MA 510 / CS 522 HW #4
Due in class Thursday 9/29

1. Recall the Vandermonde matrix X as defined in class, and define:

$$V_n(x) = det \begin{bmatrix} 1 & x_o & x_o^2 & \cdots & x_o^n \\ \vdots & & & & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & & x_{n-1}^n \\ 1 & x & x^2 & \cdots & x^n \end{bmatrix}$$

   (a) Show that $V_n(x)$ is a polynomial of degree n and that its roots are x_o,. . .,x_{n-1}. Obtain the formula:

   $$V_n(x) = (x - x_o) \cdots (x - x_{n-1})V_{n-1}(x_{n-1})$$

   Hint: Expand the last row of $V_n(x)$ by minors to show that $V_n(x)$ is a polynomial of degree n and to find the coefficient of the term x^n.

   (b) Show the following:

   $$det(X) = V_n(x_n) = \Pi_{0 \le j < i \le n}(x_i - x_j)$$

2. Assuming that $x_r = x_o + rh$, verify directly (from the definition) the following special cases:

$$f[x_o, x_1] = \frac{1}{h}\left(f(x_1) - f(x_o)\right)$$

$$f[x_o, x_1, x_2] = \frac{1}{2!h^2}\left(f(x_2) - 2f(x_1) + f(x_o)\right)$$

$$f[x_o, x_1, x_2, x_3] = \frac{1}{3!h^3}\left(f(x_3) - 3f(x_2) + 3f(x_1) - f(x_o)\right)$$

Note: These look very similar to approximations of derivatives that we will go over later in the course.

3. For this problem, we will be looking at the following function,

$$f(x) = \frac{1}{1 + 25x^2}$$

Create an m-file that will take n as an argument (the number of data points) and will will give P, the $(n-1)$ degree Lagrange interpolating polynomial for f generated by n equally spaced nodes on the interval $[-1, 1]$. You can generate the x values of the nodes with the command $x = linspace(-1, 1, n)$. You can then generate the $f(x)$ values with the following command:

```
f = 1./(1+25.*x.^2)
```

   (a) Generate plots of the data nodes, function f, and Lagrange interpolating polynomial P for $n = 5$, 10, 15, and 20. (Might be easiest to copy and paste each individual graph into one word file to then print out)

   (b) Comment on what is happening and also generate a table or bar graph to examine the error, where error will be defined as:

   $$E_n = \left| \int_{-1}^{1} f(x)dx - \int_{-1}^{1} P_n(x)dx \right|$$

   for $n = 5$, 10, 15, 20

   (c) What you are seeing is called Runge's phenomenon. It happens with certain functions when we use Lagrangian interpolation with evenly spaced nodes. Why does it happen? Look at $f(1)$, $f'(1)$, $f''(1)$, and $f^{(3)}(1)$. What happens to these derivatives, and what does that have to do with the error?

   (d) Ways to choose nodes such that this effect is minimized have been developed. One such choice of nodes are called Chebyshev nodes, where the nodes are not equally spaced and more nodes are closer to the endpoints. For the case $n = 15$, plot a graph of the Lagrangian interpolation using Chebyshev nodes and compare the error using Chebyshev nodes and equally spaced nodes. The Chebyshev nodes in the interval [-1,1] are:

   $$x_i = \cos\left(\frac{2i-1}{2n}\pi\right), \quad i = 1, \ldots, n$$