Contents

Chapter 1

Mathematical concepts and notation

Throughout this article, we will use the following notation:

The space under consideration will be an euclidean point space $\mathscr{E}$ over a vector space $\mathscr{V}$.

We will use the term **tensor** as a synonym for a linear mapping from $\mathscr{V}$ into $\mathscr{V}$. Let us denote the following sets of tensors:

$$
\begin{aligned}
\mathrm{Lin} &= \text{set of all tensors},\\
\mathrm{Lin}^{+} &= \text{set of all tensors } S \text{ with } \det S > 0,\\
\mathrm{Sym} &= \text{set of all symmetric tensors},\\
\mathrm{Skw} &= \text{set of all skew (antisymmetric) tensors},\\
\mathrm{Psym} &= \text{set of all symmetric, positive definite tensors},\\
\mathrm{Orth} &= \text{set of all orthogonal tensors},\\
\mathrm{Orth}^{+} &= \text{set of all rotations}.
\end{aligned}
$$

We will use the term **body** $\mathscr{B}$ to describe a regular region in $\mathscr{E}$. We refer to $\mathscr{B}$ as a **reference configuration**. Points $\mathbf{p} \in \mathscr{B}$ are called **material points**.

By the **deformation** of $\mathscr{B}$ we mean a smooth, one-to-one mapping f which maps $\mathscr{B}$ onto a closed region in $\mathscr{E}$, and which satisfies $\det \nabla f > 0$. The tensor $F(\mathbf{p}) = \nabla f(\mathbf{p})$ is called the **deformation gradient** and belongs to Lin^{+}.

Let $\mathscr{B}$ be a body. A **motion** of $\mathscr{B}$ is a class C^3 function

$$x : \mathscr{B} \times \mathbb{R} \to \mathscr{E}$$

with $x(\cdot, t)$, for each fixed t, a deformation of $\mathscr{B}$. We refer to

$$\mathbf{x} = x(\mathbf{p}, t)$$

as the **place** occupied by a material point $\mathbf{p}$ at a time t and we write $\mathscr{B}_t = x(\mathscr{B}, t)$ for the region of space occupied by the body at t. We define the **trajectory** of the body as a set

$$\mathscr{T} = \{(\mathbf{x}, t) | \, \mathbf{x} \in \mathscr{B}_t, t \in \mathbb{R}\}.$$

At each t, $x(\cdot, t)$ has an inverse

$$p(\cdot, t) : \mathscr{B}_t \to \mathscr{B}$$

such that

$$x(p(\mathbf{x}, t), t) = \mathbf{x}, \qquad p(x(\mathbf{p}, t), t) = \mathbf{p}.$$

Given $(\mathbf{x}, t) \in \mathscr{T}$,

$$\mathbf{p} = p(\mathbf{x}, t)$$

is the material point that occupies a place $\mathbf{x}$ at a time t. The map

$$p : \mathscr{T} \to \mathscr{B}$$

is called the **reference map** of the motion.

A **material field** is a function with domain $\mathscr{B} \times \mathbb{R}$; a **spatial field** is a function with domain $\mathscr{T}$. We can transform a material field into a spatial field, and vice versa. We define the **spatial description** Φ_s of a material field $(\mathbf{p}, t) \to \Phi(\mathbf{p}, t)$ by

$$\Phi_s(\mathbf{x}, t) = \Phi(p(\mathbf{x}, t), t),$$

and the **material description** Ω_m of a spatial field $(\mathbf{x}, t) \to \Omega(\mathbf{x}, t)$ by

$$\Omega_m(\mathbf{p}, t) = \Omega(x(\mathbf{p}, t), t).$$

Given a material field Φ we write

$$\dot{\Phi}(\mathbf{p}, t) = \frac{\partial}{\partial t} \Phi(\mathbf{p}, t)$$

for a derivative with respect to time t holding the material point $\mathbf{p}$ fixed, and

$$\nabla \Phi(\mathbf{p}, t) = \nabla_p \Phi(\mathbf{p}, t)$$

for a gradient with respect to $\mathbf{p}$ holding t fixed.

Similarly, given a spatial field Ω we write

$$\Omega'(\mathbf{x}, t) = \frac{\partial}{\partial t} \Omega(\mathbf{x}, t)$$

for the derivative with respect to time t holding the place $\mathbf{x}$ fixed, and

$$\operatorname{grad}\Omega(\mathbf{x},t) = \nabla_x \Omega(\mathbf{x},t)$$

for the gradient with respect to $\mathbf{x}$ holding t fixed.

We define the **material time derivative** $\dot{\Omega}$ of a spatial field Ω by

$$\dot{\Omega} = ((\Omega_m)^{\cdot})_s\,;$$

that is,

$$\dot{\Omega}(\mathbf{x},t) = \frac{\partial}{\partial t}\,\Omega(x(\mathbf{p},t),t)|_{\mathbf{p}=p(\mathbf{x},t)}\,.$$

Further, we define the **spatial divergence** div to be a divergence operation for a spatial field, so that grad is the underlying gradient. Thus, for a spatial vector field v, we have

$$\operatorname{div} v(\mathbf{x},t) = \operatorname{tr}\operatorname{grad} v(\mathbf{x},t).$$

We call

$$\dot{x}(\mathbf{p},t) = \frac{\partial}{\partial t}\,x(\mathbf{p},t)$$

the **velocity** of the material point $\mathbf{p}$, and $v : \mathscr{T} \to \mathscr{V}$ defined by

$$v(\mathbf{x},t) = \dot{x}(p(\mathbf{x},t),t)$$

the **spatial description of velocity**. The spatial field

$$L = \operatorname{grad} v$$

is called the **velocity gradient**. We write

$$L = D + W,$$

where D and W, respectively, denote the symmetric and skew parts of L.

Using the concept of velocity gradient previously defined, one can show that $\dot{F} = L_m F$ for the material time derivative of a deformation gradient F.

By the **system of forces** for $\mathscr{B}$ during a motion (with trajectory $\mathscr{T}$), we mean a pair (s,b) of functions

$$s : \mathscr{N} \times \mathscr{T} \to \mathscr{V}, \qquad b : \mathscr{T} \to \mathscr{V},$$

where $\mathscr{N}$ is the set of all unit vectors from $\mathscr{V}$[1].

By Cauchy's Theorem[2], there exists a spatial tensor field T (called the **Cauchy stress**) such that

[1] More precise definition is in Gurtin [gu, p. 99].
[2] See Gurtin [gu, p. 101].

- $s(n) = Tn$ for each unit vector n,
- T is symmetric,
- T satisfies the **equation of motion**

$$\operatorname{div} T + b = \rho \dot{v},$$

where ρ is the density in motion.

By the **dynamical process** we mean a pair (x, T) with

- x motion,
- T symmetric tensor field on trajectory $\mathscr{T}$ of x,
- $T(\mathbf{x}, t)$ smooth function of $\mathbf{x}$ on $\mathscr{B}_t$.

A **material body** is a body $\mathscr{B}$ together with a family $\mathscr{C}$ of dynamical processes. $\mathscr{C}$ is called the **constitutive class** of the body.

Let x and x^* be motions of $\mathscr{B}$. We say, that x and x^* are *related by a change in observer*, if

$$x^*(\mathbf{p}, t) = q(t) + Q(t)[x(\mathbf{p}, t) - o] \tag{1.1}$$

for every material point $\mathbf{p}$ and time t, where $q(t)$ is a point of the space and $Q(t)$ is a rotation.

Letting

$$L = \operatorname{grad} v, \qquad L^* = \operatorname{grad} v^*,$$

where

$$v = (\dot{x})_s, \qquad v^* = (\dot{x}^*)_s,$$

we obtain

$$L^* = QLQ^T + \dot{Q}Q^T, \qquad D^* = QDQ^T,$$

where D and D^*, respectively, are symmetric parts of L and L^*. Thus, we have $\operatorname{tr} L^* = \operatorname{tr} L$.

We say that two dynamical processes (x, T) and (x^*, T^*) **are related by a change in observer** if there exist C^3 functions

$$q : \mathbb{R} \to \mathscr{E}, \qquad Q : \mathbb{R} \to \mathrm{Orth}^+$$

such that

- (1.1) holds for all $\mathbf{p} \in \mathscr{B}$ and $t \in \mathbb{R}$,

- $T^* = QTQ^T$ in trajectory of x.

We say that **a response of a material body is independent of the observer** provided its constitutive class $\mathscr{C}$ has the following property: if a process (x, T) belongs to $\mathscr{C}$, so does every dynamical process related to (x, T) by a change in observer.

Chapter 2

Governing equations

In this work, we will solve a flow of compressible Newtonian fluid, which is a material, for which the Cauchy stress is defined by the constitutive equation of the form

$$T = -\pi I + C[L], \tag{2.1}$$

where C is a linear function of the velocity gradient

$$L = \operatorname{grad} v.$$

As considered in Gurtin [gu, p. 147], Newtonian fluid means *incompressible* Newtonian fluid. The Navier-Stokes equations are derived with the assumption $\operatorname{tr} L = 0$, which means incompressibility. In our case, we need to consider compressibility effects and cannot neglect the term $\operatorname{tr} L = \operatorname{div} v$. We will use the name Newtonian fluid for *compressible* Newtonian fluid. In order to simplify the constitutive equation, we define the **extra stress** T_0 by

$$T_0 = T + \pi I = T - \frac{1}{3}(\operatorname{tr} T)I.$$

Then the constitutive equation (2.1) takes the simple form

$$T_0 = C[L]. \tag{2.2}$$

In view of the previous, we consider **Newtonian fluid** a compressible material body consistent with the following constitutive equation: there exists a linear *response function*

$$C : \mathrm{Lin} \to \mathrm{Sym}$$

such that the constitutive class $\mathscr{C}$ is a set of all dynamical processes (x, T) which obey the constitutive equation (2.2).

In the following theorem, we will show that the response is determined by *two constants*.

Theorem[1] A necessary and sufficient condition for the response of a Newtonian fluid to be independent of the observer is that its response function C has the form

$$C[L] = 2\mu D + \lambda(\operatorname{tr} L)I \tag{2.3}$$

for every $L \in \mathrm{Lin}$, where

$$D = \frac{1}{2}(L + L^T).$$

The scalar constants μ and λ are called the first and the second **viscosity coefficients** of the fluid.

Proof. Our proof will copy the one from Gurtin, so that we include the compressibility.

(Sufficiency) Assume that (2.3) holds. Let (x, T) belong to the constitutive class $\mathscr{C}$ of the fluid. Then

$$T_0 = 2\mu D + \lambda(\operatorname{tr} L)I.$$

Let (x^*, T^*) be related to (x, T) by a change in observer. Then

$$T^* = QTQ^T, \qquad D^* = QDQ^T,$$

and

$$\operatorname{tr} T^* = \operatorname{tr}(QTQ^T) = \operatorname{tr} T.$$

Therefore

$$\begin{aligned} T_0^* &= T^* - \frac{1}{3}(\operatorname{tr} T^*)I = QTQ^T - \frac{1}{3}(\operatorname{tr} T)QQ^T = QT_0Q^T \\ &= Q(2\mu D)Q^T + \lambda \operatorname{tr}(QLQ^T)I = 2\mu D^* + \lambda(\operatorname{tr} L^*)I, \end{aligned}$$

because

$$L^* = QLQ^T + \dot{Q}Q^T, \qquad \operatorname{tr} L^* = \operatorname{tr}(QLQ^T) = \operatorname{tr} L,$$

since $\dot{Q}Q^T \in \mathrm{Skw}$.

Thus $(x^*, T^*) \in \mathscr{C}$ and the response is independent of the observer.

[1] Cf. Gurtin [gu, p. 149].

The proof of necessity is facilitated by the following Lemma and Representation Theorem:

Lemma. Let $L \in \mathrm{Lin}$ be a constant tensor. Then there exists a motion x with velocity gradient

$$\operatorname{grad} v = L. \tag{2.4}$$

Proof. Take

$$F(t) = e^{Lt}$$

so that F is a unique solution of

$$\dot{F} = LF, \qquad F(0) = I. \tag{2.5}$$

Thus

$$x(\mathbf{p}, t) = \mathbf{q} + F(t)[\mathbf{p} - \mathbf{q}]$$

defines a motion with the deformation gradient F. Further, (2.4) follows from $(2.5)_1$, since $\dot{F} = (\operatorname{grad} v)_m F$ and $L = L_m$. □

Representation Theorem for Isotropic Tensor Functions.[2] A linear function

$$G : Sym \to Sym$$

is isotropic if and only if there exist scalars μ and λ such that

$$G(A) = 2\mu A + \lambda(\operatorname{tr} A)I \tag{2.6}$$

for every $A \in \mathrm{Sym}$.

We now return to the proof of the previous theorem. To establish the *necessity* of (2.3) we assume that

$$\text{the response is independent of the observer.} \tag{2.7}$$

Let $L \in \mathrm{Lin}$ be arbitrary, let x be the motion constructed in the previous lemma, and let $T = T_0 = C[L]$ be the constant field defined by (2.2). Then, clearly, $(x, T) \in \mathscr{C}$. Let (x^*, T^*) be related to (x, T) by a change in observer. Then by (2.7), $(x^*, T^*) \in \mathscr{C}$ and

$$T_0^* = C[L^*]. \tag{2.8}$$

But

$$T_0^* = QT_0Q^T, \qquad L^* = QLQ^T + \dot{Q}Q^T;$$

[2] Cf. Gurtin [gu, p. 235].

hence (2.8) yields

$$QT_0Q^T = C[QLQ^T + \dot{Q}Q^T],$$

and we conclude from (2.2) and (2.4) that

$$QC[L]Q^T = C[QLQ^T + \dot{Q}Q^T]. \tag{2.9}$$

Clearly, this relation holds for every $L \in \mathrm{Lin}$ (the definition scope of C) and every C^3 function $Q : \mathbb{R} \to \mathrm{Orth}^+$. Fix L and take

$$Q(t) = e^{-Wt},$$

where

$$W = \frac{1}{2}\,(L - L^T).$$

Then $Q(t)$ is rotation, since W is skew, and

$$Q(0) = I, \qquad \dot{Q}(0) = -W.$$

Using this function Q in (2.9) at $t = 0$ yields

$$C[L] = C[L - W] = C[D],$$

where

$$D = \frac{1}{2}\,(L + L^T).$$

Thus C is completely determined by its restriction to Sym. Next, let Q be a constant function with values in Orth^+. Then (2.9) with $L = D$ ($D \in \mathrm{Sym}$) implies that

$$QC[D]Q^T = C[QDQ^T].$$

Since this relation must hold for every $D \in \mathrm{Sym}$ and every $Q \in \mathrm{Orth}^+$, the restriction of C to Sym is isotropic; we therefore conclude from the representation (2.6) that

$$C[D] = 2\mu D + \lambda(\mathrm{tr}\, L)I$$

for all $D \in \mathrm{Sym}$.

□

By (2.3) the constitutive equation (2.1) takes the form

$$T = -\pi I + 2\mu D + \lambda(\mathrm{tr}\, L)I. \tag{2.10}$$

We consider the equation of motion

$$\rho[v' + (\operatorname{grad} v)v] = \operatorname{div} T + b.$$

and substitute (2.10) for T. We have,

$$2 \operatorname{div} D = \operatorname{div} (\operatorname{grad} v + \operatorname{grad} v^T) = \Delta v + \operatorname{grad} \operatorname{div} v$$

and

$$\operatorname{div} (\operatorname{tr} L)I = \operatorname{grad} (\operatorname{tr} L) = \operatorname{grad} \operatorname{div} v,$$

where $\Delta = \operatorname{div} \operatorname{grad}$ is the spatial Laplacian. Thus the equation of motion reduces to

$$\rho[v' + (\operatorname{grad} v)v] = \mu\Delta v + (\lambda + \mu) \operatorname{grad} \operatorname{div} v - \operatorname{grad} \pi + b. \tag{2.11}$$

These (vector) relations are the **Navier-Stokes equations**; given μ, λ and b they constitute a nonlinear system of partial differential equations for the velocity v, density ρ and pressure π. We supplement these equations by the continuity equation

$$\rho' + \operatorname{div} (\rho v) = 0. \tag{2.12}$$

Now, we have four equations for five unknowns, so we have to add one more equation to the system. We will consider barotropic flow[3], where the pressure is a known function of the density

$$\pi = \widehat{\pi}(\rho). \tag{2.13}$$

[3]Cf. Feistauer et al. [fe, p. 33].

Chapter 3

Formulation of the problem

In what follows, we shall be concerned with a two-dimensional model describing an interaction of a viscous, compressible fluid with an airfoil. The airfoil is considered to be a rigid body with two degrees of freedom - its vertical and torsion vibrations (see Obrázok 3.1). Equations describing the airfoil motion will be presented later.

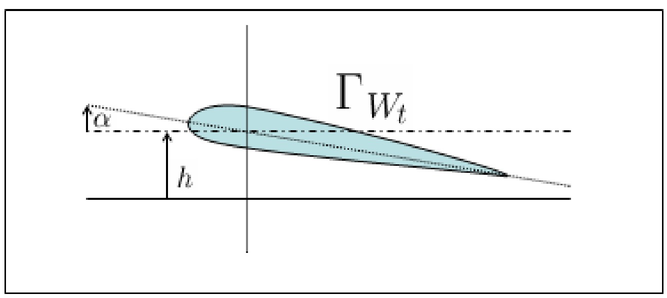

Figure 3.1: Airfoil model

This problem has a time-dependent boundary (moving airfoil) and therefore, a time-dependent computational domain (see Obrázok 3.2).

3.1 Input data of our problem

We consider the problem in the domain

$$\widetilde{\Omega} := \bigcup_{t\in[0,T]} \Omega_t \times \{t\}.$$

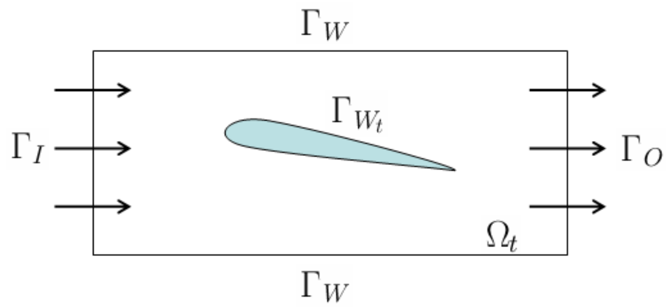

Figure 3.2: Problem setting

We split the domain boundary into four parts. Three of them are time independent, whereas the part representing the moving airfoil depends on time:

$$\begin{array}{ll} \Gamma_I := \Gamma_I \times [0,T] & \text{inlet,} \\ \Gamma_O := \Gamma_O \times [0,T] & \text{outlet,} \\ \Gamma_W := \Gamma_W \times [0,T] & \text{virtual flow wall,} \\ \Sigma := \bigcup_{t\in[0,T]} \Gamma_{W_t} \times \{t\} & \text{airfoil.} \end{array}$$

In the domain $\widetilde{\Omega}$, we consider the Navier-Stokes equations, the continuity equation and the condition of barotropic flow:

$$\begin{aligned} \rho[v' + (\operatorname{grad} v)v] = \mu\Delta v + (\lambda+\mu)\operatorname{grad}\operatorname{div} v - \operatorname{grad}\pi + b \qquad & \text{in } \widetilde{\Omega}, \\ \rho' + \operatorname{div}(\rho v) = 0 \qquad & \text{in } \widetilde{\Omega}, \\ \pi = \widehat{\pi}(\rho) \qquad & \text{in } \widetilde{\Omega}. \end{aligned} \tag{3.1}$$

Boundary conditions for the time independent part of the boundary:

$$\begin{aligned} v = v_D \qquad & \text{on } \Gamma_I \cup \Gamma_W, \\ -(\pi - \pi_{ref})n + \mu(\operatorname{grad} v)n + (\lambda+\mu)(\operatorname{div} v)n = 0 \qquad & \text{on } \Gamma_O, \\ \rho = \rho_D \qquad & \text{on } \Gamma_I. \end{aligned} \tag{3.2}$$

Initial conditions:

$$\begin{aligned} v(x,0) = v_0(x) \qquad & \text{in } \Omega_0, \\ \rho(x,0) = \rho_0(x) \qquad & \text{in } \Omega_0. \end{aligned} \tag{3.3}$$

We also need to prescribe boundary conditions on the part Σ of the boundary and initial conditions on Ω_0. This will be discussed further.

The fact that the domain occupied by the fluid depends on time causes difficulties. In order to overcome them, we can use Arbitrary Lagrangian-Eulerian

(ALE) formulation for the mathematical description of the problem with a moving boundary.

3.2 Equations of airfoil motion

In our case, the airfoil can perform its vertical and torsion vibrations. These vibrations are described by two degrees of freedom: airfoil deflection angle α and vertical displacement h. The evolution of these values for small angles of deflection is described by the following differential equations[1]

$$\begin{aligned} m\ddot{h} + D_{hh}\dot{h} + D_{h\alpha}\dot{\alpha} + S_\alpha\ddot{\alpha} + k_{hh}h &= -L_2, \\ I_\alpha\ddot{\alpha} + D_{\alpha h}\dot{h} + D_{\alpha\alpha}\dot{\alpha} + S_\alpha\ddot{h} + k_{\alpha\alpha}\alpha &= M. \end{aligned} \tag{3.4}$$

Here we use the following notation:

$m = \int_{\Pi_t} \rho \, dx$	weight of airfoil,
$S_\alpha = \int_{\Pi_t} x\rho \, dx$	static momentum,
$I_\alpha = \int_{\Pi_t} x^2\rho \, dx$	momentum of inertia,
$L_2 = -\int_{\Gamma_{W_t}} \sum_{j=1}^2 T_{2j}n_j \, dS$	aerodynamic lift,
$M = -\int_{\Gamma_{W_t}} \sum_{i,j=1}^2 T_{ij}n_jr_i^{ort}) \, dS$	aerodynamic momentum,

where Π_t is the area of the airfoil, $\Gamma_{W_t} = \partial\Pi_t$, T is the stress tensor obtained from (2.10), $r_1^{ort} = -(x_2 - x_{EA2})$ and $r_2^{ort} = x_1 - x_{EA1}$. Further,

k_{hh}	vertical stiffness,
$k_{\alpha\alpha}$	torsion stiffness,
$D_{hh}, D_{h\alpha}, D_{\alpha h}, D_{\alpha\alpha}$	components of viscous damping

are given (constant) parameters.

Equations (3.4) are supplemented with these initial conditions

$$\begin{aligned} \alpha(0) = \alpha_0, \quad \dot{\alpha}(0) = \alpha_1, \\ h(0) = h_0, \quad \dot{h}(0) = h_1. \end{aligned}$$

[1]See Růžička [ru, p. 17].

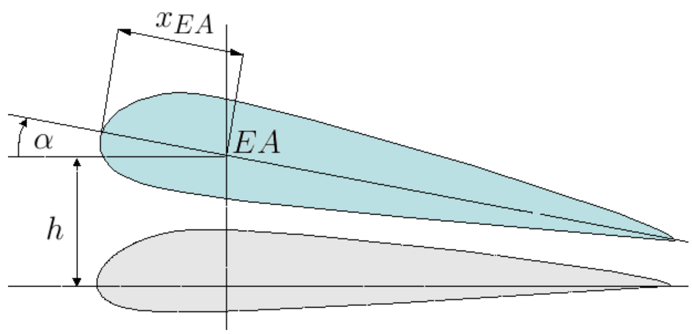

Figure 3.3: Airfoil vibrations

3.3 ALE formulation

We consider the Navier-Stokes equations in a moving domain $\widetilde{\Omega} = \Omega_t \times [0,T]$ (which means $\bigcup_{t\in[0,T]} \Omega_t \times \{t\}$ - the so-called non-cylindrical domain).
In order to simulate a fluid flow on a moving domain, we employ the *Arbitrary Lagrangian-Eulerian* (ALE) method[2].

Let Ω_0 be the original domain and Ω_t be the computational domain at a (later) time t. We introduce the ALE mapping

$$\begin{aligned} &\mathscr{A}_t : \Omega_0 \to \Omega_t \\ &X \mapsto y = y(X,t) = \mathscr{A}_t(X), \end{aligned}$$

which maps the original domain Ω_0 onto the computational domain Ω_t, such that $\mathscr{A}_t$ is continuous and bijective on Ω_0.

We define the *domain velocity* field at points X of the original domain at each time level t

$$\tilde{w}(X,t) = \frac{\partial}{\partial t}\, y(X,t) = \frac{\partial}{\partial t}\, \mathscr{A}_t(X),$$

which, in spatial coordinates has the form

$$w = \tilde{w} \circ A_t^{-1}, \qquad \text{i.e.} \quad w(y,t) = \tilde{w}(\mathscr{A}_t^{-1}(y),t).$$

For a function $f : \widetilde{\Omega} \to \mathbb{R}$, we define the *ALE derivative* of f as

$$\frac{D^{\mathscr{A}}}{Dt}\, f(y,t) = \frac{\partial}{\partial t}\, \tilde{f}(X,t),$$

[2]See, e.g. Quarteroni [qa, p. 37]; Sváček [sv, p. 6].

where $\tilde{f} = f \circ \mathscr{A}_t$ and $X = \mathscr{A}_t^{-1}(y)$.

Using the chain rule for derivative, we obtain

$$\begin{aligned}\frac{D^{\mathscr{A}}}{Dt} f(y,t) &= \frac{\partial}{\partial t} f(\mathscr{A}_t(X), t) \\ &= \frac{\partial}{\partial t} f(y,t) + \operatorname{grad} f(y,t) \cdot \frac{\partial}{\partial t} \mathscr{A}_t(X)|_{X=\mathscr{A}_t^{-1}(y)} \\ &= \frac{\partial}{\partial t} f(y,t) + \operatorname{grad} f(y,t) \cdot w(y,t).\end{aligned}$$

Using ALE derivative, we can rewrite the Navier-Stokes equations in the form

$$\begin{aligned}\rho[\tfrac{D^{\mathscr{A}}}{Dt} v + (\operatorname{grad} v)(v-w)] = \mu\Delta v + (\lambda+\mu) \operatorname{grad} \operatorname{div} v - \operatorname{grad} \pi + b \\ \tfrac{D^{\mathscr{A}}}{Dt} \rho + \operatorname{div}(\rho v) - \operatorname{grad} \rho \cdot w = 0 \\ \pi = \widehat{\pi}(\rho)\end{aligned} \tag{3.5}$$

where all equations are considered on the domain $\widetilde{\Omega}$. Note, that the continuity equation can be written in the form

$$\frac{D^{\mathscr{A}}}{Dt} \rho + \rho \operatorname{div}(v) + \operatorname{grad} \rho \cdot (v-w) = 0. \tag{3.6}$$

Boundary and initial conditions remain the same as before.

3.4 Weak formulation

First, we define the spaces of test functions. Let $q \in Q = L^2(\Omega_t)$ and $u \in V = \{u \in H^1(\Omega_t)^2 : u|_{\Gamma_D} = 0\}$, where $\Gamma_D = \Gamma_I \cup \Gamma_W \cup \Gamma_{W_t}$ is the part of the boundary, where we prescribe the Dirichlet condition.

Multiplying equation $(3.5)_1$ with any $u \in V$, integrating over Ω_t and using Green's theorem, we obtain

$$\begin{aligned}\int_{\Omega_t} \rho \frac{D^{\mathscr{A}}}{Dt} v \cdot u \, dx + \int_{\Omega_t} \rho (\operatorname{grad} v)(v-w) \cdot u \, dx = \\ -\mu \int_{\Omega_t} \operatorname{grad} v \cdot \operatorname{grad} u \, dx - (\lambda+\mu) \int_{\Omega_t} \operatorname{div} v \operatorname{div} u \, dx \\ + \int_{\Omega_t} \pi \operatorname{div} u \, dx + \int_{\Omega_t} b \cdot u \, dx \\ + \int_{\Gamma_O} [-\pi + \mu (\operatorname{grad} v) + (\lambda+\mu)(\operatorname{div} v)] \, n \cdot u \, dS\end{aligned}$$

Same proceeding with (3.6), in which we use any $q \in Q$, yields

$$\int_{\Omega_t} \frac{D^{\mathscr{A}}}{Dt}\,\rho\; q\; dx + \int_{\Omega_t} \rho \operatorname{div} v\; q\; dx + \int_{\Omega_t} \operatorname{grad} \rho \cdot (v - w)\; q\; dx = 0$$

For simplicity, we define the following forms[3]:

$$\begin{aligned} a(v,u) &= \mu\,(\operatorname{grad} v, \operatorname{grad} u) + (\lambda + \mu)(\operatorname{div} v, \operatorname{div} u), \\ b(u,q) &= (\operatorname{div} u, q), \\ \alpha(v,\rho,q) &= (v \cdot \operatorname{grad} \rho, q), \\ d(\rho,w,v,u) &= (\rho(\operatorname{grad} v)w, u), \\ e(\rho,v,q) &= (\rho \operatorname{div} v, q). \end{aligned}$$

Then, we can rewrite the previous equations in the form

$$\begin{aligned} (\rho\,\frac{D^{\mathscr{A}}}{Dt}\,v, u) + d(\rho, v-w, v, u) + a(v,u) & \\ = b(u,\pi) + (b,u) + \int_{\Gamma_O} \pi_{ref}\, n \cdot u\; dS, & \\ (\frac{D^{\mathscr{A}}}{Dt}\,\rho, q) + e(\rho, v, q) + \alpha(v-w, \rho, q) = 0, & \end{aligned} \tag{3.7}$$

where we put the so-called soft boundary condition $(3.2)_2$.

3.5 Boundary conditions

We assume that for each $t \in [0,T]$ there exists $v^* \in H^1(\Omega_t)^2$, such that

$$\begin{aligned} v^*(x,t) &= v_D(x,t), \quad x \in \Gamma_I \cup \Gamma_W \\ v^*(x,t) &= w(x,t), \quad\;\; x \in \Gamma_{W_t} \end{aligned}$$

(in the sense of traces). Then the *weak formulation* reads:

- Find v, such that $v - v^* \in V$; $\rho \in Q$
- equation $(3.7)_1$ is satisfied $\forall u \in V$.

[3]Denotation from Feistauer et al. [fe, p. 368].

The boundary condition for the density ρ prescribed on inlet Γ_I is formulated in the so-called weak integral sense[4]

$$\begin{aligned}(\frac{D^{\mathscr{A}}}{Dt}\,\rho, q) + e(\rho, v, q) + \alpha(v - w, \rho, q) - \gamma \int_{\Gamma_I} \rho v_D \cdot nq \, dS = \\ - \gamma \int_{\Gamma_I} \rho_D v_D \cdot nq \, dS \qquad \forall q \in Q,\end{aligned}$$

where γ is a suitable parameter.

3.6 Discrete problem

Let $\{\mathscr{T}_h\}_{h\in(0,T)}$ be a regular system of triangulations of the domain $\widetilde{\Omega} = \Omega_t \times \{t\}$. In a time interval $[0, T]$ we construct a partition $t_n = n\tau, n = 0, \dots, r$ with time step τ. For a function f defined in $\widetilde{\Omega}$, we set

$$\begin{aligned}\frac{D^{\mathscr{A}}}{Dt}\, f(y_n, t_n) &= \frac{\partial}{\partial t}\,\tilde{f}(X, t_n) \\ &\approx (\tilde{f}(X, t_n) - \tilde{f}(X, t_{n-1}))/\tau \\ &= (f(y_n, t_n) - f(y_{n-1}, t_{n-1}))/\tau,\end{aligned}$$

where $y_n = \mathscr{A}_{t_n}(X)$.

For simplicity, we will write $f^n = f(y_n, t_n)$ and $d_{\mathscr{A}_t} f^n = (f^n - f^{n-1})/\tau$.

The approximate solution will be sought at each time level t_n in finite dimensional spaces of finite elements X_h and Q_h.
We set $Q_h = X_h^{(m)}$, $X_h = [X_h^{(k)}]^2$, $V_h = \{v_h \in [X_h^{(k)}]^2; v_h|_{\Gamma_D} = 0\}$, where $X_h^{(p)} = \{v_h \in C(\bar{\Omega}_h); v_h|_K \in P^p(K)\ \forall K \in \mathscr{T}_h\}$ and $P^p(K)$ is a set of all polynomials on K of degree $\leq p$.

First, we approximate the spaces V and Q by V_h and Q_h respectively. We use the approximations

$$\begin{aligned}v^n &\approx v_h^n \in V_h, \\ \rho^n &\approx \rho_h^n \in Q_h, \\ \frac{D^{\mathscr{A}}}{Dt}\, v^n &\approx (v^n - v^{n-1})/\tau \approx (v_h^n - v_h^{n-1})/\tau = d_{\mathscr{A}_t} v_h^n, \\ \frac{D^{\mathscr{A}}}{Dt}\, \rho^n &\approx (\rho^n - \rho^{n-1})/\tau \approx (\rho_h^n - \rho_h^{n-1})/\tau = d_{\mathscr{A}_t} \rho_h^n\end{aligned}$$

[4]See Feistauer et al. [fe, p. 373].

Moreover, we will use the streamline diffusion test function

$$q_h + \delta q_{h\beta} \quad \text{with } q_{h\beta} = (v_h^{n-1}, \operatorname{grad} q_h)$$

for suitable constant $\delta > 0$, which will be used instead of q_h to avoid Gibb's phenomenon in the numerical solution[5].

Let $v_h^* \in X_h$ be the approximation of v^*, we can use the approximation

$$\begin{array}{ll} v_h^*(P_i,t) = v_D(P_i,t) & \forall P_i \in \Gamma_I \cup \Gamma_W, \\ v_h^*(P_i,t) = w(P_i,t) & \forall P_i \in \Gamma_{W_t}, \\ v_h^*(P_i,t) = 0 & \forall P_i \in \Omega_t. \end{array}$$

We obtain the following formulation of the discrete problem:
Find $v_h^n \in X_h$, such that $v_h^n - v_h^{*n} \in V_h$; $\rho_h^n \in Q_h$ and the following equations holds:

$$\begin{aligned} &(\rho_h^{n-1}\, d_{\mathscr{A}_t} v_h^n, u_h) + d(\rho_h^{n-1}, v_h^{n-1} - w_h^{n-1}, v_h^n, u_h) + a(v_h^n, u_h) \\ &\qquad = b(u_h, \pi_h^{n-1}) + (b_h^{n-1}, u_h) + \int_{\Gamma_O} \pi_{ref}\, u_h \cdot n \, dS \qquad \forall u_h \in V_h, \\ &(d_{\mathscr{A}_t} \rho_h^n, q_h) + e(\rho_h^{n-1}, v_h^n, q_h + \delta q_{h\beta}) \\ &\qquad + \alpha(v_h^{n-1} - w_h^{n-1}, \rho_h^n, q_h + \delta q_{h\beta}) - \gamma \int_{\Gamma_I} \rho_h^n v_D^n \cdot n q_h \, dS \\ &\qquad = -\gamma \int_{\Gamma_I} \rho_D^n v_D^n \cdot n q_h \, dS \qquad \forall q_h \in Q_h, \\ &\qquad \pi_h^n = \widehat{\pi}(\rho_h^n). \end{aligned} \tag{3.8}$$

We can write

$$v_h^n = v_h^{*n} + z_h^n, \quad \text{with } z_h^n \in V_h.$$

Assuming that $u_h^{n-1}, \rho_h^{n-1}, \pi_h^{n-1}, w_h^{n-1}$ are known, using substitution for v_h^n, we get a linear system for parameters determining the unknown functions z_h^n and ρ_h^n.

System (3.8) can be solved in two separate steps. First, we find v_h^n by solving the first equation. Using the result, we can find ρ_h^n by solving the second one.

[5] See Feistauer et al. [fe, p. 346]

Chapter 4

Existence of approximate solution

Assume homogenous Dirichlet condition $v_D = 0$ given on the whole boundary $\partial\Omega$. We obtain equations

$$\begin{aligned}
&(\rho_h^{n-1}\, d_{\mathscr{A}_t}\, v_h^n, u_h) + d(\rho_h^{n-1}, v_h^{n-1} - w_h^{n-1}, v_h^n, u_h) + a(v_h^n, u_h) \\
&\qquad = b(u_h, \pi_h^{n-1}) + (b_h^{n-1}, u_h) \qquad \forall u_h \in V_h, \\
&(d_{\mathscr{A}_t}\, \rho_h^n, q_h) + e(\rho_h^{n-1}, v_h^n, q_h + \delta q_{h\beta}) \\
&\qquad + \alpha(v_h^{n-1} - w_h^{n-1}, \rho_h^n, q_h + \delta q_{h\beta}) = 0 \qquad \forall q_h \in Q_h.
\end{aligned} \tag{4.1}$$

For this problem, we will prove existence of approximate solution on next time level under assumptions of existence of approximate solution from previous time level and constraints for time step τ and for constant δ.

Theorem[1] Let v_h^{n-1},ρ_h^{n-1} be approximate solution in time level t_{n-1}, such that $\rho_h^{n-1} \geq \rho_0$, where $\rho_0 > 0$ is a positive constant. Denote

$$K_{n-1} = \max\{\|v_h^{n-1}\|_\infty, \|v_h^{n-1} - w_h^{n-1}\|_\infty, \|\rho_h^{n-1}\|_\infty\}. \tag{4.2}$$

Further, let

$$\tau \leq \frac{\mu\rho_0}{2K_{n-1}^4}, \quad \frac{3}{2}\tau \leq \delta \leq \frac{\mu}{4\,N\,K_{n-1}^2}. \tag{4.3}$$

Then there exists a unique solution v_h^n, ρ_h^n of problem (4.1) on time level t_n.

Proof. Denote forms

[1]See. Feistauer et al. [fe, p. 371].

$$
\begin{aligned}
\tilde{a}(v,\rho,u,q) &= \frac{1}{\tau}\,(\rho_h^{n-1}\,v,u) + d(\rho_h^{n-1}, v_h^{n-1} - w_h^{n-1}, v, u) + a(v,u) \\
&\quad + \frac{1}{\tau}\,(\rho,q) + e(\rho_h^{n-1}, v, q + \delta q_\beta) + \alpha(v_h^{n-1} - w_h^{n-1}, \rho, q + \delta q_\beta)\,, \qquad (4.4) \\
F(u,q) &= b(u,\pi_h^{n-1}) + (b_h^{n-1}, u) + \frac{1}{\tau}\,(\rho_h^{n-1} v_h^{n-1}, u) + \frac{1}{\tau}\,(\rho_h^{n-1}, q)
\end{aligned}
$$

Problem (4.1) with unknowns $v = v_h^n \in V_h,\ \rho = \rho_h^n \in Q_h$ can be written in a form

$$
\tilde{a}(v,\rho,u,q) = F(u,q) \quad \forall u \in V_h, \forall q \in Q_h. \tag{4.5}
$$

To prove existence and uniqueness, we show that $\tilde{a}$ is positively definite.

We will use Cauchy's inequality and Young's inequality in a form $\alpha\beta \leq \varepsilon\alpha^2 + \beta^2/(4\varepsilon)$. For arbitrary $\varepsilon_1, \ldots \varepsilon_4 > 0$ we can write

$$
\begin{aligned}
\frac{1}{\tau}\,(\rho_h^{n-1} v, v) &\geq \frac{\rho_0}{\tau}\,\|v\|^2\,, \\
|d(\rho_h^{n-1}, v_h^{n-1} - w_h^{n-1}, v, v)| &\leq \|\rho_h^{n-1}(v_h^{n-1} - w_h^{n-1})\|_\infty\,\|\mathrm{grad}\, v\|\,\|v\| \\
&\leq \varepsilon_1 \|\mathrm{grad}\, v\|^2 + \frac{K_{n-1}^4}{\varepsilon_1}\,\|v\|^2\,, \\
|\alpha(v_h^{n-1} - w_h^{n-1}, \rho, \rho + \delta\rho_\beta)| &= |(\rho_\beta, \rho) + \delta\|\rho_\beta\|^2| \\
&\leq \varepsilon_2 \|\rho_\beta\|^2 + \frac{1}{4\varepsilon_2}\,\|\rho\|^2 + \delta\|\rho_\beta\|^2 \\
|e(\rho_h^{n-1}, v, \rho + \delta\rho_\beta)| &\leq \varepsilon_3 \|\mathrm{grad}\, v\|^2 + \frac{N\,K_{n-1}^2}{4\varepsilon_3}\,\|\rho\|^2 \\
&\quad + \varepsilon_4 \|\mathrm{grad}\, v\|^2 + \frac{N\,K_{n-1}^2 \delta^2}{4\varepsilon_4}\|\rho_\beta\|^2.
\end{aligned} \tag{4.6}
$$

Putting previous estimates together, we get

$$
\begin{aligned}
\tilde{a}(v,\rho,v,\rho) \geq &\left(\frac{\rho_0}{\tau} - \frac{K_{n-1}^4}{4\varepsilon_1}\right)\|v\|^2 \\
&+ (\mu - \varepsilon_1 - \varepsilon_3 - \varepsilon_4)\|\mathrm{grad}\, v\|^2 + (\lambda + \mu)\|\mathrm{div}\, v\|^2 \\
&+ \left(\delta - \varepsilon_2 - \frac{N\,\delta^2\,K_{n-1}^2}{4\varepsilon_4}\right)\|\rho_\beta\|^2 \\
&+ \left(\frac{1}{\tau} - \frac{1}{4\varepsilon_2} - \frac{N\,K_{n-1}^2}{4\varepsilon_3}\right)\|\rho\|^2.
\end{aligned}
$$

Let $\varepsilon_i = \mu/4$ for $i = 1, 3, 4$, $\varepsilon_2 = \delta/2$ and using (4.3) we obtain

$$\begin{aligned}\tilde{a}(v, \rho, v, \rho) \geq & \frac{\rho_0}{2\tau}\|v\|^2 + \frac{\mu}{4}\|\operatorname{grad} v\|^2 + (\lambda + \mu)\|\operatorname{div} v\|^2 \\ & + \frac{\delta}{4}\|\rho_\beta\|^2 + \frac{1}{2\tau}\|\rho\|^2.\end{aligned}$$

Now, it is obvious that form $\tilde{a}$ is positively definite. Thus problem (4.5) has exactly one solution.

□

Let us assume generally nonzero Dirichlet boundary condition for velocity given on the whole boundary $\partial\Omega$. Problem (3.8) will have a form

$$\begin{aligned}&(\rho_h^{n-1}\, d_{\mathscr{A}_t} v_h^n, u_h) + d(\rho_h^{n-1}, v_h^{n-1} - w_h^{n-1}, v_h^n, u_h) + a(v_h^n, u_h) \\ &\qquad = b(u_h, \pi_h^{n-1}) + (b_h^{n-1}, u_h) \qquad \forall u_h \in V_h, \\ &\quad (d_{\mathscr{A}_t} \rho_h^n, q_h) + e(\rho_h^{n-1}, v_h^n, q_h + \delta q_{h\beta}) \\ &\qquad + \alpha(v_h^{n-1} - w_h^{n-1}, \rho_h^n, q_h + \delta q_{h\beta}) - \gamma \int_{\Gamma_I} \rho_h^n v_D^n \cdot n q_h \, dS \\ &\qquad = -\gamma \int_{\Gamma_I} \rho_D^n v_D^n \cdot n q_h \, dS \qquad \forall q_h \in Q_h, \\ &\qquad \pi_h^n = \widehat{\pi}(\rho_h^n),\end{aligned} \tag{4.7}$$

Let $v^* \in H^1(\Omega_t)^2, v^*|_{\partial\Omega} = v_D$ be a realization of the boundary condition, we find solution $v = v_h^n, \rho = \rho_h^n$, such that $v - v^* \in V_h, \rho \in Q_h$. Let $v = v^* + z$ for $z \in V_h$, we may write problem (4.7) in a form

$$\begin{aligned}&\tilde{a}(z + v^*, \rho, u, q) - \gamma \int_{\Gamma_I} \rho v_D^n \cdot n q \, dS \\ &\qquad = F(u, q) - \gamma \int_{\Gamma_I} \rho_D^n v_D^n \cdot n q \, dS \quad \forall u \in V_h, \forall q \in Q_h,\end{aligned} \tag{4.8}$$

where we use denotation from the proof of previous theorem.

Define following forms

$$\begin{aligned}\hat{a}(z, \rho, u, q) &= \tilde{a}(z, \rho, u, q) - \gamma \int_{\Gamma_I} \rho v_D^n \cdot n q \, dS, \\ \hat{F}(u, q) &= F(u, q) - \tilde{a}(v^*, \rho, u, q) - \gamma \int_{\Gamma_I} \rho_D^n v_D^n \cdot n q \, dS.\end{aligned}$$

Problem (4.8) will have this form

$$\hat{a}(z, \rho, u, q) = \hat{F}(u, q) \quad \forall u \in V_h, \forall q \in Q_h \tag{4.9}$$

for unknowns $z \in V_h$ and $\rho \in Q_h$.

Theorem[2] Let assumptions from previous theorem hold. Moreover, assume $\gamma > 0$ and let $v_D \cdot n < 0$ on Γ_I. Let $v^* \in H^1(\Omega_t)^2$ be the realization of boundary condition for velocity. Then, there exists a unique solution $v = v_h^n$, $\rho = \rho_h^n$ of the problem (4.7), where $v - v^* \in V_h, \rho \in Q_h$.

Proof. We need to show that $\hat{a}$ is positively definite. We may write

$$\hat{a}(z, \rho, z, \rho) = \tilde{a}(z, \rho, z, \rho) - \gamma \int_{\Gamma_I} \rho^2 v_D^n \cdot n \, dS.$$

Second term in righthand side is positive (under the assumptions of the theorem), form $\tilde{a}$ is positively definite by the previous theorem. $\square$

[2]See Feistauer et al. [fe, p. 374].

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