MATH 114 HOMEWORK #3 SOLUTIONS

DUE APRIL 25, 2008

Problem 4.5.15. *The Vandermonde determinant. Given scalars a_j, $j = 1, \ldots, n$, the Vandermonde determinant $V(a_1, \ldots, a_n)$ is defined by*

$$V(a_1,\ldots,a_n) = \begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix}.$$

Use the following steps to compute $V(a_1, \ldots, a_n)$. Observe that

$$V(a_1,\ldots,a_n,x) = \begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}$$

is a polynomial of degree n (in x).

(a) *Prove that $V(a_1, \ldots, a_n, x) = V(a_1, \ldots, a_n) \prod_{j=1}^{n}(x - a_j)$.*

(b) *Use induction to prove $V(a_1, \ldots, a_n) = \prod_{i<j}(a_j - a_i)$. What is the rank of the matrix of $V(a_1, \ldots, a_n)$?*

Solution. (a) First suppose that $a_i = a_j$ for some $i \neq j$. Then both sides are plainly 0. Now suppose that all the a_i are distinct. Substituting in each a_i for x in $V(a_1, \ldots, a_n, x)$ gives us 0, so each a_i is a root of $V(a_1, \ldots, a_n, x)$. Thus we have found n roots of a degree n polynomial, so we must have

$$V(a_1,\ldots,a_n,x) = c\prod_{i=1}^{n}(x-a_i)$$

for some constant c. In order to solve for c, we note that the coefficient of the leading term is equal to

$$A_{n+1,n+1} = (-1)^{n+1+n+1} \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix} V(a_1,\ldots,a_n).$$

Hence we have

$$V(a_1,\dots,a_n,x) = V(a_1,\dots,a_n)\prod_{j=1}^{n}(x-a_j),$$

as desired.

(b) We work by induction. The base case, $n=1$, holds vacuously (since an empty product is 1 by definition). Now suppose the result holds up to n. Thus we have

$$V(a_1,\dots,a_n) = \prod_{1\le i<j\le n}(a_j-a_i).$$

By part (a), we have

$$\begin{aligned} V(a_1,\dots,a_n,a_{n+1}) &= V(a_1,\dots,a_n)\prod_{j=1}^{n}(a_{n+1}-a_j) \\ &= \prod_{1\le i<j\le k}(a_j-a_i)\prod_{j=1}^{n}(a_{n+1}-a_j) \\ &= \prod_{1\le i<j\le n+1}(a_j-a_i), \end{aligned}$$

as desired. To compute the rank, note that if the a_i are distinct, then $V(a_1,\dots,a_n)\neq 0$, so the matrix has full rank (i.e., the rank is n). More generally, if there are exactly m distinct a_i, which we may assume to be $a_1,\dots,a_m$ without loss of generality, then $V(a_1,\dots,a_m)\neq 0$, so the submatrix consisting of the first m rows and columns has full rank, so the rank of the full matrix has rank at least m. However, the full rank has only m distinct rows, so the rank of the matrix is at most m. Hence the rank is equal to the number of distinct a_i. ■

Problem 4.5.16. *Let α_j, $j=1,\dots,m$ be distinct and let $\mathscr{P}$ be the space of all trigonometric polynomials of the form $P(x)=\sum_{j=1}^{m}a_je^{i\alpha_jx}$.*

(a) *Prove that if $P\in\mathscr{P}$ has a zero of order m (that is, a point x_0 such that $P^{(\ell)}(x_0)=0$ for $\ell=0,\dots,m-1$), then P is identically zero.*

(b) *For every $k\in\mathbb{N}$, there exist constants $c_{k,\ell}$, $\ell=0,\dots,m-1$, such that if $P\in\mathscr{P}$, then $P^{(k)}(0)=\sum_{\ell=0}^{m-1}c_{k,\ell}P^{(\ell)}(0)$.*

(c) *Given $\{c_\ell\}$, $\ell=0,\dots,m-1$, there exists $P\in\mathscr{P}$ such that $P^{(\ell)}(0)=c_\ell$ for $0\le\ell\le m$.*

Solution. (a) First, we note that

$$P^{(\ell)}(x_0) = \sum_{j=1}^{m}a_j(i\alpha_j)^{\ell}e^{i\alpha_jx_0}.$$

Hence if x_0 is a zero of P of order m, then

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ i\alpha_1 & i\alpha_2 & \cdots & i\alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ (i\alpha_1)^{m-1} & (i\alpha_2)^{m-1} & \cdots & (i\alpha_m)^{m-1} \end{pmatrix} \begin{pmatrix} a_1 e^{i\alpha_1 x_0} \\ a_2 e^{i\alpha_2 x_0} \\ \vdots \\ a_m e^{i\alpha_m x_0} \end{pmatrix} = \begin{pmatrix} P^{(0)}(x_0) \\ P^{(1)}(x_0) \\ \vdots \quad P^{(m-1)}(x_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By Exercises 4.5.6 and 4.5.15, the matrix on the left has nonzero determinant, so $a_j e^{i\alpha_j x_0} = 0$ for all j. Since the exponential factor is never zero, we must have $a_j = 0$ for all j. Hence P is identically zero.

(b) Let

$$T = \begin{pmatrix} 1 & i\alpha_1 & \cdots & (i\alpha_1)^{m-1} \\ 1 & i\alpha_2 & \cdots & (i\alpha_2)^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & i\alpha_m & \cdots & (i\alpha_m)^{m-1} \end{pmatrix}.$$

Then $\det(T) \neq 0$, so T is invertible. Define

$$\begin{pmatrix} c_{k,0} \\ c_{k,1} \\ \vdots \\ c_{k,m-1} \end{pmatrix} = T^{-1} \begin{pmatrix} (i\alpha_1)^k \\ (i\alpha_2)^k \\ \vdots \\ (i\alpha_m)^k \end{pmatrix}.$$

Now,

$$\sum_{\ell=0}^{m-1} (i\alpha_j)^\ell c_{k,\ell} = (i\alpha_j)^k$$

for $j = 1, \ldots, m$, so

$$\begin{aligned} \sum_{\ell=0}^{m-1} c_{k,\ell} P^{(\ell)}(0) &= \sum_{\ell=0}^{m-1} c_{k,\ell} \sum_{j=1}^{m} a_j (i\alpha_j)^\ell \\ &= \sum_{j=1}^{m} a_j \left(\sum_{\ell=0}^{m-1} c_{k,\ell} (i\alpha_j)^\ell \right) \\ &= \sum_{j=1}^{m} a_j (i\alpha_j)^k \\ &= P^{(k)}(0), \end{aligned}$$

as desired.

(c) Since the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ i\alpha_1 & i\alpha_2 & \cdots & i\alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ (i\alpha_1)^{m-1} & (i\alpha_2)^{m-1} & \cdots & (i\alpha_m)^{m-1} \end{pmatrix}$$

is invertible, we can solve the following system of linear equations:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ i\alpha_1 & i\alpha_2 & \cdots & i\alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ (i\alpha_1)^{m-1} & (i\alpha_2)^{m-1} & \cdots & (i\alpha_m)^{m-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{m-1} \end{pmatrix}$$

for the a_i. Then $P(x) = \sum_{j=1}^m a_j e^{i\alpha_j x}$ satisfies $P^{(\ell)}(0) = c_\ell$ for $0 \le \ell < m$. ■

Problem 4.5.18. *Given that the matrices $B_1, B_2 \in \mathscr{M}(n; \mathbb{R})$ are similar in $\mathscr{M}(n; \mathbb{C})$, show that they are similar in $\mathscr{M}(n; \mathbb{R})$.*

Solution. We can find some invertible matrix $C \in \mathscr{M}(n; \mathbb{C})$ so that $B_2 = CB_1C^{-1}$, or $CB_1 = B_2C$. Now suppose $C = C_1 + iC_2$, where $C_1, C_2 \in \mathscr{M}(n; \mathbb{R})$. Then

$$(C_1 + iC_2)B_1 = B_2(C_1 + iC_2),$$

so

$$C_1B_1 + iC_2B_1 = B_2C_1 + iB_2C_2.$$

Equating real and imaginary parts, we have

$$C_1B_1 = B_2C_1, \qquad C_2B_1 = B_2C_2.$$

Now, $\det(C_1 + xC_2)$ is a nonzero polynomial in x (since i is not a root), so it has only finitely many complex roots, and hence only finitely many real roots. Choose x_0 so that $\det(C_1 + x_0C_2) \neq 0$, and let $D = C_1 + xC_2$. Then

$$DC_1 = C_2D,$$

and D is invertible. Hence B_1 and B_2 are also similar in $\mathscr{M}(n; \mathbb{R})$. ■

Problem 5.1.1. *If $\mathscr{W} \subset \mathscr{V}$ is T-invariant, then $\chi_T(\lambda) = \chi_{T_\mathscr{W}} \chi_{T_{\mathscr{V}/\mathscr{W}}}$.*

Solution. First note that if $\mathscr{W}$ is T-invariant, it is also $(T - \lambda)$-invariant: for $w \in \mathscr{W}$, $(T - \lambda)w = Tw - \lambda w \in \mathscr{W}$. Thus we have

$$\begin{aligned} \chi_T(\lambda) &= \det(T - \lambda) \\ &= \det(T - \lambda)_\mathscr{W} \det(T - \lambda)_{\mathscr{V}/\mathscr{W}} \\ &= \chi_{T_\mathscr{W}} \chi_{T_{\mathscr{V}/\mathscr{W}}}, \end{aligned}$$

as desired. ■

Problem 5.1.2. *Let $T \in \mathscr{L}(V)$ and let $\{v_j\}_{j=1}^k$ be eigenvectors of T corresponding to distinct eigenvalues $\{\lambda_j\}_{j=1}^k$. Prove that the set $\{v_j\}_{j=1}^k$ is linearly independent.*

Solution. First note that if $Tv = \lambda v$, then $T^\ell v = \lambda^\ell v$. Now suppose $\sum_{j=1}^k a_j v_j = 0$. Now, for $1 \leq \ell \leq k$, apply $\prod_{i \neq \ell}(T - \lambda_i)$ to this equation to get

$$\begin{aligned} 0 &= \prod_{i \neq \ell}(T - \lambda_i) \sum_{j=1}^k a_j v_j \\ &= \sum_{j=1}^k \prod_{i \neq \ell}(T - \lambda_i) a_j v_j \\ &= \prod_{i \neq \ell}(\lambda_\ell - \lambda_i) a_\ell v_\ell, \end{aligned}$$

since the other terms vanish since they are hit by some $T - \lambda_i$. Hence $a_\ell = 0$. Thus the v_j are linearly independent. ■

Problem 5.1.3. *Let $T \in \mathscr{L}(\mathscr{V})$ and assume that $\sigma(T)$ consists of $n = \dim \mathscr{V}$ distinct points. Prove that $\chi_T(T) = 0$.*

Solution. Suppose $\sigma(T) = \{\lambda_1, \ldots, \lambda_n\}$, and let $\{v_1, \ldots, v_n\}$ be corresponding eigenvectors. By Exercise 5.1.2, $\{v_1, \ldots, v_n\}$ are linearly independent, and there are n of them, so they form a basis for $\mathscr{V}$. Note that $\chi_T(x)$ is a polynomial of degree n with roots $\lambda_1, \ldots, \lambda_n$, so $\chi_T(x) = c(x - \lambda_1) \cdots (x - \lambda_n)$ for some nonzero constant c. Hence for each ℓ,

$$\chi_T(T) v_\ell = c \left(\prod_{i \neq \ell}(T - \lambda_i) \right) (T - \lambda_\ell) v_\ell = 0.$$

Hence $\chi_T(T)$ maps each element of a basis of $\mathscr{V}$ to 0, so it must be identically zero. Thus $\chi_T(T) = 0$, as desired. ■