MA 510 / CS 522 HW #8
Due Thursday November 17^{th}

1. *Epidemic Model.* A community with L members contains $I(t)$ infected individuals at time t and $S(t)$ is the uninfected/ susceptible individuals. Assuming L is a constant, we have $S(t) = L - I(t)$. For a mild illness, such as the common cold, everyone continues to be active and the epidemic spreads from those who are infected to those who are susceptible. Since there are $I(t)S(t)$ possible contacts between these two groups, the rate of change of $I(t)$ is proportional to $I(t)S(t)$. The model can be stated as the following IVP:

$$I' = kI(L - I), \quad I(0) = I_o, \quad 0 \leq t \leq 60$$

   where t is in days.

   (a) Use Euler's method with $h = 0.2$ to approximate the solution to above IVP on [0,60] using the parameters $L = 25,000$, $k = 0.00003$, with initial condition $I_o = 250$.

   (b) Create a plot that includes the approximations at each mesh point using $h = 0.2$, 0.1, 0.05, and 0.025. Looking at $I(30)$, what order of convergence are you seeing?
   Note: There should be a different value for the infected population at $t = 30$ for the different values of h. When we did the integration error check for Trapezoid and Simpsons rule, we looked at $(Error - Exact)/h^p$ where h^p was the expected theoretical error. The error is approximately ch^p, so when we observed a constant value c in the previous homework for different h values, we could say that the error as we expected. Do the same analysis for Euler method using the solution to $h = 0.0125$ as the 'exact' solution in the calculation. Often, if we do not have an analytical solution, we can use a finer grid as a good estimate (as long as the method converges and approaches the true solution as the grid is refined).

   (c) Estimate the average number of individuals infected by finding the average solution value. Compare this to fitting a curve to the Euler solution and using mean value theorem for integrals.

2. The IVP $y' = \sqrt{y}$ has the nontrivial solution $y(x) = (x/2)^2$. Application of Euler's method however yields a numerical solution $y(x; h) = 0$ for all x and $h = x/n$, $n = 1, 2, \ldots$. Explain this paradox. What will happen when using a method other than Euler's method?

3. Use the quadratic interpolant to $y'(t) = f(t, y(t))$ at t_n, t_{n-1}, t_{n-2} to obtain the formula:

$$y(t_{n+1}) = y(t_{n-3}) + \frac{4h}{3}\left(2y'_n - y'_{n-1} + 2y'_{n-2}\right) + \mathcal{O}(h^5)$$

   for $n \geq 3$. This is the Milne method.

4. The linear system $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_o$, where A is a symmetric matrix is solved using Euler's method. In class, we discussed a single initial value problem, but one could solve a coupled system of differential equations. For example:

$$\vec{y}\,' = \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

   (a) Letting $x_j = jh$ and the error $e(x_j; h) = y(x_j) - y^*(x_j)$, i.e. the difference between the Euler approximation y and the exact value of the solution y^* for $j = 0, 1, \ldots, n$. Prove that:

$$||e(x_j; h)||_2 \leq ||\mathbf{y}_o||_2 \max_{\lambda \in \sigma(A)} |(1 + h\lambda)^j - e^{jh\lambda}|$$

   where $\sigma(A)$ is the set of eigenvalues of A and $|| \cdot ||$ is the Euclidean matrix norm.
   Hint: You might need to use Schur decomposition, norm of unitary (orthogonal for real values) matrices, and Holder's Inequality.

   (b) Demonstrate that for every $-1 \ll x \leq 0$ and $j = 0, 1, \ldots$, it is true that

$$e^{jx} - \frac{1}{2}jx^2 e^{(j-1)x} \leq (1 + x)^j \leq e^{jx}$$

   (Hint: Prove first that $1 + x \leq e^x$, $1 + x + \frac{1}{2}x^2 \geq e^x$ for all $x \leq 0$, and then argue that, provided $|a - 1|$ and $|b|$ are small, it is true that $(a - b)^n \geq a^n - na^{n-1}b$.)

(c) Suppose that the maximal eigenvalue of A is $\lambda_{max} < 0$. Prove that, as $h \to 0$, and $jh \to x \in [0, X]$

$$||e(x_j; h)||_2 \le \frac{1}{2} x \lambda^2 e^{\lambda_{max} x} ||\mathbf{y}_o||_2 h \le \frac{1}{2} X \lambda_{max}^2 ||\mathbf{y}_o||_2 h$$