajectory of $\hat{x}(t)$:

- For $t \in [0, t_0]$, $\hat{x}_1(t) = 2t/t_0$, and $\hat{x}_2(t)$ is monotonically decreasing (since $\eta(z)$ is nonnegative) and gives $\hat{x}_2(t_0) = B - \int_0^{t_0} \eta(\hat{x}_1(t))\mathrm{d}t = B - \int_0^{t_0} \eta(2t/t_0)\mathrm{d}t = B - 2t_0 \int_0^2 \eta(z)\mathrm{d}z = B - 2t_0 \cdot \frac{1}{2} = B - t_0$. The total cost incurred during this time is at most

$$t_0(q2^2 + qB^2 + r\frac{4}{t_0^2}) = \frac{4q}{B\sqrt{q/r}} + \frac{qB^2}{B\sqrt{q/r}} + 4rB\sqrt{q/r} = \sqrt{rq}(4/B + 5B). \tag{17}$$

- For $t \in [t_0, t_1 + t_0]$, $\hat{x}_1(t) = 2$, and $\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_2(t) = -\alpha$; $\hat{x}_2(t_0 + s) = B - t_0 - \alpha s$. For our choice of $t_1 = \frac{1}{\alpha}(B - 2t_0)$, we have $\hat{x}_2(t_1 + t_0) = t_0$. The total cost incurred during the time is at most $t_1(4 + B^2) \leq (4/B + B)/\alpha \leq 4(1/B + B)/\alpha$.

- One can verify that, for $t \in [t_1 + t_0, t_1 + 2t_0]$, $\hat{x}_1(t) \in [0, 2]$ and $\hat{x}_2(t_1 + 2t_0) \in [0, t_0]$. Moreover, $\hat{x}(t_1 + 2t_0) = (0, 0)$. The total cost incurred is at most $q(4 + t_0^2)t_0 + 4r/t_0$. The choice of B as in (14) ensures $t_0 \leq 1$, so that we can upper bound the incurred cost by $5qt_0 + 4r/t_0$, which is at most

$$\sqrt{rq}(5/B + 4B). \tag{18}$$

- For the remainder of the trajectory, we remain at $(0,0)$ and incur no cost.

Since Eq. (14) ensures $B \geq 1$, the total cost is at most

$$J_T(\hat{u}; x_0) \leq (4/B + B)/\alpha + 9\sqrt{rq}(1/B + B) \leq 18B(\sqrt{rq} + 1/\alpha). \tag{19}$$

In sum, we have $J_T(\hat{u}; x_0) < J_T(\tilde{u}; x_0)$ as soon as $18B(\sqrt{rq} + 1/\alpha) \leq B^2\sqrt{rq}$; that is, as soon as (14) holds.

3.4 Connection to Feedback Linearization

Feedback linearizable systems are perhaps the most widely studied and well-characterized class of systems in the nonlinear geometric control literature [6, Chapter 9]. Roughly speaking, a system is feedback linearizable if it can be transformed into a linear system using state feedback and a coordinate transformation. Specifically, we say that the system (1) is feedback linearizable if F is of the form $F(x,u) = f(x) + g(x)u$ where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and there exists a change of coordinates $\xi = \Phi(x)$, where $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, such that in the new coordinates the dynamics of the system are of the form

$$\dot{\xi} = \hat{A}\xi + \hat{B}[\hat{f}(\xi) + \hat{g}(\xi)u] := \hat{F}(\xi, u),$$

where $\hat{A} \in \mathbb{R}^{n\times n}$ and $\hat{B} \in \mathbb{R}^{n\times m}$ define a controllable pair $(\hat{A}, \hat{B})$, $\hat{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\hat{g}\colon \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is such that $\hat{g}(\xi)$ is invertible for each $\xi \in \mathbb{R}^n$. We will let $\hat{g}_i(\xi)$ denote the i-th column of $\hat{g}(\xi)$. We emphasize that this *global* transformation is distinct from the local Jacobian linearizations employed earlier. In this case the application of the feedback rule $u(\xi, v) = \hat{g}^{-1}(\xi)[-\hat{f}(\xi) + v]$, where $v \in \mathbb{R}^m$ is a new 'virtual' input, results in $\dot{\xi} = \hat{A}\xi + \hat{B}v$. In essence, feedback linearization reveals a linear structure underlying the global geometry of the system. Clearly $\hat{F}$ satisfies Assumption 4, and the following proposition provides sufficient conditions for Assumptions 5 and 6 to hold in these coordinates:

Proposition 1. *Suppose that* (1) *is feedback linearizable, and let $\hat{f}$, $\hat{g}$, $\hat{A}$ and $\hat{B}$ be as defined above. Assume that there exists $\hat{L} > 0$ such that $\|\frac{d}{d\xi}\hat{f}(\xi)\| < \hat{L}$ for each $\xi \in \mathbb{R}^n$. Further assume that there exists a constant $G \in \mathbb{R}^{n\times m}$ such that $\hat{g}(\xi) = G$ for each $\xi \in \mathbb{R}^n$. Then there exists $\gamma > 0$ such that along each trajectory $(\tilde{\xi}(\cdot), \tilde{u}(\cdot))$ of $\hat{F}$ the associated Jacobian linearization $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ is γ-stabilizable. Furthermore the drift term $\tilde{d}(t) = \hat{F}(\tilde{\xi}(t), \tilde{u}(t)) - \tilde{A}(t)\tilde{\xi}(t) - \tilde{B}(t)\tilde{u}(t)$ satisfies $\|\tilde{B}^{\dagger}(t)\tilde{d}(t)\|_2 \leq 2\hat{L}\|\tilde{\xi}(t)\|_2$.*

Proof of the proposition is given in the report [29]. However to understand the necessity of these assumptions (for our analysis) note that along a given solution $(\tilde{\xi}(\cdot), \tilde{u}(\cdot))$ we have $\tilde{A}(t) = \hat{A} + \hat{B}[\frac{d}{d\xi}\hat{f}(\tilde{\xi}(t)) + \sum_{i=1}^{m} \frac{d}{d\xi} g_i(\tilde{\xi}(t))\tilde{u}_i(t)]$ and $\tilde{B}(t) = \hat{B}\hat{g}(\tilde{\xi}(t))$. Thus, we see that the terms in $\tilde{A}(t)$ and $\tilde{B}(t)$ may grow unbounded for large values of $\tilde{\xi}(t)$ and $\tilde{u}(t)$ without the assumptions in Proposition 1. Without uniform upper bounds for these terms, it is difficult to establish that the assumptions in Theorem 1 hold.

Remark 1. *Suppose that the representations of the state running and terminal costs, $\hat{Q} := Q \circ \Phi^{-1}$ and $\hat{V} := V \circ \Phi^{-1}$, are convex in the linearizing coordinates and satisfy pointwise bounds as in Assumption 3. Further assume that the assumptions made of $\hat{F}$ in Proposition 1 hold. Then the conclusions of Theorem 1 can be applied to the representation of $J_T(\cdot, \xi_0)$ in the linearizing coordinates by rescaling $\hat{Q}$ and R appropriately.*

Thus, the global linearizing coordinates provide a useful tool for verifying the sufficient conditions in Theorem 1. Moreover, they provide insight into what goes wrong in situations like the second example presented in Section 3.3. Indeed, a set of linearizing coordinates can be constructed using input-output linearization [6] and the output $y_1 = x_1$. This results in the coordinates

$$\xi_1 = y = x_1$$
$$\xi_2 = \dot{y} = -x_1 + x_2 + 10\eta(x_1),$$

and the dynamics

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = (10\eta'(x_1) - 1)(-x_1 + x_2 + 10\eta(x_1)) + bu.$$

Note that this representation of the dynamics satisfies the hypotheses of Proposition 1. However, the corresponding state running cost $\tilde{Q}$ is clearly nonconvex due to the terms involving η in the diffeomorphism which transforms $(\xi_1, \xi_2) = \Phi(x_1, x_2)$. In this sense the state costs are *incompatible with the underlying geometry of the control system*, as the nonconvexity in the linearizing coordinates leads to the undesirable stationary point of J_T identified above.

4 First-Order Stability Guarantees for Receding Horizon Control

In *receding horizon control* (`RHC`) or *model predictive control*, a planner solves $\inf_{\tilde{u}(\cdot)} J_T(\tilde{u}, x(t))$, where $x(t)$ is the current state of the real world system, and applies the resulting open loop predictive control until a new state measurement is received and the process can be repeated. As discussed above, most formal stability guarantees require that an (approximate) globally optimal solution is found for each (generally nonconvex) planning problem. Applying Theorem 1, we provide the first stability guarantees for a formal model of nonlinear RHC which only requires that approximate stationary points can be found.

4.1 First-Order Receding Horizon Control

Many practical implementations of RHC use a technique known as *warm starting*, where the predictive control returned during each optimization phase is used to construct the 'initial guess' for the subsequent planning problem. This approach has proven highly effective for systems which require rapid re-planning to maintain stability [30].

To model this approach, we define the *first-order receding horizon control* strategy, denoted `FO-RHC`, as follows. First a prediction horizon $T \geq 0$ and a replanning interval $\delta \in (0, T]$ are chosen and a sequence of replanning times $t_k = k\delta$ for $k \in \mathbb{N}$ are defined. Next, the process takes in an initial condition of the physical system $x_0 \in \mathbb{R}^n$ and a warm-start control $\bar{u}_0 \in \mathcal{U}_T$ specified by the user. We let $(\mathbf{x}(\cdot, x_0, \bar{u}_0), \mathbf{u}(\cdot, x_0, \bar{u}_0))$ denote the resulting trajectory produced by the control scheme described below.

At each t_k for $k \in \mathbb{N}$ a warm-start routine generates an initial guess $\tilde{u}_k(\cdot) = \bar{u}_k(\cdot; x_0, \bar{u}_0) \in \mathcal{U}_T$ for the problem $J_T(\cdot, \mathbf{x}(\cdot, x_0, \bar{u}_0))$; a simple choice for such a routine is presented momentarily. The local search method then optimizes the problem using the chosen initial guess, and produces the predictive control $\tilde{u}_k(\cdot) = \tilde{u}(\cdot; x_0, \bar{u}_0)$. Note that both of these quantities depend on both the initial condition of the system and the initial warm-start control specified by the user. The predictive control is constructed via $\tilde{u}_k(\cdot) = u_T^{\texttt{plan}}(\cdot, \mathbf{x}(t_k, x_0, \bar{u}_0), \bar{u}_k) \in \mathcal{U}_T$, where the map $u_T^{\texttt{plan}}$ is used to model how the chosen search algorithm selects a predictive control given for a given initial condition and warm-start input. Finally, the actual control $\mathbf{u}(t, x_0, \bar{u}_0) = \tilde{u}_k(t - t_k)$ is applied on the interval $[t_k, t_{k+1})$, and then the process repeats. We make the following assumptions:

Assumption 8. *We assume that, for any $\hat{x}_0 \in \mathbb{R}^n, \bar{u} \in \mathcal{U}_T$, the planned solution $\tilde{u} = u_T^{\texttt{plan}}(\cdot, \hat{x}_0, \bar{u}) \in \mathcal{U}_T$ satisfies the following two conditions with parameter $\epsilon_0 > 0$:*

1. *$J_T(\tilde{u}; \hat{x}_0) \leq J_T(\hat{u}; \hat{x}_0)$; and,*
2. *$\tilde{u}$ is an $\epsilon_0 J_T(\hat{x}_0, \hat{u})^{1/2}$-*`FOS` *of $J_T(\cdot\,; \hat{x}_0)$.*

The rationale for the first condition is that many popular trajectory optimization methods are *descent methods*, and therefore only decrease the value of the functional J_T. The second condition is reasonable because such methods converge to approximate first-order stationary points, even for nonconvex landscapes [17, 18]. The normalization by $J_T(\hat{x}_0, \hat{u})^{1/2}$ affords geometric stability in Theorem 2 by ensuring the optimization terminates close enough to a stationary point for each planning problem as the system trajectory approaches the origin.

It remains to specify how the warm-starts $\bar{u}_k$ are produced for $k \geq 1$. We propose selecting $\tilde{u}_k$ with δ-delay, continuing until time T, and then applying zero input:

$$\bar{u}_{k+1}(t, x_0, \bar{u}_0) = \begin{cases} \tilde{u}_k(t + \delta, x_0, \bar{u}_0) & t \in [0, T - \delta] \\ 0 & t \in (T - \delta, T] \end{cases}$$

While more sophisticated warm-starts may be adopted in practice, the above is preferable for the present analysis because (a) it does not require further system knowledge, and (b) is ammenable to transparent stability guarantees.

4.2 Sufficient Conditions for Exponential Stability of `FO-RHC`

Finally, we apply Theorem 1 and its assumptions to provide sufficient conditions for the stability of `FO-RHC`. In order to obtain exponential convergence, we will require that, given a desired replanning interval $\delta > 0$, the prediction horizon $T > 0$ is sufficiently large and the optimality parameter $\epsilon_0 > 0$ in Assumption 8 is sufficiently small:

Theorem 2. *Let the assumptions in Theorem 1 hold. Further assume that the search algorithm chosen for* `FO-RHC` *satisfies the conditions in Assumption 8. Then for any prediction horizon $T > 0$, replanning interval $\delta \in (0, T]$ and optimality parameter $\epsilon_0 < \sqrt{2\alpha_Q \mathcal{C}_2}$, and each initial condition for the physical system $x_0 \in \mathbb{R}^n$ and initial warm-start decision variable $\bar{u}_0 \in \mathcal{U}_T$ the system trajectory $(\mathbf{x}(\cdot, x_0, \bar{u}_0), \mathbf{u}(\cdot, x_0, \bar{u}_0))$ generated by the corresponding* `FO-RHC` *scheme satisfies*

$$\|\mathbf{x}(t_k, x_0, \bar{u}_0)\|_2 \leq \sqrt{M(\delta, T, \epsilon_0)} e^{\eta(\delta, T, \epsilon_0) t_k} \|x_0\|_2,$$

for each $k \in \mathbb{N}$ where we define

$$\begin{aligned} M(\delta,\epsilon_0) &:= \mathcal{C}_0^\delta \mathcal{C}_1 \left(1 - \tfrac{\alpha_Q}{2}\mathcal{C}_2\epsilon_0^2\right)^{-1} \\ \eta(\delta,T,\epsilon_0) &:= \tfrac{1}{2\delta}\ln\left(e^{-\delta/\mathcal{C}_1} + \mathcal{T}(\delta,T) + \mathcal{E}(\delta,\epsilon_0)\right) \\ \mathcal{E}(\delta,\epsilon_0) &:= \tfrac{1}{2}\mathcal{C}_0^\delta \mathcal{C}_2 e^{2L_F\delta}((\delta+1)\alpha_Q + \alpha_V)\epsilon_0^2 \\ \mathcal{T}(\delta,T) &:= \mathcal{C}_0^\delta \mathcal{C}_1(\delta\alpha_Q + \alpha_V)e^{-\frac{T}{\mathcal{C}_1} + \delta(\frac{1}{\mathcal{C}_1} + 2L_F)}. \end{aligned}$$

To interpret the above constants, first note that for a fixed replanning interval $\delta > 0$ we have $\lim_{\epsilon_0 \to 0} M(\delta,\epsilon_0) = \bar{\mathcal{C}}_0^\delta \mathcal{C}_1$ and $\lim_{\substack{T\to\infty \\ \epsilon_0 \to 0}} \eta(\delta,T,\epsilon_0) = \frac{1}{2\mathcal{C}_1}$. Thus, in the limiting case `FO-RHC` recovers the exponential rate of convergence predicted by Theorem 1. Next, note that $\eta(\delta,T,\epsilon_0)$ will only be negative if $e^{-\frac{\delta}{\mathcal{C}_1}} + \mathcal{T}(\delta,T,\epsilon_0) + \mathcal{E}(\delta,\epsilon_0) < 1$. Thus, for our estimate on the rate of convergence to be exponentially decaying we require that T is (at least) as big as $T \geq \ln(\bar{\mathcal{C}}_0^\delta \mathcal{C}_1(\delta\alpha_Q + \alpha_V)) + \delta(1 + \mathcal{C}_1 2L_F)$.

5 Future Directions

There are many important directions for future work. First, it should be determined whether the strong assumptions required for Theorem 1 can be relaxed. Concretely, in the context of control affine systems of the form $F(x,u) = f(x) + g(x)u$ it remains to be determined whether similar convergence results can be obtained when g is not constant. The primary shortcoming of our current proof technique is that we completely 'cancel out' the disturbance term $\tilde{d}(t)$ when finding a sub-optimal stabilizing control in Lemma 4, and in some cases this cancellation may be too 'costly' to obtain a useful upper bound. However, the disturbance term may actually be useful at certain points if it helps drive the system towards the origin, making complete cancellation unnecessary. It is interesting to note that feedback linearizing controllers, which can also cancel out 'useful' nonlinearities, often receive a similar criticism. Thus, we suspect that in general further structural assumptions on the system dynamics (e.g. Lagrangian dynamics) could lead to positive results. Moreover, there are systematic procedures for globally stabilizing other classes of systems (e.g strict-feedback, strict-feedforward). Can we also draw inspiration from these global control strategies to better understand when the local structure of stationary points promotes stabilizing behavior in the context of RHC? Finally, it would be interesting to understand if specific trajectory optimization algorithms, such as iLQR or iLQG, enjoy stronger convergence and stability properties than those which can be ensured by arbitrary and possibly worst-case first-order stationary points.

Proof of Lemma 3. Under Assumptions 1 and 2, we have that $\|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x}(t)\| = \|F(\tilde{x}(t),\tilde{u}(t))\| \leq L_F(\|\tilde{x}(t)\| + \|\tilde{u}(t)\|)$. Hence, $|\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{x}(t)\|^2| = |\langle \tilde{x}(t), \frac{\mathrm{d}}{\mathrm{d}t}\tilde{x}(t)\rangle| \leq L_F\|\tilde{x}(t)\|^2 + L_F\|\tilde{u}(t)\|\|\tilde{x}(t)\|$, which at most $\frac{3L_F}{2}\|\tilde{x}(t)\|^2 + \frac{L_F}{2}\|\tilde{u}(t)\|^2$ by the AM-GM inequality. Assumption 3 then implies this is at most $\frac{c_1}{2\alpha_Q}(Q(\tilde{x}(t)) + R(\tilde{u}(t)))$, where we define $c_1 := L_F(3 + \frac{\alpha_Q}{\alpha_R})$. Thus, given any times $s_1 \leq s_2 \in [0,T]$, we have

$$\begin{aligned} \left|\|\tilde{x}(s_1)\|^2 - \|\tilde{x}(s_2)\|^2\right| &\leq \int_{t=s_1}^{s_2} \left|\tfrac{\mathrm{d}}{\mathrm{d}t}\|\tilde{x}(t)\|^2\right| \mathrm{d}t \\ &\leq \tfrac{c_1}{2\alpha_Q}\int_{t=s_1}^{s_2} (Q(\tilde{x}(t)) + R(\tilde{u}(t))\mathrm{d}t \leq \tfrac{c_1}{2\alpha_Q}\tilde{\mathcal{V}}(s_1). \end{aligned} \tag{20}$$

To conclude, fix a time $s \in [0,T]$. We consider two cases: **Case 1:** There is a time $\tau \in [s,T]$ such that $\|\tilde{x}(\tau)\|^2 \le \frac{1}{2}\|\tilde{x}(s)\|^2$. Invoking (20),

$$\begin{aligned}\tfrac{1}{2}\|\tilde{x}(s)\|^2 &\ge \left|\|\tilde{x}(s)\|^2 - \|\tilde{x}(\tau)\|^2\right| \\ &\ge \left|\|\tilde{x}(s)\|^2 - \|\tilde{x}(s\tau)\|^2\right| \ge \tfrac{c_1}{2\alpha_Q}\tilde{\mathcal{V}}(s).\end{aligned}$$

Case 2: There is no such time τ, so $\|\tilde{x}(t)\|^2 \ge \frac{1}{2}\|\tilde{x}(s)\|^2$ for all $t \in [s,T]$. In this case, $\tilde{\mathcal{V}}(s) \ge \alpha_Q \int_{t=s}^{T} \|\tilde{x}(t)\|^2 \mathrm{d}t + \alpha_V\|\tilde{x}(T)^2\| \ge (\alpha_Q(T-s)+\alpha_V)\cdot\|\tilde{x}(s)\|^2/2$. Inverting, and combining both cases,

$$\|\tilde{x}(s)\|^2 \le \frac{1}{\alpha_Q}\max\{c_1, \tfrac{2}{T-s+\alpha_V/\alpha_Q}\}\tilde{\mathcal{V}}(s) \tag{21}$$

Finally, for any $s' \in [0,s]$, arguing as in (20), and applying (21) and some simplifications (incuding $\tilde{\mathcal{V}}(s') \ge \tilde{\mathcal{V}}(s)$),

$$\begin{aligned}\|\tilde{x}(s)\|^2 &\le \|\tilde{x}(s')\|^2 + \left|\int_{t=s'}^{s} (\tfrac{\mathrm{d}}{\mathrm{d}t}\|\tilde{x}(t)\|^2)\mathrm{d}t\right| \\ &\le \|\tilde{x}(s')\|^2 + \frac{c_1}{2\alpha_Q}\tilde{\mathcal{V}}(s') \\ &\le \frac{1}{\alpha_Q}\underbrace{(2c_1 + 2\tfrac{1}{T-(s')+\frac{\alpha_V}{\alpha_Q}})}_{:=\mathcal{C}_0(s')}\tilde{\mathcal{V}}(s'),\end{aligned}$$

which can be specialized to the desired cases. □

Proof of Lemma 4. In view of Lemma 2, to obtain a bound on $\tilde{\mathcal{V}}(s)$ it suffices to bound $\mathcal{V}^{\mathtt{jac},\star}(s) := \inf_{\bar{u}_{[s,T]}} J^{\mathtt{jac}}_{T-s}(\cdot,\ \tilde{x}(s), \tilde{u}_{[s,T]})$. Moreover, we can bound $\mathcal{V}^{\mathtt{jac},\star}(s)$ by bounding $J^{\mathtt{jac}}_{T-s}(\bar{u}_{[s,T]};\ \tilde{x}(t), \tilde{u}_{[s,T]})$ for any (possibly suboptimal) control $\bar{u}_{[s,T]}$; for simplicity, let us drop the $[s,T]$-subscript going forward. We select $\bar{u}(t) = \bar{u}_1(t)+\bar{u}_2(t)$, where $\bar{u}_1(t)$ satisfies $\tilde{B}(t)\bar{u}_1(t) = -\tilde{d}(t)$, and where $\bar{u}_2$ witnesses γ-stabilizability at time s as in Assumption 5.

With this choice of $\bar{u}(t)$ the dynamics of $\bar{x}(t)$ in $J^{\mathtt{jac}}_{T-s}$ are $\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}(t) = \tilde{A}(t)\bar{x}(t) + \tilde{B}(t)\bar{u}(t) + \tilde{d}(t) = \tilde{A}(t)\bar{x}(t) + \tilde{B}\bar{u}_2(t)$, and, writing out $J^{\mathtt{jac}}_{T-s}$ explicitly, we obtain

$$\mathcal{V}^{\mathtt{jac},\star}(s) \le \int_{t=s}^{T} (Q(\bar{x}(t)) + R(\bar{u}(t))\mathrm{d}t + V(\bar{x}(T)). \tag{22}$$

By the elementary bound $\|\bar{u}(t)\|^2 \le 2\|\bar{u}_1(t)\|^2 + 2\|\bar{u}_2(t)\|^2$, the following holds for constant $c = \max\{\beta_V, 2\beta_R, \beta_Q\}$,

$$\mathcal{V}^{\mathtt{jac},\star}(s) \le 2\beta_R \int_{t=s}^{T} \|\bar{u}_1(t)\|^2\mathrm{d}t \tag{23}$$

$$+\, c\left(\int_{t=s}^{T} (\|\bar{x}(t)\|^2 + \|\bar{u}(t)\|^2)\mathrm{d}t + \|\bar{x}(T)\|^2\right). \tag{24}$$

To bound (24), we observe that $\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}(t) = \tilde{A}(t)\bar{x}(t) + \tilde{B}(t)\bar{u}(t) + \tilde{d}(t) = \tilde{A}(t)\bar{x}(t) + \tilde{B}(t)\bar{u}_2(t)$, which corresponds to the $\hat{x}(t)$ dynamics in the definion of γ-stabilizability; thus, (24) is at most $c\cdot\gamma\|\tilde{x}(s)\|^2$.

To bound (23), we use (6) to bound $\|\bar{u}_1(t)\|^2 \leq 2L_x^2\|\tilde{x}(t)\|^2 + 2L_u^2\|\tilde{u}(t)\|^2 \leq 2c'(Q(\tilde{x}(t)) + R(\tilde{u}(t))/\beta_R$, where $c' = \beta_R \max\{\frac{L_x^2}{\alpha_Q}, \frac{L_u^2}{\alpha_R}\}$. Hence, in view of Lemma 2, and the principle of optimality:

$$
\begin{aligned}
2\beta_R \textstyle\int_{t=s}^{T} \|\bar{u}_1(t)\|^2 \mathrm{d}t &\leq 4c'\beta_R \textstyle\int_{t=s}^{T}(Q(\tilde{x}(t)) + R(\tilde{u}(t))\mathrm{d}t \\
&\leq 4c'\beta_R \tilde{\mathcal{V}}(s) \leq 4c'\beta_R(\mathcal{V}^{\texttt{jac},\star}(s) + \tfrac{\epsilon^2}{2\alpha_R})
\end{aligned}
$$

Putting the bounds together and rearranging:

$$
(1 - 4\beta_R c')\mathcal{V}^{\texttt{jac},\star}(s) \leq c \cdot \gamma\|\tilde{x}(s)\|^2 + \tfrac{2c'\beta_R}{\alpha_R}\epsilon^2
$$

Under Assumption 7, we have $4\beta_R c' \leq 1/2$, so that

$$
\mathcal{V}^{\texttt{jac},\star}(s) \leq 2c \cdot \gamma\|\tilde{x}(s)\|^2 + \tfrac{4c'\beta_R}{\alpha_R}\epsilon^2. \tag{25}
$$

We recognize $2c\gamma \leq \mathcal{C}_1$, and $\frac{4c'\beta_R}{\alpha_R} = \mathcal{C}_2 - \frac{1}{2\alpha_R}$, and invoke Lemma 2 to obtain the desired bound. $\square$

Proof of Theorem 2. We first assume that $\alpha_V > 0$ and bound the rate of covegence in terms of $\mathcal{C}_0$; the steps of the proof can be repeated by replacing $\mathcal{C}_0$ with $\mathcal{C}_0^{\delta}$ throughout and then applying the tighter of the two bounds.

For each $k \in \mathbb{N}$ we will let $\tilde{J}_k = J_T(\tilde{u}_k(\cdot; x_0, \bar{u}_0), x_0)$ and $\tilde{J}_k = J_T(\tilde{u}_k(\cdot; x_0, \bar{u}_0), x_0)$. Respectively, these are the cost incurred by the k-th planning solution and the k-th warm-start initial guess. For simplicity, we drop the dependence on x_0 and $\bar{u}_0$ from here on. We also let $(\tilde{x}_k(\cdot), \tilde{u}(\cdot))$ and $(\bar{x}_k(\cdot), \bar{u}_k(\cdot))$ denote the corresponding system trajectories. Our goal is to show that the sequence of losses $\left\{\tilde{J}_k\right\}_{k=1}^{\infty}$ is geometrically decreasing. Indeed, using property 1) of Assumption 8 we have

$$
\begin{aligned}
\tilde{J}_T^{k+1} &\leq \bar{J}_T^{k+1} = \bar{J}_T^{k+1} - \tilde{J}_T^{k} + \tilde{J}_T^{k} \qquad (26)\\
&= \int_{T-\delta}^{T} Q(\bar{x}_{k+1}(\tau)) + R(\bar{u}_{k+1}(\tau))d\tau + V(\bar{x}_{k+1}(T)) \\
&- \int_{0}^{\delta} Q(\tilde{x}_k(\tau)) + R(\tilde{u}_k(\tau))d\tau - V(\tilde{x}_k(T)) + \tilde{J}_T^{k} \\
&\leq \int_{T-\delta}^{T} \alpha_Q\|\bar{x}_{k+1}(\tau)\|_2^2 + \alpha_V\|\bar{x}_{k+1}(T))\|_2^2 \\
&+ \left(e^{-\frac{\delta}{\mathcal{C}_1}} + \frac{\alpha_Q}{2}\mathcal{C}_2\epsilon_0^2\right)\tilde{J}_k
\end{aligned}
$$

where the second inequality follows from Assumption 3, the fact that $\bar{u}_{k+1}(t) = 0$ for each $t \in [T-\delta, T]$, and applying the second inequality in (9) which shows that $J_{T-\delta}(\tilde{u}_k|_{[\delta,T]}, \tilde{x}_k(\delta)) \leq e^{-\frac{\delta}{\mathcal{C}_1}}\tilde{J}_k + \frac{\alpha_Q}{2}\mathcal{C}_2\epsilon_0^2\tilde{J}_k)$, where we have also used property 2) of Assumption 8. The second fact also implies that $|\dot{\bar{x}}_{k+1}(t)| = |F(x_{k+1}(t), 0)| \leq L_F|\bar{x}(\bar{x}(t))|$ (by Assumption 2) for each $t \in [T-\delta, T]$. Thus, by a standard application of a Gronwall-type inequality for each $t \in [T-\delta, T]$ we have will will have

$$
\begin{aligned}
\|\bar{x}_{k+1}(t)\|_2^2 &\leq e^{2L_F\delta}\|\bar{x}_{k+1}(T-\delta)\|_2^2 \qquad (27)\\
&\leq e^{2L_F\delta}\mathcal{C}_0\left(\mathcal{C}_1 e^{-\frac{T-\delta}{\mathcal{C}_1}}\tilde{J}_k + \frac{1}{2}\mathcal{C}_2\epsilon_0^2\tilde{J}_k\right) \qquad (28)
\end{aligned}
$$

where we have used the fact that $\tilde{x}_k(T-\delta) = \bar{x}_{k+1}(T-\delta)$, property 2 from Assumption 8 and Theorem 1. Combining the above observations with the final inequality in (26),integrating, and rearranging terms, and simplifying provides

$$\tilde{J}_{k+1} \;\; \leq \;\; \left(e^{-\frac{\delta}{\mathcal{C}_1}} + \mathcal{C}_0\left[\mathcal{C}_1 e^{2L_F\delta}(\delta\alpha_Q + \alpha_V)e^{-\frac{T-\delta}{\mathcal{C}_1}} + \frac{1}{2}e^{2L_F\delta}((\delta+1)\alpha_Q + \alpha_V)\mathcal{C}_2\epsilon_0^2\right]\right)\tilde{J}_k \quad (29)$$

Which simplifies to

$$\tilde{J}_{k+1} \leq \left(e^{-\frac{\delta}{\mathcal{C}_1}} + [\mathcal{T}(\delta,T) + \mathcal{E}(\delta,\epsilon_0)]\right)\tilde{J}_k. \quad (30)$$

This geometric decay implies that

$$J_T(\tilde{u}_k(0), \tilde{x}_k(0)) \leq e^{2\eta(\delta,T,\epsilon_0)t_k} J_T(\tilde{u}_0(0), \tilde{x}_0(0)) \quad (31)$$

This can then be converted into the desired bound on the state trajectory by applying Lemmas 3 and 4. □

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