MA 510 / CS 522 HW #7
Due Thursday November 10^{th}

1. **Richardson's Extrapolation**
   Suppose that two approximations of $O(h^{2k})$ for $f'(x_o)$ are $D_{k-1}(2h)$ and $D_{k-1}(h)$ that satisfy:

$$f'(x_o) = D_{k-1}(2h) + c_1(2)^{2k}h^{2k} + c_2(2)^{2k+2}h^{2k+2} + \cdots$$

$$f'(x_o) = D_{k-1}(h) + c_1 h^{2k} + c_2 h^{2k+2} + \cdots$$

   Then an improved approximation has the form $f'(x_o) = D_k(h) + O(h^{2k+2})$, where $D_k(h)$ is a linear combination of $D_{k-1}(2h)$ and $D_{k-1}(h)$. Note that a spatial change of $2h$ to h can be rewritten as h to $h/2$.

   Here, we wish to approximate $f'(x)$ numerically by generating a table of approximations $D(j,k)$ for $k \le j$ using $f'(x) \sim D(n,n)$ as the final answer. The approximations $D(j,k)$ are stored in a lower-triangular matrix. The first column is:

$$D(j,0) = \frac{f(x + 2^{-j}h) - f(x - 2^{-j}h)}{2^{-j+1}h}$$

   and the elements in row j are:

$$D(j,k) = D(j,k-1) + \frac{D(j,k-1) - D(j-1,k-1)}{4^k - 1}, \qquad \text{for } 1 \le k \le j$$

   Using Richardson Extrapolation to determine a higher order derivative approximation
   Use Richardson extrapolation to evaluate $f'(x)$ for the function $f(x) = sin(cos(1/x))$ at $x = 1/\sqrt{2}$. Stop computing higher derivative approximations when $|D(j,k) - f'(1/\sqrt{2})| < TOL = 10^-13$. Use a starting value of $h = 0.1$.

2. Composite Integration $\int_a^b f(x)dx$

   (a) Create function files for composite trapezoid and composite Simpson's rules. The main file should be able to call these functions that have an input including the function along with the upper bound (b), lower bound a and either n or h.

   (b) Approximate the following integrals:

$$\int_0^5 \sqrt{1+x^2}dx$$

$$\int_0^\pi 2 + \cos(x)\sin(3x)dx$$

   Note that the exact value for the first integral is $(5/2)\sqrt{26} - (1/2)\ln(-5 + \sqrt{26})$ and the exact value for the second is 2π.

   (c) Create Tables similar to Table 1 and Table 2 to describe error and convergence. Discuss the behavior of the error with regards to varying the total number of intervals n or $NTot$ (or as h decreases by a factor of 2).

Table 1: Errors for Numerical Integration

Ntot	Trap	Simp
10		
20		
40		
80		
160		

Table 2: Order of Convergence for Numerical Integration

NTot	(trap error)/h^2	(Simp error)/h^4
10		
20		
40		
80		
160		

3. Determine constants a, b, c d, and e that will produce a quadrature formula as follows that has degree of precision 4.

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(0) + cf(1) + df'(-1) + ef'(1)$$