MATHEMATICS 271 L01/02 WINTER 2013
MIDTERM SOLUTIONS

[6] 1. **Use the Euclidean algorithm** to find $\gcd(78, 59)$. Then use your work to write $\gcd(78, 59)$ in the form $78a + 59b$ where a and b are integers.
Solution: We have

$$\begin{aligned} 78 &= 1 \times 59 + 19 \\ 59 &= 3 \times 19 + 2 \\ 19 &= 9 \times 2 + 1 \\ 2 &= 2 \times 1 + 0, \end{aligned}$$

so $\gcd(78, 59) = 1$, and
$\gcd(78, 59) = 1 = 19 - 9 \times 2 = 19 - 9 \times (59 - 3 \times 19)$
$= 28 \times 19 - 9 \times 59 = 28 \times (78 - 19) - 9 \times 59 = 28 \times 78 - 37 \times 59.$

Another way is to use the "table method" as follows.

		78	59
	78	1	0
	59	0	1
$R_1 - R_2$	19	1	-1
$R_2 - 3R_3$	2	-3	4
$R_3 - 9R_4$	1	28	-37

Thus, $\gcd(78, 59) = 1$ and $\gcd(78, 59) = 78 \times 28 + 59 \times (-37)$, that is, $a = 28$ and $b = -37$.

[4] 2. You are given that A and B are sets, and that $(2, 3) \in A \times B$, but $(2, 4) \notin A \times B$.
(a) Find an ordered pair that definitely is in $B \times A$. Explain.
Solution: The pair $(3, 2)$ is definitely an element of $B \times A$. Since $(2, 3) \in A \times B$, we know $2 \in A$ and $3 \in B$, and hence $(3, 2) \in B \times A$.

(b) Find another ordered pair ($\neq (2, 4)$) that definitely is **not** in $A \times B$. Explain.
Solution: The pair $(1, 4)$ is definitely not an element of $A \times B$. Since $(2, 3) \in A \times B$, we know $2 \in A$. Since $(2, 4) \notin A \times B$, we know $2 \notin A$ or $4 \notin B$, but $2 \in A$ as seen above, and so we know that $4 \notin B$. Since $4 \notin B$, the pair $(1, 4)$ is not an element of $A \times B$.

Note that $(a, b) \notin A \times B$ if and only if $a \notin A$ **or** $b \notin B$. Thus, if we know $b \notin B$ then we can conclude that $(a, b) \notin A \times B$ (it does not matter if $a \in A$ or $a \notin A$). In this case, we know $4 \notin B$, so we can conclude $(1, 4) \notin A \times B$, it does not matter whether 1 is an element of A or not, and in fact we do not know whether 1 is an element of A.

Also, note that the empty set is **not** an element of every set. It is true that the empty set is a **subset** of every set.

[11] 3. Two of the following three statements are true, and one is false. Prove the true statements. Write out and prove the **negation** of the false statement. Use no properties of even and odd integers other than their definitions.
(a) $\forall n \in \mathbb{Z}\ \exists$ a prime number p so that pn is even.
Solution: This statement is true and here is a proof. Let $n \in \mathbb{Z}$. Choose $p = 2$. Then p is a prime, and $pn = 2n$ where $n \in \mathbb{Z}$ and so pn is even.

(b) $\forall n \in \mathbb{Z}\ \exists$ a prime number p so that pn is odd.
Solution: This statement is false. Its negation is: "$\exists n \in \mathbb{Z}$ so that $\forall$ prime numbers p, pn is even". For example, let $n = 2$. Then $n \in \mathbb{Z}$ and for any prime number p, $pn = 2p$ is clearly even.

(c) $\forall n \in \mathbb{Z}\ \exists$ a prime number p so that $p + n$ is odd.
Solution: This statement is true and here is a proof. Let $n \in \mathbb{Z}$.
Case 1: n is odd, that is, $n = 2k + 1$ for some $k \in \mathbb{Z}$. We choose $p = 2$. Then p is a prime, and $p + n = 2 + 2k + 1 = 2\,(k + 1) + 1$ where $k + 1 \in \mathbb{Z}$, and so $p + n$ is odd.
Case 2: n is even, that is, $n = 2m$ for some $m \in \mathbb{Z}$. We choose $p = 3$. Then p is a prime, and $p + n = 3 + 2m = 2\,(m + 1) + 1$ where $m + 1 \in \mathbb{Z}$, and so $p + n$ is odd.

[13] 4. Let $\mathcal{S}$ be the statement:

for all sets A, B and C, if $A \cap B = \emptyset$ then $(A - C) \cap (B - C) = \emptyset$.

(a) Write out the contrapositive of $\mathcal{S}$.
Solution: The contrapositive of $\mathcal{S}$ is
for all sets A, B and C, if $(A - C) \cap (B - C) \neq \emptyset$ then $A \cap B \neq \emptyset$.

(b) Prove $\mathcal{S}$, using contradiction or the contrapositive.
Solution (by contradiction): Suppose that there are sets A, B and C so that $A \cap B = \emptyset$ and $(A - C) \cap (B - C) \neq \emptyset$. Since $(A - C) \cap (B - C) \neq \emptyset$, there exist an element $x \in (A - C) \cap (B - C)$. It follows that $x \in (A - C)$ and $x \in (B - C)$, which implies that $x \in A$ and $x \in B$. Since $x \in A$ and $x \in B$, we know $x \in A \cap B$. Thus, there exists an element $x \in A \cap B$, which contradicts the assumption that $A \cap B = \emptyset$. Hence, for all sets A, B and C, if $A \cap B = \emptyset$ then $(A - C) \cap (B - C) = \emptyset$.

Solution (using the contrapositive): We prove the contrapositive of $\mathcal{S}$ (stated in part (a)). Suppose that A, B and C are sets so that $(A - C) \cap (B - C) \neq \emptyset$. Since $(A - C) \cap (B - C) \neq \emptyset$, there exist an element $x \in (A - C) \cap (B - C)$. It follows that $x \in (A - C)$ and $x \in (B - C)$, which implies that $x \in A$ and $x \in B$. Since $x \in A$ and $x \in B$, we know $x \in A \cap B$. Thus, there exists an element $x \in A \cap B$, and so $A \cap B \neq \emptyset$. Thus, the contrapositive of $\mathcal{S}$ is true and since $\mathcal{S}$ is logically equivalent to the contrapositive of $\mathcal{S}$, $\mathcal{S}$ is true.

(c) Write out the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.
Solution: The converse of $\mathcal{S}$ is:
for all sets A, B and C, if $(A - C) \cap (B - C) = \emptyset$ then $A \cap B = \emptyset$.
The converse of $\mathcal{S}$ is false. For example, in the case $A = B = C = \{1\}$, we have $(A - C) \cap (B - C) = \emptyset \cap \emptyset = \emptyset$, but $A \cap B = \{1\} \neq \emptyset$.
(d) Write out the *negation* of statement $\mathcal{S}$. Is it true or false? Explain.
Solution: The negation of $\mathcal{S}$ is:
there exist sets A, B and C so that $A \cap B = \emptyset$ but $(A - C) \cap (B - C) \neq \emptyset$.
The negation of $\mathcal{S}$ is false since it has the opposite truth value of $\mathcal{S}$, which is true as proven in (b).

[6] 5. **Use mathematical induction** (or well ordering), to prove that $2^n \geq 5n - 7$ for all integers $n \geq 3$.
Solution:
Basis step: $(n = 3)$
$2^n = 2^3 = 8 \geq 8 = 15 - 7 = 5 \times 3 - 7 = 5n - 7$. Thus, the statement is true when $n = 3$.
Inductive step: Let $k \geq 3$ be an integer and suppose that

$$2^k \geq 5k - 7. \qquad \text{[IH]}$$

We want to prove that $2^{k+1} \geq 5\,(k+1) - 7$.
Now,

$$\begin{aligned} 2^{k+1} &= 2 \times 2^k \\ &\geq 2\,(5k - 7) && \text{by [IH]} \\ &= 10k - 14 \\ &= 5\,(k+1) - 7 + (5k - 12) \\ &\geq 5\,(k+1) - 7. && \text{Note that } k \geq 3 \text{, so } 5k - 12 \geq 5 \times 3 - 12 = 3 \geq 0 \end{aligned}$$

Thus, $2^n \geq 5n - 7$ for all integers $n \geq 3$.