MA 510 / CS 522 HW #9
Due Thursday December 1^{st}

1. In class we showed that when Heun's method is used to solve the IVP $y' = f(t)$ over $[a,b]$, with $y(a) = y_o = 0$, we get the trapezoidal rule approximation for the definite integral of $f(t)$ when taken over the interval $[a,b]$.

   We derived the Runge-Kutta method of order N=4 from the perspective of a Taylor method of order 4 where we approximated each of the derivatives. Show that this is equivalent to the Simpson's approximation with step size $h/2$ for the definite integral of $f(t)$ taken over the interval $[a,b]$. That is, show:
$$y(b) \approx \frac{h}{6} \sum_{k=0}^{M-1} \left(f(t_k) + 4f(t_{k+1/2}) + f(t_{k+1})\right)$$
   where $h = (b-a)/2$ and $t_k = a + kh$, and $t_{k+1/2} = a + (k + 1/2_h$.

2. The ODE $y' = 2y/t + t^2 e^t$ on $1 \leq t \leq 2$ with $y(1) = 0$ has a solution $y(t) = t^2(e^t - e)$. Approximate the solution to this ODE with $h = 0.2$ using:

   (a) Taylor's method of order two

   (b) Runge-Kutta order 2

   (c) Implicit Euler method

   Create a plot (or plots) that include each of the approximations at each mesh point and the exact solution at each mesh point. Comment on: (1) error for each approximation, (2) computation time for each method, (3) sensitivity to time steps. In the third part, continue to increase and decrease time step h to understand conditional stability of explicit methods and unconditional stability of the Implicit (forward) Euler method.