MA 510 / CS 522 HW #6
Due Thursday November 3^{rd}
The 5th problem has been added

1. Find the discrete least squares polynomials of degrees 1, 2, and 3 for the following data: $x =$[1, 1.1, 1.3, 1.5, 1.9, 2.1], $y =$[1.84, 1.96, 2.21, 2.45, 2.94, 3.18]. Compute the error in each polynomial approximation of degree n, $E_n = \sum_{k=1}^{m}(y_i - p_n(x_i))^2$.

2. Define the means $\bar{x}$ and $\bar{y}$ for the points $\{(x_k, y_k)\}_{k=1}^{N}$ by:

$$\bar{x} = \frac{1}{N}\sum_{k=1}^{N} x_k, \qquad \bar{y} = \frac{1}{N}\sum_{k=1}^{N} y_k$$

   Show that the point ($\bar{x}$,$\bar{y}$) lies on the least-squares line determined by the given set of points.

3. **Least Squares Approximation Using Orthogonal Polynomials**

$$f(x) = e^{-x}cos(x), \qquad x \in [-1, 1]$$

   (a) Use the Legendre polynomials to find 3^{rd} and 4^{th} degree polynomial least squares approximations to the function above. (More details on Legendre Orthogonal Polynomial below)

   (b) **Plot** the function and the approximations on the same plot and comment on error.

   (c) The Legendre polynomial is only orthogonal on [-1,1]. Write another version of code that can find a least squares polynomial approximation of degree n on any interval [a,b] using a change of variables. To complete the code, use an interval $x \in [0, 2]$ and $n = 4$.

   Remember that for orthogonal polynomials, the structure of the matrix $\tilde{A}$ will be simplified. You can use this fact when writing your code.

   The Legendre polynomials ϕ_n^L are orthogonal on [-1,1] with respect to the weight function $w(x) = 1$, i.e.:

$$\int_{-1}^{1} w(x)\phi_j^L(x)\phi_k^L(x)dx = \begin{cases} 0, & \text{when } j \neq k \\ \alpha_k > 0, & \text{when } j = k \end{cases}$$

   A Legendre polynomial $\phi_n^L(x)$ can be determined by using $\phi_n^L(1) = 1$ for each n and a recursive relation is used when $n \geq 2$. In your code, use the general formula below to calculate the Legendre polynomials of any order n given $\phi_0^L = 1$ and $\phi_1^L = x - B_1$ for $x \in [-1, 1]$:

$$B_k = \frac{\int_{-1}^{1} xw(x)[\phi_{k-1}^L(x)]^2 dx}{\int_{-1}^{1} w(x)[\phi_{k-1}^L(x)]^2 dx}, \qquad \text{for } k \geq 1$$

$$C_k = \frac{\int_{-1}^{1} w(x)[\phi_{k-1}^L(x)]^2 dx}{\int_{-1}^{1} w(x)[\phi_{k-2}^L(x)]^2 dx}, \qquad \text{for } k \geq 2$$

$$\phi_k^L(x) = (x - B_k)\phi_{k-1}^L(x) - C_k\phi_{k-2}^L(x), \qquad \text{for } k \geq 2$$

   Last step: using this formula, the Legendre polynomials do not satisfy $\phi_n^L(1) = 1$ for all n. Scale each one by the proper value to ensure that this is satisfied.
   (Note: you can check the Legendre polynomials generated from your code with the known Legendre polynomials)

4. Define the *Chebyshev polynomials of the second kind* $S_n(x)$ as follows:

$$S_n(x) = \frac{1}{n+1}T'_{n+1}(x)$$

   for $n \geq 0$ and T_{n+1} the Chebyshev polynomial of degree $n + 1$.

(a) Show that $\{S_n(x) | n \geq 0\}$ is an orthogonal family on $[-1, 1]$ with respect to the weight function $w(x) = \sqrt{1 - x^2}$.

(b) Show that the family $\{S_n(x)\}$ satisfies the same triple recursion relation as the family $\{T_n(x)\}$.

5. Approximating Derivatives

   (a) Derive a method for approximating $f'''(x_o)$ whose error term is $\mathcal{O}(h^2)$, by expanding the function f in a Taylor polynomial about x_o using $x_o \pm h$, x_o, and $x_o \pm 2h$.

   (b) The partial derivative $f_x(x, y)$ of $f(x, y)$ with respect to x is obtained by holding y fixed and differentiating with respect to x. Similar for f_y when holding x fixed. Using a Taylor series expansion of a function of two variables, determine the $O(h^2)$ numerical approximation formulas and associated truncation error for f_x and f_y.